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Analytical and Graphical Design of PID Compensators on the Nyquist plane

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Abstract: This paper focuses on the design of PID compensator to exactly satisfy the gain margin, the phase margin and the gain or phase crossover frequency specifications. The design problem is numerically solved using the so called PID inversion formulae method. A graphical interpretation of the solution on the Nyquist plane is presented. This could be suitable on education environment to deeply understand the design of PID compensators. Simulations results show the effectiveness of the presented method.

Keywords: PID Compensators, Linear Control, Compensator Design.

1. INTRODUCTION

The proportional-integral-derivative (PID) compensators are widely used in the industrial processes to meet most of the control objectives. The gain and phase margin (GPM) specifications are important measure of the robustness of the controlled systems, and many methods are known in the literature to meet these design requirements. Some of them are described in Lee (2004), Kim et al. (2005) and Aström and Hagglund (2006). These methods are normally based on numerical or graphical trial-and-error solutions which use Bode plots or fuzzy neural network (FNN). In this paper a new and exact graphical procedure on Nyquist plane to meet the gain margin, the phase margin and the phase or gain crossover frequency specifications is shown. This procedure is similar to ones described in Wang et al. (1999) and is based on so called PID inversion formulae, see Zanasi et al. (2011), Ntogramatzidis and Ferrante (2011), Zanasi and Cuoghi (2011).

The paper is organized as follows. In section II, the graphical properties of PID compensator on Nyquist plane are described. In section III a new design method based on the use of the PID inversion formulas is presented. In section IV numerical and graphical solutions of design problems based on GMP specifications are given. Numerical examples and conclusions end the paper.

2. PID COMPENSATORS: THE GENERAL STRUCTURE

Let us consider the classical form of the PID compensator

$$C(s) = K_P + sK_D + \frac{K_I}{s}, \qquad (1)$$

where the proportional, derivative and integrative terms K_P, K_D and K_I are supposed to be real and positive. The frequency response of $C(s)$

$$C(j\omega) = K_P + j\left(\omega K_D - \frac{K_I}{\omega}\right), \qquad (2)$$

can be written as

$$C(j\omega) = X + jY(\omega),$$

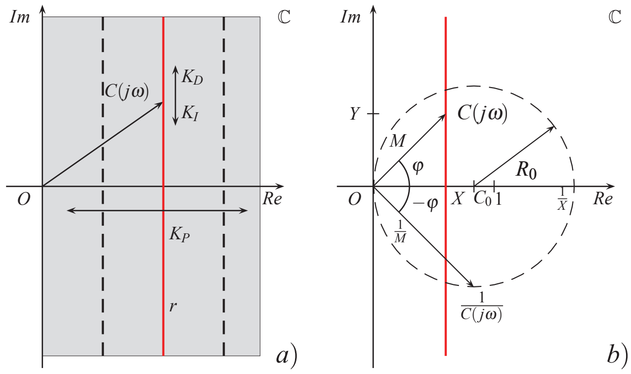

Fig. 1. Nyquist plot of functions $C(j\omega)$ and $C^{-1}(j\omega)$.

where $X = K_P$ and $Y(\omega) = \omega K_D - \frac{K_I}{\omega}$. The parameter X is always positive, while the function $Y(\omega)$ is positive for $\omega > \sqrt{\frac{K_I}{K_D}}$ and negative for $0 < \omega < \sqrt{\frac{K_I}{K_D}}$.

Let $\mathscr{C}(K_P)$ and $\mathscr{C}^-(K_P)$ denote, respectively, the set of all the PID compensators $C(s)$ having the same parameter K_P

$$\mathscr{C}(K_P) = \left\{ C(s) \text{ as in (1)} \;\middle|\; K_I > 0, K_D > 0 \right\}, \qquad (3)$$

and the set of all the inverse functions $C(s)^{-1}$

$$\mathscr{C}^-(K_P) = \left\{ \frac{1}{C(s)} \;\middle|\; C(s) \in \mathscr{C}(K_P) \right\}. \qquad (4)$$

It can be easily shown that the graphical representation of each element of $\mathscr{C}(K_P)$ on the Nyquist plane is a vertical straight line r which passes through point $(K_P, 0)$, see Fig. 1.

