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Describing Function and Event Based Analysis of a General Class of Piecewise Feedback Control Systems ⋆

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Abstract: The analysis of nonlinear systems can be greatly improved using computer software. In this work the Sridhar nonlinearity is described, and a new parameterization is given that allows to systematically obtain a large number of the classical piecewise nonlinearities as particular cases. The calculation of limit cycles using the describing function (DF) approach for the case of autonomous nonlinear control systems and an event-based simulation method are presented and compared. Finally, a user-friendly, graphically oriented interactive tool based on Sysquake is presented. In this way the students can easily assimilate some concepts of introductory courses in nonlinear control such as the behavior of piecewise linear systems, stable and unstable limit cycles, event-based simulation and the DF method.

Keywords: Control education, piecewise linear systems, describing function, limit cycles, event-based control simulation, frequency response identification.

1. INTRODUCTION

Classical control theory has mainly emphasized the use of linear analysis and design tools. This is because most control loops aim to stabilize the system at an equilibrium point; this is usually called a regulation problem. Since the behavior of a system around an equilibrium point can be described by a linear system, the controller can be developed using linear tools.

Furthermore, the linear systems theory of dynamical systems is a body of mathematical knowledge that is very well structured, and supplies useful operative tools, that are simple to handle (for instance, the transfer function or the linear description in the state space). Most of the students of control engineering only know these linear methods, and with them most of the practical problems in control analysis and design can be dealt with.

The consideration of the nonlinear effects in control systems deserves a special attention. Nonlinear systems present behavior modes far richer than the ones displayed by linear systems. A more detailed description of the nature of limit cycles can be found in Atherton (1982) and Khalil (2002).

When an unstable open-loop plant is to be stabilized linear control gives misleading results. Even if the plant can be stabilized at the operating point with a linear control plant, this control will behave adequately only locally, due to the nonlinearity associated with the actuator saturation (Stein (1989)). Fortunately, for many interesting problems in control systems raised by the presence of nonlinearities, a quite simple and powerful analysis can be worked out. For many nonlinear problems the control engineer has the Describing Function (DF) as a tool to deal with them (Gelb and Van der Velde (1968), Schwartz and Gran (2001)). This method has been used for decades by practical engineers. Even if the results are only approximate, they are very easy to reach and supply clues about the oscillatory and the global behavior at a very low cost.

The analysis of nonlinear systems can be greatly improved using computer software. In this paper a user-friendly, graphically oriented interactive tool based on Sysquake (Piguet (2000), Guzmán et al. (2023)) is also presented. Nonlinear systems whose nonlinearity is given by a general class of piecewise linear systems are considered. The tool can help the students to understand the behavior of piecewise linear systems, the DF method, and the basis of bifurcations theory in control systems.

The paper is organized as follows: In Section 2, Sridhar nonlinearity is described, and a new parameterization is given that allows to systematically obtain a large number of the classical piecewise nonlinearities as particular cases. In Section 3 the DF of the Sridhar nonlinearity is calculated by a procedure of parallel combination of simpler nonlinearities that can be obtained from more common DF's tables. The calculation of limit cycles using the DF method is briefly reviewed in Section 4 for the case of autonomous nonlinear control systems. In Section 5,

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an event-based simulation and limit cycle determination method is proposed that considers at each moment the selected parameterization of the Sridhar nonlinearity. To compare both methods, Section 6 briefly describes the main features and functionality of an interactive software tool developed for the general class of piecewise feedback control systems presented in the paper. Finally, some conclusions and further lines of work are given in Section 7.

2. SRIDHAR NONLINEARITY

In 1960 Sridhar proposed and determined the DF of the nonlinearity shown in Figure 1 (Sridhar (1960)). The input-output characteristic of this piecewise linear type of nonlinearity consists of straight-line segments. Probably does not occur as the characteristic of a single nonlinearity in practice, although it is quite conceivable that might represent the combined characteristic of multiple nonlinearities in cascade. The main reason for choosing this nonlinearity is that many other piecewise linear types of nonlinearities to which the DF analysis is applicable are modifications of this general nonlinearity. It is assumed that the input-output characteristics of all the nonlinearities considered here are odd functions with respect to the input.

