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On the State-Space Analysis in the Synthesis of Time-Optimal Control for a Class of Linear Systems

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***Abstract*—The paper deals with a new approach for synthesis of time-optimal control for a class of linear systems. It is based on the decomposition of the time-optimal control problem into a class of decreasing order problems, and the properties and relations between problems within this class. First, the problems state-spaces properties are analyzed, and then the optimal control is obtained by using a multi-stage procedure avoiding the switching hyper surface description. The emphasis in this paper is on the state-space analysis stage of the approach proposed.**

I. INTRODUCTION

THE linear time-optimal control problem has a half-a-century history. Fundamental theoretical results have been obtained and a great number of papers have been published in this field. However, in the last decade the interest towards this problem considerably declines. It may be stated that despite the more than 40-year intensive research, the synthesis of time-optimal control for high order systems is still an open problem. An approach to go further in the solution of the time-optimal synthesis problem is to refine the well-known state-space method, removing the factors that restrict its application to low order systems only. Some new state-space properties of a class of linear systems make possible to develop an efficient time-optimal synthesis approach requiring no description of the switching hyper surface [1], [2]. This paper presents the main results in the state-space analysis of the considered class of time-optimal control problems.

The following time-optimal synthesis problem for a linear system of order $\boldsymbol{k}$ is considered. The system is described by

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$$
\begin{aligned}
&\dot{\boldsymbol{x}}_k = A_k \boldsymbol{x}_k + B_k u_k,\\
&\boldsymbol{x}_k = \begin{bmatrix} x_1 & x_2 & . & . & . & x_k \end{bmatrix}^{\mathrm{T}},\ \boldsymbol{x}_k \in R^k,\\
&A_k = \mathrm{diag}(\lambda_1, \quad \lambda_2, \quad . \quad . \quad . \quad , \lambda_k),\\
&\qquad \lambda_i \in R,\ \lambda_i \le 0,\ {\scriptstyle i,\, j\, =\, \overline{1,k}},\ \lambda_i \ne \lambda_j \ \text{if} \ {\scriptstyle i\,\ne\, j},\\
&B_k = \begin{bmatrix} b_1 & b_2 & . & . & . & b_k \end{bmatrix}^{\mathrm{T}},\ b_i \in R,\ b_i \ne 0,\ {\scriptstyle i\,=\,\overline{1,k}},\\
&\qquad\qquad \overline{1,k} = 1, 2, \ \ldots, k.
\end{aligned}
\tag{1}
$$

The initial and the target states of the system are

$$\boldsymbol{x}_k(0) = \begin{bmatrix} x_{10} & x_{20} & . & . & . & x_{k0} \end{bmatrix}^{\mathrm{T}} \tag{2}$$

and

$$\boldsymbol{x}_k(t_{kf}) = [\underbrace{0 \quad 0 \quad . \quad . \quad . \quad 0}_{k}]^{\mathrm{T}} \tag{3}$$

where t_{kf} is unspecified. The admissible control $u_k(t)$ is a piecewise continuous function that takes its values from the range

$$-u_0 \le u_k(t) \le u_0,\ u_0 = const > 0\,. \tag{4}$$

We suppose that $u_k(t)$ is continuous on the boundary of the set of allowed values (4) and in the points of discontinuity τ we have

$$u(\tau) = u(\tau + 0)\,. \tag{5}$$

The problem is to find an admissible control $u_k = u_k(\boldsymbol{x}_k)$ that transfers the system (1) from its initial state (2) to the target state (3) in minimum time, i.e. minimizing the performance index

$$J_k = \int_0^{t_{kf}} dt = t_{kf}\,. \tag{6}$$

We shall refer to this problem as **Problem *A(k)*** and to the set {Problem *A(n)*, Problem *A(n-1)*, ..., Problem *A(1)*}, $n \ge 2$, as **class of problems *A(n), A(n-1), …, A(1)***.

The following relations exist between the systems of Problem *A(k)* and Problem *A(k-1)*, $k = \overline{n,2}$:

$$A_k = \left[\begin{array}{c:c} A_{k-1} & 0_{((k-1)\times 1)} \\ \hdashline 0_{(1\times(k-1))} & \lambda_k \end{array}\right],\ B_k = \left[\begin{array}{c} B_{k-1} \\ \hdashline b_k \end{array}\right],\ \boldsymbol{x}_k(0) = \left[\begin{array}{c} \boldsymbol{x}_{k-1}(0) \\ \hdashline x_{k0} \end{array}\right]. \tag{7}$$

For Problem *A(k)*, $k = \overline{n,1}$, denote:

$u_k^o(t)$ - the optimal control which is a piecewise constant function taking the values $+u_0$ or $-u_0$ and having at most $(k-1)$ discontinuities [3]-[6];

t_{kf}^o - the minimum of the performance index;

L_{kk-1} - the set of all state space points for which the optimal control has no more than $(k-2)$ discontinuities;

S_k - the switching hyper surface. Note that S_k is time-invariant and includes the state space origin. As it is well known, the switching hyper surface S_k is identical with the set L_{kk-1} [4, ch. 14].