Property 1. *The shape of the frequency response of each element of set $\mathscr{C}^-(K_P)$ is a circle with center $C_0 = \frac{1}{2K_P}$ and radius $R_0 = \frac{1}{2K_P}$ which intersects the real axis in point 0 and point $\frac{1}{K_P}$.*

This graphical property hinges on the fact that the Nyquist diagram of $C^{-1}(j\omega)$ is a circle

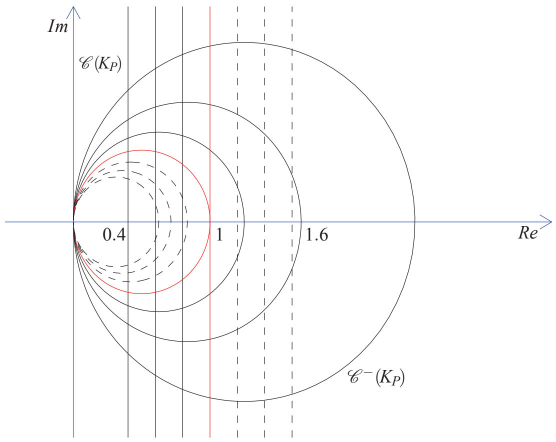

Fig. 2. The Nyquist diagrams of frequency responses of set $\mathscr{C}(K_P)$ and $\mathscr{C}^-(K_P)$ for $K_P = 0.4 : 0.2 : 1.6$.

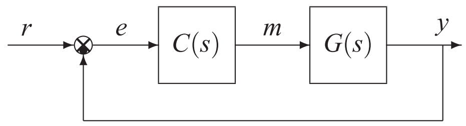

Fig. 3. Unity feedback control structure.

$$C^{\text{-}1}(j\omega) = \frac{1}{C(j\omega)} = C_0 + R_0 e^{j\theta(\omega)}, \qquad (5)$$

where $C_0 = R_0 = \frac{1}{2X}$, $\theta(\omega) = -\arctan\frac{Y(\omega)}{X} \in [0, 2\pi]$ and $X = k_p$.

Proof: the frequency response (2) can be written in polar form as follows

$$C(j\omega) = M(\omega) e^{j\varphi(\omega)},$$

where $M(\omega) = \frac{X}{\cos\varphi(\omega)}$ and $\varphi(\omega) = \arctan\frac{Y(\omega)}{X}$. It follows that $C^{\text{-}1}(j\omega)$ can be expressed in the form

$$\begin{aligned} C^{\text{-}1}(j\omega) &= \frac{1}{C(j\omega)} = \frac{\cos\varphi(\omega)}{X} e^{-j\varphi(\omega)} \\ &= \frac{1}{2X}[1 + \cos(2\varphi(\omega))] - j\frac{1}{2X}\sin(2\varphi(\omega)) \\ &= \frac{1}{2X} + \frac{1}{2X} e^{-j2\varphi(\omega)}, \end{aligned}$$

for $Y(\omega) \in [-\infty, +\infty]$. The last relation clearly shows that the shape of $C^{\text{-}1}(j\omega)$ in the complex plane is a circle with center $C_0 = \frac{1}{2X}$ and radius $R_0 = \frac{1}{2X}$.

The Nyquist diagrams of frequency responses of sets $\mathscr{C}(K_P)$ and $\mathscr{C}^-(K_P)$ for different values of parameter K_P are shown in Fig. 2.

3. PID COMPENSATORS $C(s, K_D)$ AND $C(s, K_I)$ MOVING A POINT A TO A POINT B

Consider the block-diagram shown in Fig. 3, where $G(s)$ denotes the transfer function of the LTI plant to be controlled. Let $C(j\omega_0) = M_0 e^{j\varphi_0}$ denote the value of the frequency response $C(j\omega) = M(\omega) e^{j\varphi(\omega)}$ at frequency ω_0, where $M_0 = M(\omega_0)$ and $\varphi_0 = \varphi(\omega_0)$. To study how $C(j\omega_0)$ affects $G(j\omega)$ at frequency ω_0, let us consider two generic points $A = M_A e^{j\varphi_A}$ and $B = M_B e^{j\varphi_B}$ of the complex plane. Referring to Fig. 4, we say that *point A is controllable to point B* (or equivalently that *point A can be moved to point B*) if a value $C(j\omega_0)$ exists such that $B = C(j\omega_0) \cdot A$, that is if and only if the following conditions hold:

$$M_B = M_A M_0, \qquad \varphi_B = \varphi_A + \varphi_0. \qquad (6)$$

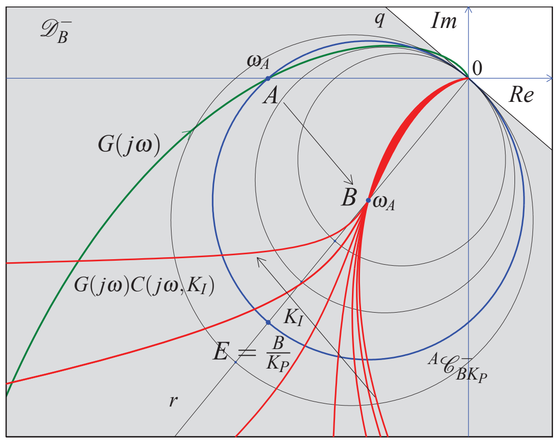

Fig. 4. Controllable domain $\mathscr{D}_B^-$ and graphical design of compensators $C(j\omega, K_I)$ moving point A to B.

Definition 1. Given a point $B \in \mathbb{C}$, let us define "*controllable domain of the PID compensator $C(s)$ to point B*" the set $\mathscr{D}_B^-$ defined as follows:

$$\mathscr{D}_B^- = \left\{ A \in \mathbb{C} \,\middle|\, \exists K_P, K_I, K_D > 0, \exists\, \omega \geq 0 : C(j\omega) \cdot A = B \right\}.$$

It can be easily shown that the domain $\mathscr{D}_B^-$ on Nyquist plane is the half-plane which includes point B and is delimited by the straight line q passing through point O and perpendicular to segment $\overline{B0}$, see the gray region in Fig. 4.

Definition 2. Given a point $B \in \mathbb{C}$, let $\mathscr{C}_B(K_P)$ and $\mathscr{C}_B^-(K_P)$ denote the sets of PID compensators defined as follows

$$\mathscr{C}_B(K_P) = \left\{ B \cdot C(s) \,\middle|\, C(s) \in \mathscr{C}(K_P) \right\}, \qquad (7)$$

$$\mathscr{C}_B^-(K_P) = \left\{ \frac{B}{C(s)} \,\middle|\, C(s) \in \mathscr{C}(K_P) \right\}, \qquad (8)$$

with $\mathscr{C}(K_P)$ defined in (3). Moreover, let $\mathscr{C}_{BK_P}(s) \in \mathscr{C}_B(K_P)$ and $\mathscr{C}_{BK_P}^-(s) \in \mathscr{C}_B^-(K_P)$ denote particular elements of the two sets $\mathscr{C}_B(K_P)$ and $\mathscr{C}_B^-(K_P)$ chosen arbitrarily.

Definition 3. ***(PID Inversion Formulae)*** Given two points $A = M_A e^{j\varphi_A}$ and $B = M_B e^{j\varphi_B}$ of the complex plane $\mathbb{C}$, the PID inversion formulae are defined as follows:

$$\left\{ \begin{aligned} X(A,B) &= \frac{M_B}{M_A}\cos(\varphi_B - \varphi_A), \\ Y(A,B) &= \frac{M_B}{M_A}\sin(\varphi_B - \varphi_A). \end{aligned} \right. \qquad (9)$$

These formulae are similar to the ones used in Phillips (1985) and to the *Inversion Formulae* introduced and used in Marro and Zanasi (1998) and Zanasi and Cuoghi (2011) for lead and lag compensators design.