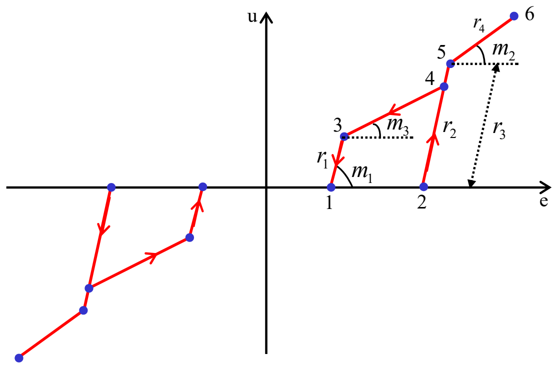

Fig. 1. Sridhar nonlinearity

2.1 Parameterization

Let the input to the nonlinear element be e and the output by u. Then u can be represented as a function of e and $\dot{e}$; i.e., $u = f(e, \dot{e})$ where $f(e, \dot{e})$ is the input-output characteristic shown in Figure 1. Sridhar nonlinearity is fully defined by the following parameters and constraints between them: slopes m_1 and m_2 ($0 \leq m_2 \leq m_1 \leq \infty$), segment lengths r_1, r_2 and r_3 ($0 \leq r_1 \leq r_2 \leq r_3$) and coordinates of points 1 ($a = e_1, 0$) and 2 ($b = e_2, 0$) of the nonlinearity ($-e_2 \leq e_1 \leq e_2$).

The rest of the coordinates that define points 3 (e_3, u_3), 4 (e_4, u_4) and 5 (e_5, u_5) are easily determined from the parameters previously indicated. To calculate the coordinates of point 6 (e_6, u_6), the length r_4 of the segment joining point 5 and point 6 ($e6, u6$) is assumed to be 0.5. This length does not affect the nature of the nonlinearity in any way. The slope m_3 of the segment joining points 3 and 4 of the nonlinearity satisfies by geometric construction of the nonlinearity the constraint $m_3 < m_1$. It must also be true in this nonlinearity that $m_2 \leq m_3$. That is, the slopes m_1, m_2 and m_3 satisfy the following relationship: $0 \leq m_2 \leq m_3 < m_1 \leq \infty$. In this way the parameter vector $p = (a, b, r_1, r_2, r_3, m_1, m_2, m_3)$ completely defines the nonlinearity f.

2.2 Particular cases

A wide range of non-linearities types can be obtained by varying the components of the vector p.

$$(type, e_v, d) = \text{NonLinearityType}(p) \tag{1}$$

where *type* is an integer that uniquely determines a particular nonlinearity, e_v is a vector whose dimension and components depend on the value that *type* takes, and d is a vector that has the same dimension as the vector e_v. The dimension of the vector e_v indicates the number of points of the nonlinearity where the signal u or its derivative are discontinuous. When the input signal e passes through these points, a *state event* can occur depending on the value of the derivative of e and the type of nonlinearity.

If a component of the vector d is $1(-1)$ it means that the event associated in e_v is activated when the threshold is crossed in an increasing (decreasing) direction. On the other hand, if a component of the vector d is 0 it means that the event associated in e_v is activated when the threshold is crossed in any direction. From the given definition of the vector d it follows that if all its components are 0 then the associated nonlinearity is single valued. The existence of non-zero components means that nonlinearity is multi valued and the effect of hysteresis is taken into consideration since the existence of an event depends on the derivative of the error signal e.