II. Preliminary Results

In this section we present some preliminary results proved in [1], [2], along with the idea of the proposed approach.

Let $k \geq 2$. Suppose we are in the initial point $\boldsymbol{x}_k(0)$ of the Problem *A(k)* state-space and the obviously easier Problem *A(k-1)* has been solved, i.e. we have the optimal control $u_{k-1}^o(t)$ and the minimum of the performance index t_{k-1f}^o of Problem *A(k-1)*. Applying the optimal control $u_{k-1}^o(t)$ of Problem *A(k-1)* to the system of Problem *A(k)* with initial state $\boldsymbol{x}_k(0)$ we obtain the trajectory

$$\boldsymbol{x}_k(t) = e^{A_k t}\boldsymbol{x}_k(0) + \int_0^t e^{A_k(t-\tau)} B_k u_{k-1}^o(\tau)d\tau,\ t \in [0,\ t_{k-1f}^o]. \quad (8)$$

The following result is valid for this trajectory [1], [2].

Theorem 1. *The state trajectory of system (1) starting from the initial point* $\boldsymbol{x}_k(0)$ *and generated by the optimal control* $u_{k-1}^o(t)$, $t \in [0,\ t_{k-1f}^o]$, *either entirely lies on the switching hyper surface* S_k, *or is above or below* S_k, *nowhere intersecting it.*

According to this theorem all points of trajectory (8) have the same relation to the switching hyper surface S_k of Problem *A(k)*, including the initial point $\boldsymbol{x}_k(0)$ and the final point

$$\boldsymbol{x}_k(t_{k-1f}^o) = e^{A_k t_{k-1f}^o}\boldsymbol{x}_k(0) + \int_0^{t_{k-1f}^o} e^{A_k(t_{k-1f}^o-\tau)} B_k u_{k-1}^o(\tau)d\tau. \quad (9)$$

It is shown in [1], [2] that

$$\boldsymbol{x}_k(t_{k-1f}^o) \in Ox_k, \quad (10)$$

and its last, *k*th coordinate denoted by x_{kw} is given by

$$x_{kw} = e^{\lambda_k t_{k-1f}^o} x_{k0} + \int_0^{t_{k-1f}^o} e^{\lambda_k(t_{k-1f}^o-\tau)} b_k u_{k-1}^o(\tau)d\tau,\quad k=\overline{n,2}. \quad (11)$$

Another property of the class *A(n), A(n-1), ..., A(1)* is also studied in [1], [2], which makes possible the synthesis of optimal control for Problem *A(k)*, $k=\overline{n,2}$.

Theorem 2. *There exists no piecewise constant control* $u(t)$ *with an amplitude* u_0 *and* ***k*** *non zero intervals of constancy,* $1 \leq k \leq (n-1)$, *transferring the system*

$$\dot{x}_i = \lambda_i x_i + b_i u,\ \lambda_i \in R,\ b_i \in R,\ i,j=\overline{1,n},\\ b_i \neq 0,\ \lambda_i \neq \lambda_j \ \text{ when } i \neq j \quad (12)$$

from any point of any axis $Ox_1, Ox_2, \ldots, Ox_n$ *in the system state space to the origin O, and vice-versa – from the origin O to a point of any axis* $Ox_1, Ox_2, \ldots, Ox_n$ *in the state space.*

From this theorem and the properties of the switching hyper surface S_k it follows

Corollary 1 [1], [2]. *The unique time optimal control that transfers the system of Problem A(k), where* $n \geq k \geq 2$, *from every point of the positive or negative part of any state space axis* $Ox_1, Ox_2, \ldots, Ox_k$ *to the origin O, has exactly* ***k*** *non zero intervals of constancy, and the positive, respectively the negative, part of any axis* $Ox_1, Ox_2, \ldots, Ox_k$ *is above or below the switching hyper surface* S_k.

In accordance with this corollary the term $x_{k+} \in \{-1,\ +1\}$, $k=\overline{2,n}$, is introduced in [1], [2] to indicate the relation of the axis Ox_k to the switching hyper surface S_k and the optimal control values for the points of the positive and negative semi-axis Ox_k. Thus for

$$\boldsymbol{x}_k(0) = [x_{10}\quad x_{20}\quad .\ .\ .\quad x_{k0}]^{\mathrm{T}} = \\ = [\underbrace{0\quad 0\quad .\ .\ .\quad 0}_{k-1}\ x_{k0}]^{\mathrm{T}},\quad x_{k0} : \mathrm{sign}(x_{k0}) = x_{k+}.$$

we have $u_k^o(0) = u_0$.