Property 2. *(From A to B) Given a point $B \in \mathbb{C}$ and chosen a point $A = G(j\omega_A) \in \mathscr{D}_B^-$, the sets $C(s, K_D)$ and $C(s, K_I)$ of* *all the PID compensators $C(s)$ that move point A to point B is obtained from (1) using, respectively, the parameters*

$$K_P = X, \qquad K_I = K_D\omega_A^2 - Y\omega_A, \tag{10}$$

for all $K_D > \frac{Y}{\omega_A}$, or the parameters

$$K_P = X, \qquad K_D = \frac{Y}{\omega_A} + \frac{K_I}{\omega_A^2}, \tag{11}$$

for all $K_I > 0$, where coefficients $X = X(A,B)$ and $Y = Y(A,B)$ are obtained using (9).

Proof: For $\omega = \omega_A$, relation (2) can be rewritten as

$$C(j\omega_A) = K_P + j\left(\omega_A K_D - \frac{K_I}{\omega_A}\right). \tag{12}$$

From relation $B = C(j\omega_A)\cdot A$ and (6) it is evident that point $A = G(j\omega_A) = M_A\, e^{j\varphi_A}$ can be moved to point $B = M_B\, e^{j\varphi_B}$ if and only if

$$C(j\omega_A) = \frac{M_B}{M_A} e^{j(\varphi_B - \varphi_A)} = X + jY. \tag{13}$$

Solving equations (13) with respect to X and Y, one obtains the *PID Inversion Formulae* (9). Equations (10) and (11) follow directly from (12) and (13).

Property 3. *The parameter K_P can be determined on the Nyquist plane as shown in Fig. 4:*

1. *(1) draw the unique circle ${}^{A}\mathscr{C}_B^-$ passing through points A and O having its diameter on the straight line r which passes through points 0 and B;*
2. *(2) the circle ${}^{A}\mathscr{C}_B^-$ intersects the straight line r in points O and $E = B/K_P$;*
3. *(3) the parameter K_P is equal to the modulus of point B over the modulus of point E: $K_P = |B|/|E|$.*

Proof: It follows directly from Property 1 because the circle ${}^{A}\mathscr{C}_B^-$ on the Nyquist plane is the inverse of the frequency response $\mathscr{C}_{BK_P}(j\omega)$ of function $\mathscr{C}_{BK_P}(s) \in \mathscr{C}_B(K_P)$ and because the intersections of circle ${}^{A}\mathscr{C}_B^-$ with the straight line r occur in points O and $E = B/K_P$.

4. SYNTHESIS OF PID COMPENSATORS

Let us consider the case of given steady-state specifications that impose the value of the integrative term $K_I > 0$. This case occurs for example for type-0 systems and design specification on velocity error, or for type-1 systems and design specification on acceleration error. The fact that the integrative term K_I has been fixed do not reduce the admissible domain $\mathscr{D}_B^-$ to point B, that is still the gray region shown in Fig. 4. The other two degrees of freedom K_P and K_D of the regulator can be imposed to meet the phase margin ϕ_m and the gain crossover frequency ω_p.

Design Problem A: (K_I,ϕ_m,ω_p). Given the control scheme of Fig. 3, the transfer function $G(s)$, the steady-state specifications that impose the value of the integrative term $K_I > 0$ and design specifications on the phase margin ϕ_m, and gain crossover frequency ω_p, design a PID compensator $C(s)$ such that the loop gain transfer function $C(j\omega)G(j\omega)$ passes through point $B_p = e^{j(\pi+\phi_m)}$ for $\omega = \omega_p$.

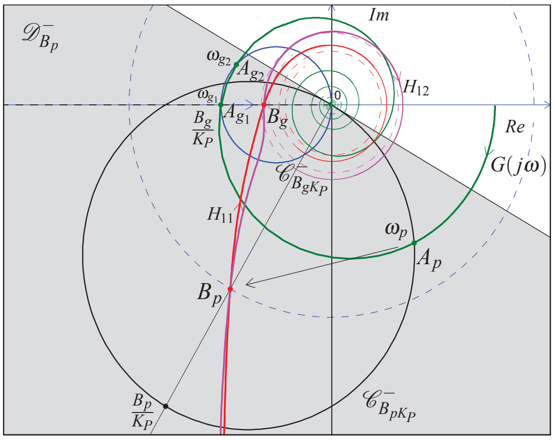

Fig. 5. Graphical solution of Design Problem B on the Nyquist plane.