If $1 \leq type \leq 6$ then $a = b$ and the nonlinearity is single valued (see Figure 2). For example, if $type = 4$, a single-valued nonlinearity occurs with dead-zone and saturation and $e_v = [e_5, e_1, -e_1, -e_5]$, which means that state events occur when the error signal e reaches some of the values given as components of e_v. Section 5 explains the treatment of events in detail. In the other cases, it is a non-linearity with memory or double-valued. If $7 \leq type \leq 12$ then $e_1 = -e_2$ and the nonlinearity has hysteresis (see Figure 3). If $13 \leq type \leq 21$ then $-e_2 < e_1 < e_2$ and the nonlinearity has hysteresis plus dead-zone (see Figure 4). Sridhar nonlinearity corresponds to $type = 21$.

3. DF CALCULATION OF THE SRIDHAR NONLINEARITY

Consider the feedback system shown in Figure 5 containing a single static nonlinearity delayed $N_D(e) = N(e)\,exp(-Ls)$ and a linear dynamic system given by the transfer function $G(s) = G_0(s)exp(-\tau s)$ (where $G_0(s)$ is strictly proper). If a limit cycle exists in the autonomous system, that is with $r(t) = 0$, with the output $y(t)$ approximately sinusoidal, then the input $e(t)$ to the nonlinearity might also be expected to be near sinusoidal. This is the concept of harmonic balance, in this case balancing the first harmonic only. Since the output of the nonlinearity may not be in phase with the sinusoidal input (multi-valued nonlinearity) the DF may be complex. Since Goldfarb's original work on DF (Goldfarb (1956)), a considerable number of papers was published in which the DF of nonlinearities was derived. It appears however that

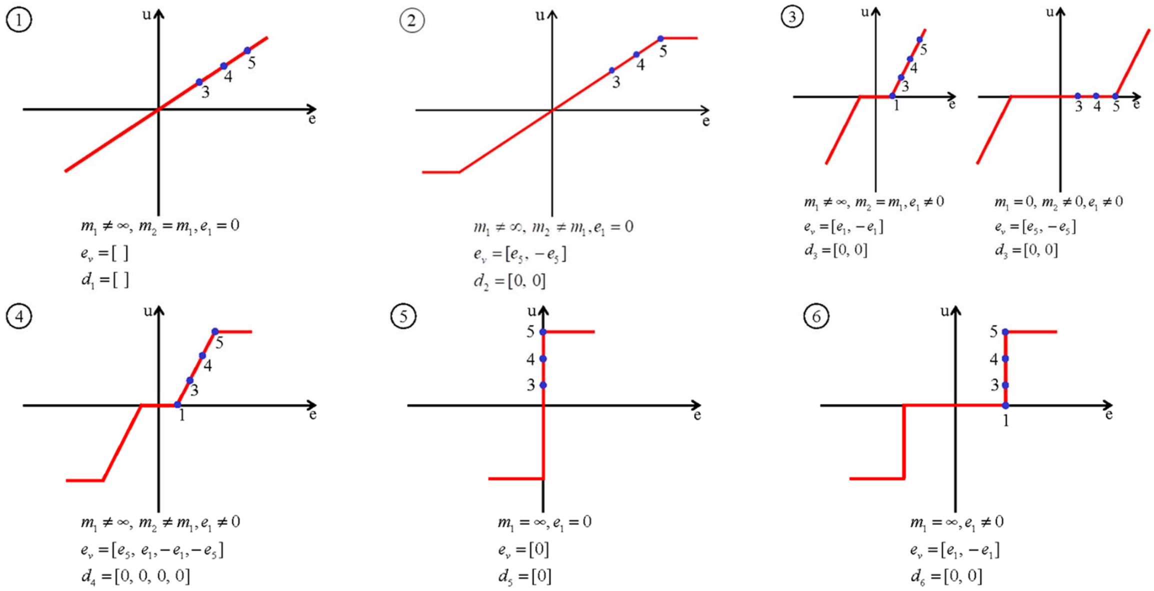

Fig. 2. Single valued nonlinearity as a particular case of the Sridhar nonlinearity $(a = b)$