The time-optimal synthesis problem for the initial point $\boldsymbol{x}_k(0)$ can be solved based on the solution of problem *A(k-1)* and the relation of the final point (9) of trajectory (8) to the switching hyper surface S_k [1], [2].

Theorem 3. *If the solution of Problem A(k-1),* $k=\overline{n,2}$, *is found, then the optimal control of Problem A(k) for initial state* $\boldsymbol{x}_k(0)$ *can be determined as*

$$u_k^o(0) = u_k(\boldsymbol{x}_k(0)) = \begin{cases} +u_0 & \text{if } \ x_{k+}x_{kw} > 0 \\ u_{k-1}^o(0) & \text{if } \ x_{k+}x_{kw} = 0 \\ -u_0 & \text{if } \ x_{k+}x_{kw} < 0 \end{cases}, \quad (13)$$

where x_{kw} *is given by (11).*

Based on this theorem, the following time-optimal synthesis algorithm is proposed [1], [2].

A. Algorithm for synthesis of optimal control for the initial state of Problem A(k), $k=\overline{n,2}$

Step 1. Solve Problem *A(k-1)* to find $u_{k-1}^o(t)$ and t_{k-1f}^o

Step 2. Compute x_{kw} from (11)

Step 3. Determine $u_k^o(0) = u_k(\boldsymbol{x}_k(0))$ according to (13).

If $x_{kw} = 0$, the solution of Problem *A*(*k*-1) is also the solution of Problem *A(k)*, i.e. $u_{k-1}^o(t) = u_k^o(t)$, $t_{k-1f}^o = t_{kf}^o$, and vice-versa: if Problem *A(k)* and Problem *A(k-1)* have the same solution, i.e. $u_{k-1}^o(t)=u_k^o(t)$, $t_{k-1f}^o=t_{kf}^o$, then $x_{kw}=0$. Depending on the value of x_{kw}, there are three possibilities:

- all $\boldsymbol{x}_k(0)$ for which $x_{kw}=0$ lie on the switching hyper surface S_k;
- all $\boldsymbol{x}_k(0)$ corresponding to $x_{kw}>0$ are above or below S_k and the optimal control for these points is $x_{k+}u_0$;
- all $\boldsymbol{x}_k(0)$ for which $x_{kw}<0$ are also above or below S_k, but in opposite to the area for $x_{kw}>0$, and the corresponding optimal control is $(-1)x_{k+}u_0$.

Thus, the solution of the optimal control problem *A(n)* is reduced to the solution of the sub-problem *A(n-1)*. In turn, the solution of Problem *A(n-1)* requires the solution of Problem *A(n-2)*, etc. reaching Problem *A(1)*. The final solution requires turning back to Problem *A(n)* using the obtained $x_{n+}, x_{n-1+}, \ldots, x_{2+}$. The determination of these quantities, called "axes initialization", is the first stage of the state-space analysis.

For solving the axes initialization problem we need the following result [2].

Theorem 4. *There exists no piecewise constant control* $u(t)$ *with amplitude* u_0 *and* $\boldsymbol{k}$ $(1\le k\le n)$ *non-zero constancy intervals, transferring the system (12) from the state space origin O to the same state space origin.*

III. MAIN RESULT

We shall show that in the state-space of Problem *A(k)*, $k\ge 2$, there exists a countless set of points such that if the initial point of Problem *A(k)* belongs to this set and we have the solution of Problem *A(k-1)*, then it is possible to determine the relation of the coordinate axis Ox_k to the switching hyper surface S_k.

Denote by l_k^o, $0\le l_k^o\le k$, the number of non-zero constancy intervals of the optimal control $u_k^o(t)$ of Problem *A(k)*, $k=\overline{n,1}$.

The basic result for the axes initialization can be formulated in the following way [2].

Theorem 5. *If the initial state of Problem A(k),* $2\le k\le n$*, is*

$$\boldsymbol{x}_k(0)=\begin{bmatrix}x_{10} & x_{20} & . & . & . & x_{k0}\end{bmatrix}^{\mathrm{T}}= \\ =\int_0^{t_0} e^{A_k(t_0-\tau)}B_k u_0 d\tau,\ 0<t_0<\infty, \tag{14}$$

and the solution $\{u_{k-1}^o(t),\ t_{k-1f}^o\}$ *of Problem A(k-1) is found, then:*

1. *a)* $u_{k-1}^o(t)$ *has exactly* $(k-1)$ *non-zero constancy intervals and its value for the initial state is* $-u_0$*, i.e.*

$$u_{k-1}^o(t):\ l_{k-1}^o=k-1,\quad u_{k-1}^o(0)=-u_0; \tag{15}$$

b) $u_k^o(t)$ *has exactly k non-zero constancy intervals and its value for the initial state is* $-u_0$*, i.e.*

$$u_k^o(t):\ l_k^o=k,\quad u_k^o(0)=-u_0; \tag{16}$$

2.