Solution A: *If point $A = G(j\omega_A)$ belongs to the admissible domain $\mathscr{D}_B^-$ shown in Fig. 4, the solution follows directly from (11) of Property 2 with parameter K_I imposed by the steady-state specifications.*

Let us now consider the case of steady-state specifications that do not constrain the value of K_I. The degree of freedom in the Property 2 can be utilized in order to satisfy another specification. In literature different methods can be found to impose the degree of freedom, such as to obtain real zeros of the compensator, Aström and Hagglund (2006). Let us consider the following design problems.

Design Problem B: (ϕ_m , G_m , ω_p). Given the control scheme of Fig. 3, the transfer function $G(s)$ and design specifications on the phase margin ϕ_m, gain margin G_m and gain crossover frequency ω_p, design a PID compensator $C(s)$ such that the loop gain transfer function $C(j\omega)G(j\omega)$ passes through point $B_p = e^{j(\pi+\phi_m)}$ for $\omega = \omega_p$ and passes through point $B_g = -1/G_m$.

Solution B: *Let $A_p = G(j\omega_p)$ denote the value of $G(j\omega)$ at the desired gain crossover frequency $\omega = \omega_p$, and let $B_p = e^{j(\pi+\phi_m)}$ and $B_g = -1/G_m = M_{B_g}\, e^{j\varphi_{B_g}}$ denote the points corresponding to the desired phase margin ϕ_m and gain margin G_m, respectively. The set $C_p(s,\omega_g)$ of all the PID compensators $C(s)$ which solve Design Problem B is obtained from (1) using the parameters*

$$K_P = X_p > 0, \qquad K_D = \frac{Y_p\omega_p - Y_g\omega_g}{\omega_p^2 - \omega_g^2} > 0, \tag{14}$$

$$K_I = \frac{Y_p\omega_p\omega_g^2 - Y_g\omega_g\omega_p^2}{\omega_p^2 - \omega_g^2} > 0, \tag{15}$$

where the coefficients $X_p = X(A_p,B_p)$, $Y_p = Y(A_p,B_p)$, $X_g = X(A_g,B_g)$ and $Y_g = Y(A_g,B_g)$ are obtained using the inversion formulas (9) with $A_g = G(j\omega_g) = M_{A_g}(\omega_g)\, e^{j\varphi_{A_g}(\omega_g)}$, for all the frequencies ω_g satisfying the relation

$$K_P = X_g(\omega_g) = \frac{M_{B_g}}{M_{A_g(\omega_g)}}\cos(\varphi_{B_g} - \varphi_{A_g}(\omega_g)). \tag{16}$$

A solution $C_p(s,\omega_g)$ of Design Problem B exists only if: 1) the set S_{ω_g} of all the $\omega_g > \omega_p$ satisfying (16) is not empty; 2) $A_p \in \mathscr{D}_{B_p}^-$ and $A_g \in \mathscr{D}_{B_g}^-$; 3) the parameters K_I and K_D in (14) and (15) are real and positive.

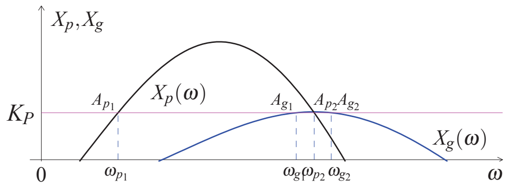

Fig. 6. Functions $X_g(\omega)$ (blue) and $X_p(\omega)$ (black).

Proof: The design specifications define the position of points $B_p = e^{j(\pi+\phi_m)}$, $A_p = G(j\omega_p)$ and $B_g = -1/G_m$. According to Property 2, the compensators $C_p(s, K_D)$ which move point $A_p \in \mathscr{D}^-_{B_p}$ to point B_p are obtained using the parameters K_P and K_I in (10). The free parameter K_D can now be used to force the loop gain frequency response $C_p(j\omega, K_D)G(j\omega)$ to pass through point B_g. This condition can be satisfied only if a frequency ω_g exists such that compensator $C_p(s, K_D)$ moves point $A_g = G(j\omega_g) \in \mathscr{D}^-_{B_g}$ to point B_g, that is only if (16) holds. The frequencies $\omega_g \in S_{\omega_g}$ satisfying (16) are acceptable only if the compensator $C_g(s, K_D)$ which moves point A_g to point B_g, obtained using Property 2, is equal to the compensator $C_p(s, K_D)$. This condition is satisfied only if the two compensators share the same K_I and K_D, that is only if

$$K_I = \omega_p^2 K_D - Y_p\omega_p = \omega_g^2 K_D - Y_g\omega_g. \qquad (17)$$

Solving (17) with respect to K_D one obtains the expression of K_D given in (14), that can be substituted in (17) obtaining (15). The solutions are acceptable only if $K_P, K_D, K_I > 0$.