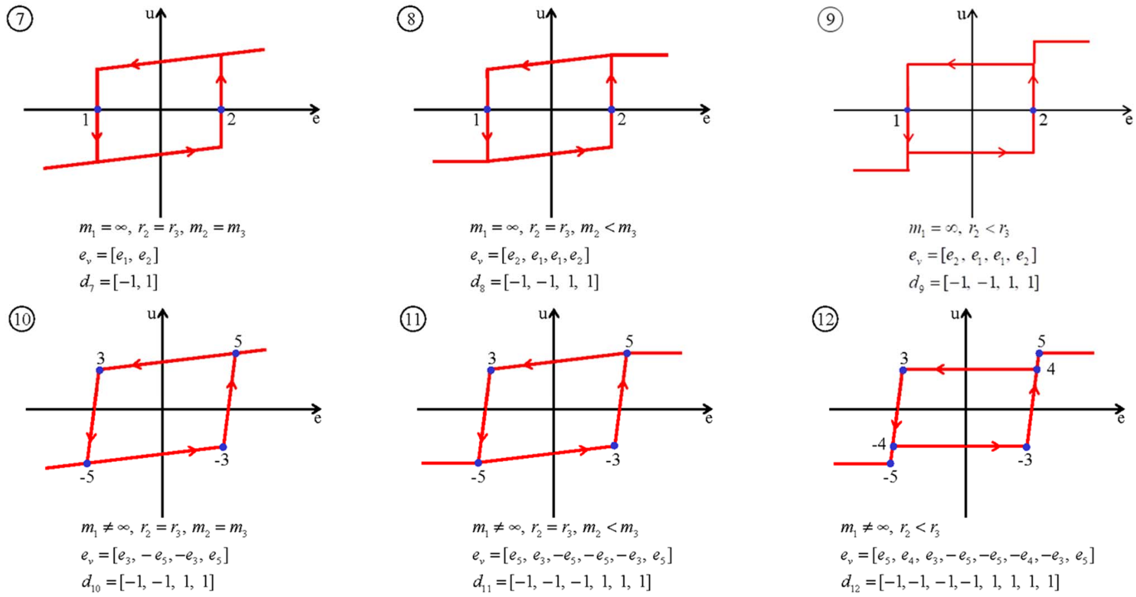

Fig. 3. Nonlinearity with hysteresis as a particular case of the Sridhar nonlinearity $(a = -b)$

little effort had been made to classify the nonlinearities until the Sridhar's paper.

In Sridhar's paper, the DF of its nonlinearity is determined and as particular cases, those corresponding to specific values of its parameters, with a different parameterization than the one presented here are calculated (top-down approach). However, it is not studied how the inclusion of the nonlinearity in the control loop affects the stability of the system, nor the temporal response of the system. In Gelb and Van der Velde (1968) the DFs of a large part of the cases of the Sridhar nonlinearity are collected in a much more systematic and compact way in a table and those that do not appear can be obtained as a composition of other nonlinearities. An example of the synthesis of a complex nonlinearity from simpler forms is the construction of the non-linearity type 15 from non-linearities types 14 and 6.

The other component of the N_D nonlinearity is the time delay L. As this phenomenon is linear, its DF is exactly its transfer function: $\exp(-jL\omega) = \cos(\omega L) - j\sin(\omega L)$. The approximate DF of two series-connected nonlinearities can be computed directly by multiplication of the DFs of the individual nonlinearities.

4. DETERMINATION OF LIMIT CYCLES USING THE DF APPROACH

DF provides an approximate method to determine the stability of an autonomous nonlinear feedback system (see Figure 5).