$$\boldsymbol{x}_k^{o+}=-\int_0^{t_0+t_{k-1f}^o} e^{-A_k\tau}B_k u_k^{o+}(\tau)d\tau \tag{17}$$

where

$$u_k^{o+}(t)=\begin{cases} u_0 & \text{when} \quad 0\le t<t_0, \\ u_{k-1}^o(t-t_0) & \text{when} \quad t_0\le t\le t_0+t_{k-1f}^o, \end{cases} \tag{18}$$

is a non-zero point of the coordinate axis Ox_k*;*

3. *a)* x_{k+} *can be determined as*

$$x_{k+}=\operatorname{sign}(x_k^{o+}) \tag{19}$$

or as

$$x_{k+}=\operatorname{sign}(-x_{kw})=-\operatorname{sign}(x_{kw}) \tag{20}$$

where x_k^{o+} *is the* ***k****th coordinate of the point* $\boldsymbol{x}_k^{o+}$ *and* x_{kw} *is the* ***k****th coordinate of (9) given by (11)*

b)

$$\frac{x_k^{o+}}{-x_{kw}}=\begin{cases} 1 & \text{when } \lambda_k=0, \\ e^{-\lambda_k(t_0+t_{k-1f}^o)}>\max\{1,-\lambda_k t_0\} & \text{when } \lambda_k<0. \end{cases} \tag{21}$$

Proof. To prove the first part of the theorem, let analyze (14). Taking into account the relations (7) between Problem *A(k)* and Problem *A(k-1)* we have

$$\boldsymbol{x}_k(0)=\begin{bmatrix}x_{10} & x_{20} & . & . & . & x_{k0}\end{bmatrix}^{\mathrm{T}}=\int_0^{t_0} e^{A_k(t_0-\tau)}B_k u_0 d\tau= \\ =\int_0^{t_0} e^{\begin{bmatrix}A_{k-1} & 0\\ 0 & \lambda_k\end{bmatrix}(t_0-\tau)}\begin{bmatrix}B_{k-1}\\ b_k\end{bmatrix}u_0 d\tau=\int_0^{t_0}\begin{bmatrix}e^{A_{k-1}(t_0-\tau)}B_{k-1}\\ e^{\lambda_k(t_0-\tau)}b_k\end{bmatrix}u_0 d\tau= \\ =\begin{bmatrix}\int_0^{t_0} e^{A_{k-1}(t_0-\tau)}B_{k-1}u_0 d\tau\\ \int_0^{t_0} e^{\lambda_k(t_0-\tau)}b_k u_0 d\tau\end{bmatrix}=\begin{bmatrix}\boldsymbol{x}_{k-1}(0)\\ x_{k0}\end{bmatrix}, \tag{22}$$

$$0<t_0<\infty,\quad 2\le k\le n.$$

Thus, the initial state of Problem *A(k-1)* is

$$\boldsymbol{x}_{k-1}(0)=\begin{bmatrix}x_{10} & x_{20} & . & . & . & x_{k-10}\end{bmatrix}^{\mathrm{T}}= \\ =\int_0^{t_0} e^{A_{k-1}(t_0-\tau)}B_{k-1}u_0 d\tau,\quad 0<t_0<\infty,\ 2\le k\le n. \tag{23}$$

Assume the optimal control $u_{k-1}^o(t)$ of Problem *A(k-1)* with initial state (23) has no $(k-1)$ non-zero constancy intervals, but at most $(k-2)$ such intervals. Then the piecewise constant function $u_k^{o+}(t)$ transfers the system of Problem *A(k-1)* from the state space origin to the same state space origin and according to the assumption has at most $[1+(k-2)]=(k-1)$ non zero constancy intervals with amplitude $u_0>0$. However, this contradicts Theorem 4 and thus the assumption made is not true. Therefore, the optimal control $u_{k-1}^o(t)$ of Problem *A(k-1)* with initial state (3.108) has exactly $(k-1)$ non-zero constancy intervals, i.e. we have

$$l_{k-1}^o = k-1. \tag{24}$$

Let assume now the value of $u_{k-1}^o(t)$ in the first constancy interval is u_0. Then $u_k^{o+}(t)$ will have again exactly $(k-1)$ non-zero constancy intervals, which contradicts Theorem 4. Therefore, the assumption is not true and

$$u_{k-1}^o(0) = -u_0. \tag{25}$$

Equations (24) and (25) give (15) and thus part 1a of the theorem is proved. Part 1b can be proved in a similar way.