The solution of equation (16) can be obtained graphically, by plotting $X_g(\omega)$ and by finding all the frequencies $\omega_g \in S_{\omega_g}$ for which $X_g(\omega_g)$ intersects the horizontal line $X_p = K_P$, see Fig. 6. In the example of Fig. 6 it is $S_{\omega_g} = \{\omega_{g1}, \omega_{g2}\}$ and therefore there are two solutions: $C_p(s, \omega_{g1})$ and $C_p(s, \omega_{g2})$. The loop gain frequency responses $H_{11}(j\omega) = C_p(j\omega, \omega_{g1})G(j\omega)$ (red line) and $H_{12}(j\omega) = C_p(j\omega, \omega_{g2})G(j\omega)$ (magenta line) on the Nyquist plane are shown in Fig. 5. The two solutions satisfy the design specifications and are acceptable only if $K_D > 0$ and $K_I > 0$.

Property 4. *The frequencies $\omega_g \in S_{\omega_g}$ satisfying (16) can be graphically determined on the Nyquist plane as shown in Fig. 5:*

(1) *draw the circle $\mathscr{C}^-_{B_pK_P}(j\omega)$ on the Nyquist plane and determine the parameter K_P of compensator $C_p(s, K_D)$ as described in Property 3 when $A = A_p$ and $B = B_p$;*

(2) *draw the circle $\mathscr{C}^-_{B_gK_P}(j\omega)$ having its diameter on the segment defined by points O and $\frac{B_g}{K_P}$;*

(3) *the intersections A_{g1}, A_{g2} of circle $\mathscr{C}^-_{B_gK_P}(j\omega)$ with $G(j\omega)$ correspond to the frequencies ω_{g1}, ω_{g2} belonging to set S_{ω_g}.*

Proof: The circles $\mathscr{C}^-_{B_pK_P}(j\omega)$ (black line) and $\mathscr{C}^-_{B_gK_P}(j\omega)$ (blue line) shown in Fig. 6 represent, respectively, the frequency responses of functions $\mathscr{C}^-_{B_pK_P}(s) \in \mathscr{C}^-_{B_p}(K_P)$ and $\mathscr{C}^-_{B_gK_P}(s) \in \mathscr{C}^-_{B_g}(K_P)$ with $K_P = X_g$ given in (16). These two circles can be easily determined on the Nyquist plane because the points A_p, B_p and B_g are known from the design specifications and K_P is given by the graphical construction described in Property 3. A frequency ω_g satisfying (16) exists only if

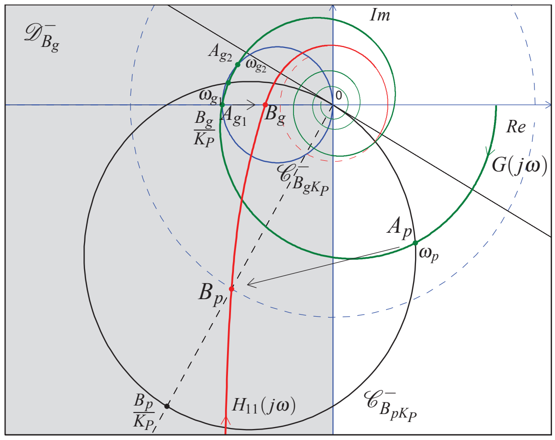

Fig. 7. Graphical solution of Design Problem C.

$$G(j\omega_g)C_{K_P}(j\omega_g) = B_g, \qquad (18)$$

where $C_{K_P}(s)$ is the PID compensator (1) with the value of parameter K_P determined as described above. Relation (18) can also be rewritten as follows

$$G(j\omega) = \frac{B_g}{C_{K_P}(j\omega)} = \mathscr{C}^-_{B_gK_P}(j\omega), \qquad (19)$$

with $\omega = \omega_g$, and therefore it can be solved graphically on the Nyquist plane by finding the intersections $\omega_g \in S_{\omega_g}$ of $G(j\omega)$ with $\mathscr{C}^-_{B_gK_P}(j\omega)$.