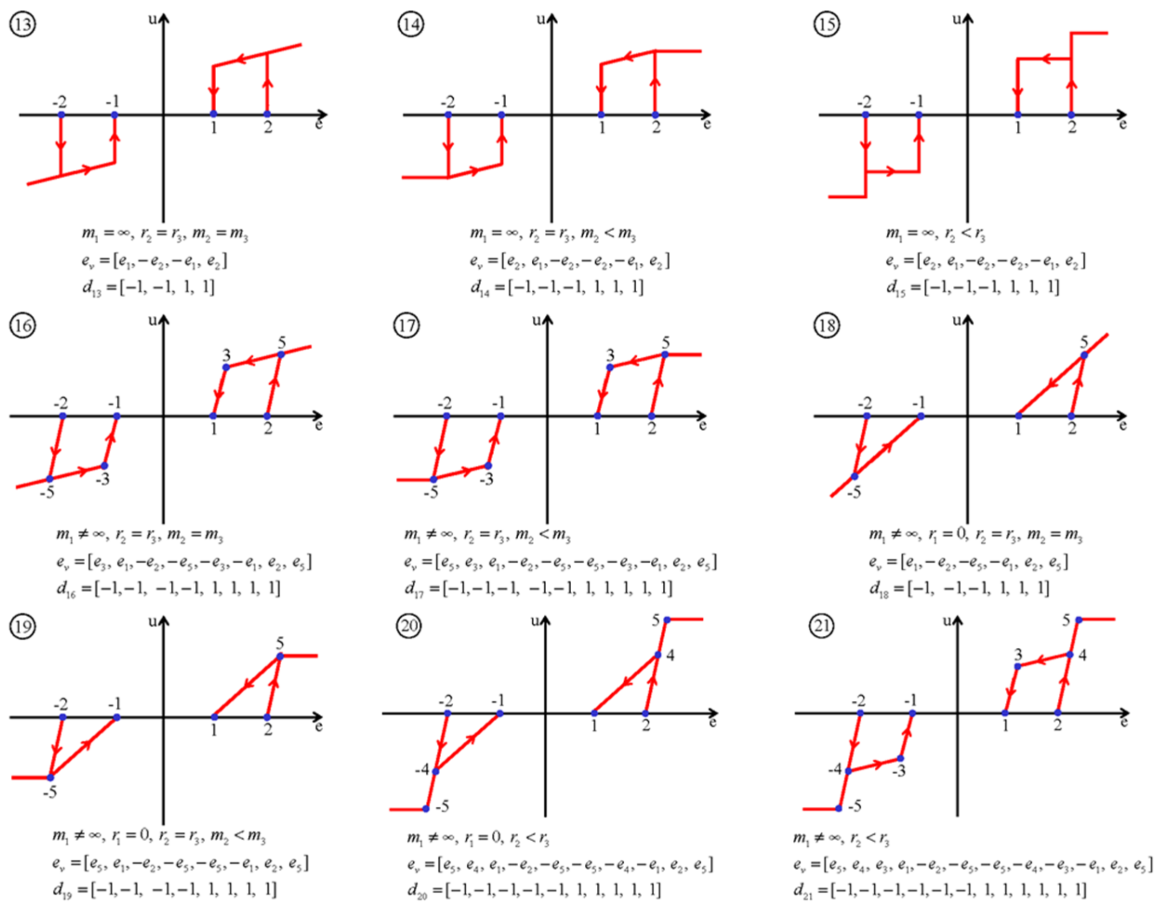

Fig. 4. Nonlinearity with hysteresis plus dead-zone as a particular case of the Sridhar nonlinearity $(-b < a < b)$

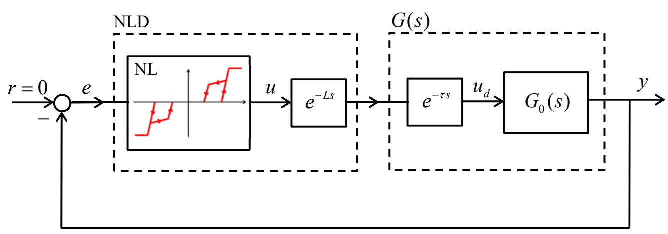

Fig. 5. Autonomous piecewise feedback control loop

To study the possibility of limit cycles, the nonlinearity NL_D is replaced by its DF, $NL_D(A)$. Thus, the open-loop gain to a sinusoid is $NL_D(A)\,G(j\omega)$ and a limit cycle will exist if