Suppose now the initial state of Problem *A(k)* is

$$\boldsymbol{x}_k^{o+}(0) = \boldsymbol{x}_k^{o+} = -\int_0^{t_0+t_{k-1f}^o} e^{-A_k\tau} B_k u_k^{o+}(\tau)d\tau. \tag{26}$$

Then the control $u_k^{o+}(t)$ transfers the system of Problem *A(k)* from $\boldsymbol{x}_k^{o+}(0)$ to the state

$$\begin{aligned}\boldsymbol{x}_k^{o+}(t_0+t_{k-1f}^o) &= e^{A_k(t_0+t_{k-1f}^o)}\boldsymbol{x}_k^{o+}(0) + \\ &+ \int_0^{t_0+t_{k-1f}^o} e^{A_k(t_0+t_{k-1f}^o-\tau)} B_k u_k^{o+}(\tau)d\tau\end{aligned} \tag{27}$$

at the moment $(t_0+t_{k-1f}^o)$. Taking into account (26) we have

$$\begin{aligned}\boldsymbol{x}_k^{o+}(t_0+t_{k-1f}^o) &= e^{A_k(t_0+t_{k-1f}^o)}\left(-\int_0^{t_0+t_{k-1f}^o} e^{-A_k\tau} B_k u_k^{o+}(\tau)d\tau\right) + \\ &+ \int_0^{t_0+t_{k-1f}^o} e^{A_k(t_0+t_{k-1f}^o-\tau)} B_k u_k^{o+}(\tau)d\tau = \\ &= -\int_0^{t_0+t_{k-1f}^o} e^{A_k(t_0+t_{k-1f}^o-\tau)} B_k u_k^{o+}(\tau)d\tau + \\ &+ \int_0^{t_0+t_{k-1f}^o} e^{A_k(t_0+t_{k-1f}^o-\tau)} B_k u_k^{o+}(\tau)d\tau = 0.\end{aligned} \tag{28}$$

From (28) it follows

Corollary 2. *The control* $u_k^{o+}(t)$, *which is a piecewise constant function with **k** non-zero constancy intervals of amplitude* u_0, *transfers the system of Problem A(k) from the state* $\boldsymbol{x}_k^{o+}(0)$ *to the state space origin.*

Let express the initial state $\boldsymbol{x}_{k-1}(0)$ of Problem *A(k-1)* by the corresponding optimal control $u_{k-1}^o(t)$. We can write

$$\begin{aligned}\boldsymbol{x}_{k-1}(t_{k-1f}^o) &= e^{A_{k-1}t_{k-1f}^o}\boldsymbol{x}_{k-1}(0) + \\ &+ \int_0^{t_{k-1f}^o} e^{A_{k-1}(t_{k-1f}^o-\tau)} B_{k-1} u_{k-1}^o(\tau)d\tau = 0\end{aligned} \tag{29}$$

and therefore

$$\begin{aligned}\boldsymbol{x}_{k-1}(0) &= -e^{-A_{k-1}t_{k-1f}^o}\int_0^{t_{k-1f}^o} e^{A_{k-1}(t_{k-1f}^o-\tau)} B_{k-1} u_{k-1}^o(\tau)d\tau = \\ &= -\int_0^{t_{k-1f}^o} e^{-A_{k-1}\tau} B_{k-1} u_{k-1}^o(\tau)d\tau.\end{aligned} \tag{30}$$

We can also express (30) by the function $u_k^{o+}(t)$:

$$\begin{aligned}\boldsymbol{x}_{k-1}(0) &= -\int_0^{t_{k-1f}^o} e^{-A_{k-1}\tau} B_{k-1} u_{k-1}^o(\tau)d\tau = \\ &= -e^{A_{k-1}t_0}\int_0^{t_{k-1f}^o} e^{-A_{k-1}(t_0+\tau)} B_{k-1} u_{k-1}^o(\tau)d\tau = \\ &= -e^{A_{k-1}t_0}\int_{t_0}^{t_0+t_{k-1f}^o} e^{-A_{k-1}\tilde{\tau}} B_{k-1} u_k^{o+}(\tilde{\tau})d\tilde{\tau}.\end{aligned} \tag{31}$$

$$\tilde{\tau} = (t_0+\tau)$$

Having in mind (23) we obtain

$$\begin{aligned}\boldsymbol{x}_{k-1}(0) &= \int_0^{t_0} e^{A_{k-1}(t_0-\tau)} B_{k-1} u_0 d\tau = -\int_0^{t_{k-1f}^o} e^{-A_{k-1}\tau} B_{k-1} u_{k-1}^o(\tau)d\tau = \\ &= -e^{A_{k-1}t_0}\int_{t_0}^{t_0+t_{k-1f}^o} e^{-A_{k-1}\tilde{\tau}} B_{k-1} u_k^{o+}(\tilde{\tau})d\tilde{\tau},\end{aligned} \tag{32}$$