$$NL_D(A)\,G(j\omega) = -1 \tag{2}$$

This condition means that the first harmonic is balanced around the closed-loop system assuming its passage through the nonlinearity is accurately described by $NL_D(A)$. Since $G(j\omega)$ is a complex function of ω and $N_D(A)$ may be a complex function of A, a solution to (2) will yield both the frequency ω_0 and amplitude A_0 of an assumed sinusoidal limit cycle. Figure 7 illustrates the procedure to solve (2) graphically on a Nyquist diagram, where $G(j\omega)$ and $C_D(A) = -1/NL_D(A)$ loci are plotted and shown intersecting for $\omega = \omega_0$ and $A = A_0$. When the $G(j\omega)$ and $C_D(A)$ loci do not intersect, the DF method predicts that no limit cycle will exist if the Nyquist stability criterion is satisfied for $G(j\omega)$ with respect to any point on the $C_D(A)$ locus.

A further point of interest when a solution to (2) exists is whether the predicted limit cycle is stable. This is obviously important if the control system is designed to have a limit cycle operation. Figure 7 also presents an example of the counterclockwise rotation of the $C_D(A)$ loci when a delay $L > 0$ in cascade with the non-linearity is introduced. Such feature allows to estimate points in the Nyquist map for any frequency going from 0 to the oscillation frequency ω_0 induced by the non-linearity when $L = 0$. With all these data, obtained by modifying the value of L, it is possible to depict the spectrum of the process in the range of frequencies $[0, \omega_0]$ which allows estimating the parameters of a transfer function selected by the user from a template (Sanchez et al. (2018), Sanchez et al. (2021a), Sanchez et al. (2021b)).

5. DETERMINATION OF LIMIT CYCLES USING AN EVENT-BASED CONTROL SIMULATION

The closed-loop system represented in Figure 5 can be described by the following state equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t - L - \tau) \\ e(t) &= -Cx(t) \\ u(t) &= f(e(t), \dot{e}(t)) \end{aligned} \tag{3}$$

where L is the delay associated with the DF and τ the delay associated with $G(s)$. $ABC = [A, B, C]$ represents the state space representation of the dynamical system.

A hybrid automaton corresponds to each type of nonlinearity f that has been presented in section 2. This allows dynamic simulation of each type of nonlinearity to be easily implemented. Although theoretically it is possible to define a single hybrid automaton that implements the set of nonlinearities that can be configured as particular cases of Sridhar nonlinearity, it has been preferred to associate a simpler hybrid automaton for each type of nonlinearity. This makes its implementation and maintenance easier. As an example, the hybrid automata associated with *type* 15 nonlinearity is presented (see Figure 6).

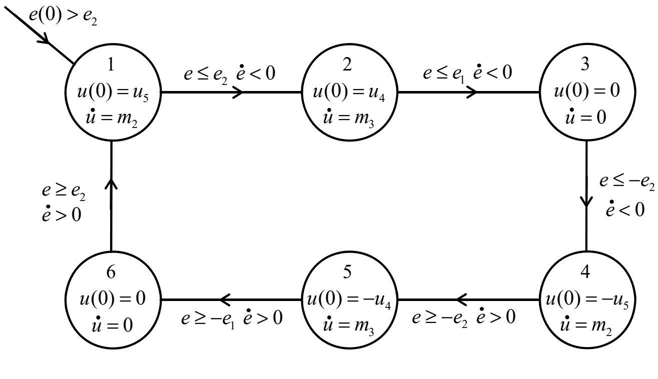

Fig. 6. Hybrid automaton of the nonlinearity *type* 15

The initial condition of the error signal e is always initialized to state 1 which corresponds to the rightmost linear segment of the nonlinearity ($u(0) = f(e(0)) = u_6$). The activation of the different events between states (edge labels) as well as the update of the output signal u of the nonlinearity and its derivative at the time of the transition between states $\dot{u} \equiv \frac{du}{de}$ are also defined.

The *trajectory* function calculates the dynamic evolution of the system

$$par_{out} = trajectory(par_{in}) \tag{4}$$

with

$$\begin{aligned} par_{out} &= (t, t_v, e, u, x_e, i_v) \\ par_{in} &= (x_{0e}, type, e_v, p, ABC, L, \tau, t_{max}, p_{aux}) \end{aligned} \tag{5}$$

Output parameter of the trajectory function, par_{out}: t is the vector of time instants simulation, t_v is the vector of time instants in which the events are activated, e is the vector of the error signal in the time instants simulation, u is the vector of the nonlinearity output in the time instants simulation, and x_e is the extended state vector defined as follows: $x_e = (x, x_e(end-1), x_e(end))$ being x the state vector of the dynamic system. $x_e(end-1)$ is the value of the output of the nonlinear element u at the time of the event activation, and $x_e(end)$ is an integer that indicates the state of the automaton that is reached at the time of the event activation. That is, the last two components of x_e are constant between events and are changed only when an event occurs. The value of the i_v variable is an integer that indicates which event was triggered.

Input parameter of the trajectory function, par_{in}: the state of the dynamical system ABC is integrated from 0 to t_{max} with options to treat the state event when the trajectory detects the limits of the nonlinearity type defined in the e_v vector, p_{aux} is an auxiliary parameter vector that defines certain values that configure the integration in the numerical solver such as the maximum absolute error ($AbsTol$), the maximum time step ($MaxStep$), the minimum time step ($MinStep$), the maximum relative error ($RelTol$), and the refinement factor ($Refine$). x_{0e} is the initial value of x_e, where $x_{0e}(end-1) = u_6$ and $x_{0e}(end) = 1$. The rest of the input arguments have already been previously defined. The output arguments of the *trajectory* function are calculated with an *ode*45 solver with options for the treatment of delays and state events.

$$(t, t_v, e, u, x_e, i_v) = ode45(fun, [0, t_{max}], x_{0e}, options) \tag{6}$$

6. INTERACTIVE TOOL FOR THE SIMULATION OF THE SRIDHAR NONLINEARITY

Sridhar Non-Linearity Delayed is a free of charge standalone executable for Windows and Mac OS computers (SridharNLD (2024)). This means that it is available to any user (student or instructor) who may want to use it. The tool has been developed using Sysquake (Piguet (2000)), an integrated development environment with a programing language like the one used in MATLAB. It provides the interactive calculation of limit cycles for an autonomous nonlinear control system that incorporates Sridhar nonlinearity or some of its simplifications. It also calculates the limit cycle using an event-based dynamic simulation, which allows comparisons between both techniques.

This section provides a brief presentation of the interactive elements that the tool incorporates as well as a description of its functionality. The main characteristics of *SridharNLD* is its interactivity and simplicity. Users can interact with the tool by menus, text fields, sliders, buttons and different items in the figures displayed on the main window of the tool (see Figure 7). Any action carried out on these elements is immediately reflected on all elements shown on the screen. This interactivity allows users to immediately perceive the effects of their actions.

The *SridharNLD* main window is organized in the following zones: Parameters setting, Non-Linear element, Limit cycle parameters (frequency and amplitude oscillation) obtained by the DF approach and using an event-based control simulation, Symbolic transfer function (TF) chosen, Pole-zero map of the TF, State events sequence, Time response (output plus control signal) and Nyquist diagram with the critical locus of the non-linearity. The menus that the interactive tool incorporates are the following:

IntParameters, allows modifying the parameters of the *ode45* solver. In general, it will not be necessary to use it, but in some cases it will be. Particularly when events are very close. It has a "reset" that returns the values to its initial conditions.

Tfs, allows selecting up to 17 types of TFs. The fields that allow modifying the parameters of the chosen TF are also displayed. The type of TF can be also modified in the pole-zero diagram by adding or deleting integrators, poles and zeros.