$$0 < t_0 < \infty, \quad 2 \le k \le n.$$

On the other hand, we have

$$\begin{aligned}\boldsymbol{x}_k^{o+} &= -\int_0^{t_0+t_{k-1f}^o} e^{-A_k\tau} B_k u_k^{o+}(\tau)d\tau = \\ &= -\int_0^{t_0+t_{k-1f}^o} e^{-\begin{bmatrix} A_{k-1} & 0 \\ 0 & \lambda_k \end{bmatrix}\tau}\begin{bmatrix} B_{k-1} \\ b_k \end{bmatrix} u_k^{o+}(\tau)d\tau = \\ &= \begin{bmatrix} -\int_0^{t_0+t_{k-1f}^o} e^{-A_{k-1}\tau} B_{k-1} u_k^{o+}(\tau)d\tau \\ -\int_0^{t_0+t_{k-1f}^o} e^{-\lambda_k\tau} b_k u_k^{o+}(\tau)d\tau \end{bmatrix} = \\ &= \begin{bmatrix} -\int_0^{t_0} e^{-A_{k-1}\tau} B_{k-1} u_0 d\tau - \int_{t_0}^{t_0+t_{k-1f}^o} e^{-A_{k-1}\tau} B_{k-1} u_k^{o+}(\tau)d\tau \\ -\int_0^{t_0+t_{k-1f}^o} e^{-\lambda_k\tau} b_k u_k^{o+}(\tau)d\tau \end{bmatrix}.\end{aligned} \tag{33}$$

Expressing $\left(-\int_{t_0}^{t_0+t_{k-1f}^o} e^{-A_{k-1}\tau} B_{k-1} u_k^{o+}(\tau)d\tau\right)$ in (33) by (32) we obtain

$$\boldsymbol{x}_k^{o+} = \begin{bmatrix} -\int_0^{t_0} e^{-A_{k-1}\tau} B_{k-1} u_0 d\tau - \int_{t_0}^{t_0+t_{k-1f}^o} e^{-A_{k-1}\tau} B_{k-1} u_k^{o+}(\tau)d\tau \\ -\int_0^{t_0+t_{k-1f}^o} e^{-\lambda_k\tau} b_k u_k^{o+}(\tau)d\tau \end{bmatrix} =$$

$$=\begin{bmatrix} -\int_0^{t_0} e^{-A_{k-1}\tau}B_{k-1}u_0 d\tau + e^{-A_{k-1}t_0}\int_0^{t_0} e^{A_{k-1}(t_0-\tau)}B_{k-1}u_0 d\tau \\ -\int_0^{t_0+t_{k-1f}^o} e^{-\lambda_k\tau}b_k u_k^{o+}(\tau)d\tau \end{bmatrix} =$$

$$=\begin{bmatrix} [\underbrace{0 \quad \dots \quad 0}_{k-1}]^{\mathrm{T}} \\ -\int_0^{t_0+t_{k-1f}^o} e^{-\lambda_k\tau}b_k u_k^{o+}(\tau)d\tau \end{bmatrix}. \qquad (34)$$

It follows from (34) that $\boldsymbol{x}_k^{o+}$ is a point of the axis Ox_k. Assume $\boldsymbol{x}_k^{o+}$ coincides with the state space origin, i.e. $\boldsymbol{x}_k^{o+} = [\underbrace{0 \quad \dots \quad 0}_{k}]^{\mathrm{T}}$. Then, according to Corollary 2, the control $u_k^{o+}(t)$ will transfer the system of Problem *A(k)* from the state space origin to the same state space origin, which contradicts Theorem 4. Hence, the assumption made is not true and therefore $\boldsymbol{x}_k^{o+}$ belongs to the axis Ox_k but is different from the state-space origin.

This completes the proof of part 2 of Theorem 5.

From the above result it follows

Corollary 3. *The control $u_k^{o+}(t)$ is the optimal control for the point $\boldsymbol{x}_k^{o+}$. The value of $u_k^{o+}(t)$ in the first constancy interval is $u_0 > 0$.*

Denoting by x_k^{o+} the ***k***th coordinate of the vector $\boldsymbol{x}_k^{o+}$, i.e.

$$x_k^{o+} = -\int_0^{t_0+t_{k-1f}^o} e^{-\lambda_k\tau}b_k u_k^{o+}(\tau)d\tau, \quad 2 \le k \le n, \qquad (35)$$

and taking into account the definition of x_{k+} and (34), we can write

$$x_{k+} = \mathrm{sign}\left(x_k^{o+}\right), \quad 2 \le k \le n. \qquad (36)$$

Thus the validity of (19) is proved.

According to the proof of part 1 of the theorem, the optimal control $u_k^o(t)$ of Problem *A(k)* with initial state (14) has exactly k non-zero constancy intervals and $u_k^o(0) = -u_0$. In other words, the initial state $\boldsymbol{x}_k(0)$ of Problem *A(k)* does not belong to the switching hyper surface S_k. On the other hand, it follows from Theorem 1 that the trajectory of the system of Problem *A(k)* starting from $\boldsymbol{x}_k(0)$ and generated by the optimal control $u_{k-1}^o(t)$, $t \in [0,\ t_{k-1f}^o]$, of Problem *A(k-1)* either entirely lies on the switching hyper surface S_k, or is above or below S_k, nowhere intersecting it. Hence, the optimal control bringing the system from $\boldsymbol{x}_k(t_{k-1f}^o)$ into the state-space origin has for $\boldsymbol{x}_k(t_{k-1f}^o)$ the same value as $u_k^o(0) = -u_0$.

It is shown in [1], [2] that the point $\boldsymbol{x}_k(t_{k-1f}^o)$ belongs to the coordinate axis Ox_k:

$$\boldsymbol{x}_k(t_{k-1f}^o) = \begin{bmatrix} \boldsymbol{x}_{k-1}(t_{k-1f}^o) \\ e^{\lambda_k t_{k-1f}^o}x_{k0} + \int_0^{t_{k-1f}^o} e^{\lambda_k(t_{k-1f}^o-\tau)}b_k u_{k-1}^o(\tau)d\tau \end{bmatrix} =$$

$$= \begin{bmatrix} \boldsymbol{x}_{k-1}(t_{k-1f}^o) \\ x_{kw} \end{bmatrix} = [\underbrace{0 \quad 0 \quad . \quad . \quad . \quad 0}_{k-1}\ x_{kw}]^{\mathrm{T}}, \qquad (37)$$

$$k = \overline{n,2},$$

where

$$x_{kw} = e^{\lambda_k t_{k-1f}^o}x_{k0} + \int_0^{t_{k-1f}^o} e^{\lambda_k(t_{k-1f}^o-\tau)}b_k u_{k-1}^o(\tau)d\tau, \quad k = \overline{n,2}. \qquad (38)$$

Hence, the optimal control value for the point $-\boldsymbol{x}_k(t_{k-1f}^o)$ is u_0. Taking into account the definition and analysis of x_{k+} [1], [2] and (37),(38), we can write

$$x_{k+} = \mathrm{sign}(-x_{kw}) = -\mathrm{sign}(x_{kw}) \qquad (39)$$

which completes the proof of section 3a of Theorem 5.

We can represent (35) as:

$$x_k^{o+} = -\int_0^{t_0+t_{k-1f}^o} e^{-\lambda_k\tau}b_k u_k^{o+}(\tau)d\tau =$$

$$= -e^{-\lambda_k(t_0+t_{k-1f}^o)}\int_0^{t_0+t_{k-1f}^o} e^{\lambda_k(t_0+t_{k-1f}^o-\tau)}b_k u_k^{o+}(\tau)d\tau. \qquad (40)$$

Since

$$-x_{kw} = -\left(e^{\lambda_k t_{k-1f}^o}x_{k0} + \int_0^{t_{k-1f}^o} e^{\lambda_k(t_{k-1f}^o-\tau)}b_k u_{k-1}^o(\tau)d\tau\right) \qquad (41)$$

and

$$x_{k0} = \int_0^{t_0} e^{\lambda_k(t_0-\tau)}b_k u_0 d\tau, \quad 0 < t_0 < \infty, \quad 2 \le k \le n, \qquad (42)$$

we have

$$-x_{kw} = -\left(e^{\lambda_k t_{k-1f}^o}\int_0^{t_0} e^{\lambda_k(t_0-\tau)}b_k u_0 d\tau + \int_0^{t_{k-1f}^o} e^{\lambda_k(t_{k-1f}^o-\tau)}b_k u_{k-1}^o(\tau)d\tau\right). \qquad (43)$$

Taking into account (18) we obtain

$$-x_{kw} = -\left(\int_0^{t_0} e^{\lambda_k(t_0+t_{k-1f}^o-\tau)}b_k u_k^{o+}d\tau + \int_{t_0}^{t_0+t_{k-1f}^o} e^{\lambda_k(t_0+t_{k-1f}^o-\tilde{\tau})}b_k u_k^{o+}(\tilde{\tau})d\tilde{\tau}\right) =$$

$$\tilde{\tau} = t_0 + \tau$$

$$= -\left(\int_0^{t_0+t_{k-1f}^o} e^{\lambda_k(t_0+t_{k-1f}^o-\tau)}b_k u_k^{o+}(\tau)d\tau\right). \qquad (44)$$