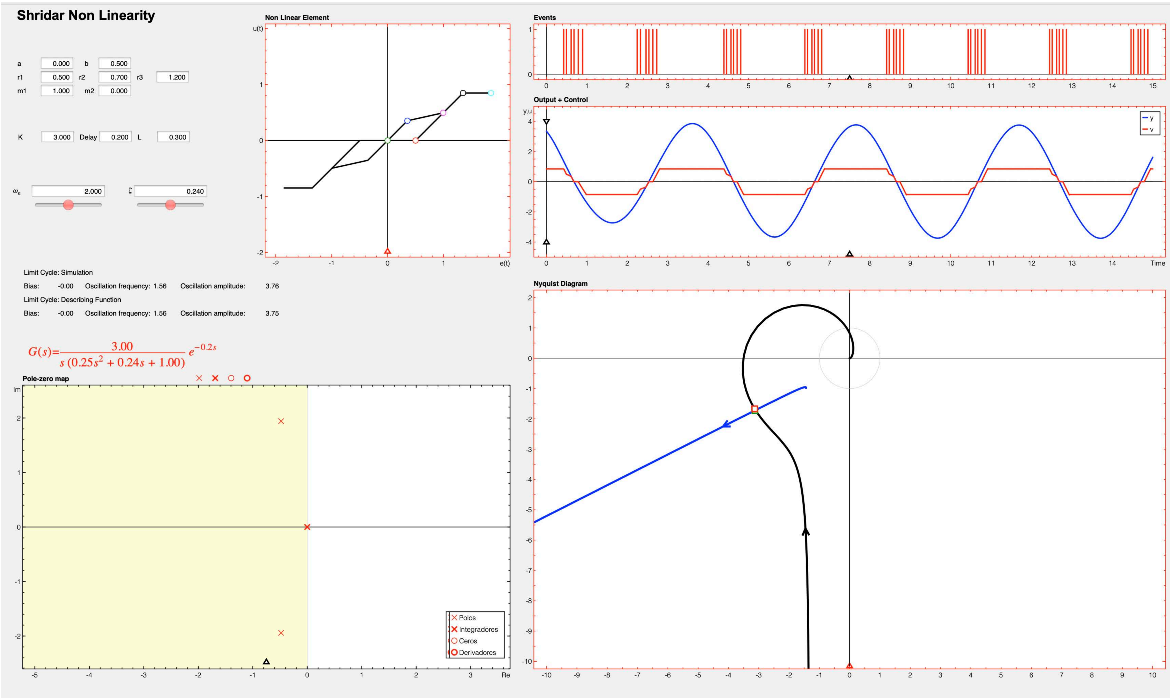

Fig. 7. Graphical user interface of the *SridharNLD* interactive tool.

NL Type, allows selecting the different types of nonlinearities that can be configured. It is always possible to modify the nonlinearity with the associated parameters that define it in the corresponding fields or by interacting directly on the nonlinearity graph.

SimulParameters, allows changing some parameters that affect the simulation, such as the tolerance value for determining the self-oscillation frequency ω_0 and the number of points that are calculated from $G(j\omega)$

Examples, allows having examples packaged to show specific issues.

7. CONCLUSIONS

In this paper, a new educational tool for the study of a general class of autonomous piecewise feedback control systems (Sridhar nonlinearity) is presented. Two approaches for the prediction of limit cycles are described and compared: the FD method and an event-based simulation. The main characteristic of the *SridharNLD* tool, developed in Sysquake, is its interactivity. The following extensions of the software capabilities are planned: 1) Sridhar asymmetric nonlinearity. 2) How does the introduction in the error channel of a controller $C(s)$ affect to the closed loop response? A case of particular interest is when the controller is a PID controller, and the interest is to study its behavior against changes in the set point or load disturbances. In both cases it will be necessary to use the dual input describing function (DIDF) and calculate, in addition to the amplitude and the oscillation frequency, the bias B that is introduced because of the asymmetry of the nonlinearity or the incorporation of a controller $C(s)$.

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