According to the proof of part 3a of the theorem $x_{kw} \neq 0$, which allows to write

$$\frac{x_k^{o+}}{-x_{kw}} = \frac{-e^{-\lambda_k(t_0+t_{k-1f}^o)} \int_0^{t_0+t_{k-1f}^o} e^{\lambda_k(t_0+t_{k-1f}^o-\tau)} b_k u_k^{o+}(\tau)d\tau}{-\int_0^{t_0+t_{k-1f}^o} e^{\lambda_k(t_0+t_{k-1f}^o-\tau)} b_k u_k^{o+}(\tau)d\tau} =$$

$$= e^{-\lambda_k(t_0+t_{k-1f}^o)}. \tag{45}$$

Since $\lambda_k \le 0$ and $t_0 \in (0, \infty)$, we have

$$\frac{x_k^{o+}}{-x_{kw}} = \begin{cases} 1 & \text{when } \lambda_k = 0, \\ e^{-\lambda_k(t_0+t_{k-1f}^o)} > 1 & \text{when } \lambda_k < 0. \end{cases} \tag{46}$$

Taking into account that $e^t > t \quad \forall t > 0$, we can estimate the exponential term in (46) and thus

$$\frac{x_k^{o+}}{-x_{kw}} = \begin{cases} 1 & \text{when } \lambda_k = 0, \\ e^{-\lambda_k(t_0+t_{k-1f}^o)} > \max\{1, -\lambda_k t_0\} & \text{when } \lambda_k < 0. \end{cases} \tag{47}$$

This completes the proof of the last part of Theorem 5.

Based on this theorem, the following axes initialization algorithm for the class of problems *A(n), A(n-1), ..., A(1)* can be proposed.

A. Algorithm for axes initialization in the class of problems A(n), A(n-1), ..., A(1)

Step 1. Choose t_0, such that $0 < t_0 < \infty$;

Step 2. Formulate Problem *A(n)*, $n \ge 2$, with initial state

$$\boldsymbol{x}_n(0) = [x_{10} \quad x_{20} \quad . \quad . \quad . \quad x_{n0}]^T = \int_0^{t_0} e^{A_n(t_0-\tau)} B_n u_0 d\tau; \tag{48}$$

Step 3. Based on relations (7) define the class of problems *A(n), A(n-1), ..., A(1)*;

Step 4. Solve problem *A(1)* to find $u_1^o(t)$ and t_{1f}^o;

Step 5. Set $k = 2$;

Step 6. Compute the ***k***th coordinate x_k^{o+} of $\boldsymbol{x}_k^{o+}(0)$ according to (35) or compute the ***k***th coordinate x_{kw} of $\boldsymbol{x}_k(t_{k-1f}^o)$ in accordance with (11).

Step 7. Determine x_{k+} as $x_{k+} = \text{sign}(x_k^{o+})$ or $x_{k+} = -\text{sign}(x_{kw})$;

Step 8. Check ***k***. If:

- $k < n$, go to step 9;
- $k = n$, then the axes initialization is complete ($x_{n+}, x_{n-1+}, ..., x_{2+}$ are obtained);

Step 9. Solve Problem *A(k)* to find $u_k^o(t)$ and t_{kf}^o, using the proposed in [1], [2] algorithm (here we need the computed $x_{k+}, x_{k-1+}, ..., x_{2+}$);

Step 10. Increase ***k*** with 1 and go back to step 6.

Thus the first stage of the time-optimal control synthesis procedure is also based on the proposed in [1], [2] synthesis algorithm. The special feature in this case is that the initial state of Problem *A(n)* belongs to a specific set of points in the state spaces of the considered class of problems.

Using the initialization and synthesis algorithms we can solve the time-optimal control problem for any initial state of Problem *A(n)*, $n \ge 2$.

B. General algorithm for solving the time-optimal control synthesis Problem A(n), $n \ge 2$

Stage 1. Solve the axes initialization problem applying the proposed initialization algorithm to the class of problems *A(n), A(n-1), ..., A(1)* for a specific initial point;

Stage 2. Solve the time-optimal control synthesis problem for the given initial state of Problem *A(n)* using the synthesis algorithm [1], [2].

IV. Concluding Remarks

In this paper a new approach to the time-optimal control synthesis problem for a class of linear systems is presented. In contrast to the existing time-optimal control synthesis methods, the new approach does not require the description of the switching hyper surface and thus enables the synthesis of time-optimal control for high order systems of the given class.

The presented approach is based on the state-space properties of the considered class of problems and consists of two main stages. The first one comprises the state-space analysis called axes initialization while at the second one the optimal control is obtained. Both stages use a multi-step time-optimal control synthesis procedure for the problems of the considered class.

This paper is focused on the first stage of the time-optimal synthesis procedure and presents the main results in the state-space analysis of the considered class of problems. In particular, it is shown how the relation of the problems state-space coordinate axes to the respective problems switching hyper surfaces and optimal control values can be determined. This makes possible the efficient design and implementation of time-optimal control for high order linear systems.

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