Design Problem C: (ϕ_m , G_m , ω_g). Given the control scheme of Fig. 3, the transfer function $G(s)$ and the design specifications on the phase margin ϕ_m, gain margin G_m and phase crossover frequency ω_g, design a PID compensator $C(s)$ such that the loop gain transfer function $C(j\omega)G(j\omega)$ passes through point $B_g = -1/G_m$ for $\omega = \omega_g$ and passes through point $B_p = e^{j(\pi+\phi_m)}$.

Solution C: *Let $A_g = G(j\omega_g)$ denote the value of $G(j\omega)$ at the desired phase crossover frequency ω_g, and let $B_p = e^{j(\pi+\phi_m)}$ and $B_g = -1/G_m = M_{B_g} e^{j\varphi_{B_g}}$ denote the points corresponding to the desired phase and gain margins. The set $C_g(s, \omega_p)$ of all the PID compensators $C(s)$ which solve Design Problem C is obtained from (1) using the parameters*

$$K_P = X_g > 0, \qquad K_D = \frac{Y_p\omega_p - Y_g\omega_g}{\omega_p^2 - \omega_g^2} > 0, \qquad (20)$$

$$K_I = \frac{Y_p\omega_p\omega_g^2 - Y_g\omega_g\omega_p^2}{\omega_p^2 - \omega_g^2} > 0, \qquad (21)$$

where $X_p = X(A_p, B_p)$, $Y_p = Y(A_p, B_p)$, $X_g = X(A_g, B_g)$ and $Y_g = Y(A_g, B_g)$ are obtained using the inversion formulas (9) with $A_p = G(j\omega_p) = M_{A_p}(\omega_p)\, e^{j\varphi_{A_p}(\omega_p)}$, for all the frequencies ω_p satisfying the relation

$$K_P = X_p(\omega_p) = \frac{M_{B_p}}{M_{A_p(\omega_p)}} \cos(\varphi_{B_p} - \varphi_{A_p}(\omega_p)). \qquad (22)$$

A solution $C_g(s, \omega_p)$ exists only if:
1) the set S_{ω_p} of all the $\omega_p > \omega_g$ satisfying (22) is not empty; 2) $A_p \in \mathscr{D}^-_{B_p}$ and $A_g \in \mathscr{D}^-_{B_g}$; 3) the parameters K_I and K_D in (20) and (21) are real and positive.

The proof is quite similar to the one given for Solution B.

Design Problem D: $(\phi_m\ ,\ G_m)$. Given the control scheme of Fig. 3, the transfer function $G(s)$ and the design specifications on the phase margin ϕ_m and gain margin G_m, design a PID compensator $C(s)$ such that the loop gain transfer function $C(j\omega)G(j\omega)$ passes through points $B_p = e^{j(\pi+\phi_m)}$ and $B_g = -1/G_m$.

Solution D: *Let* $B_p = e^{j(\pi+\phi_m)}$ *and* $B_g = -1/G_m = M_{B_g} e^{j\varphi_{B_g}}$ *denote the points corresponding to the desired phase margin* ϕ_m *and gain margin* G_m*. The set* $C_{K_P}(s, \omega_p, \omega_g)$ *of all the compensators* $C(s)$ *which solve the Design Problem D is obtained as follows:*

a) *find all the pairs* $(\omega_p,\ \omega_g) \in S_{\gamma\omega}$ *of frequencies which solve the equation*

$$K_P = X_p(\omega_p) = X_g(\omega_g), \tag{23}$$

where the parameter $K_P > 0$ *is chosen arbitrarily,* $S_{K_P\omega}$ *is the set of all the pairs* $(\omega_p,\ \omega_g)$ *satisfying (23) with* $\omega_g > \omega_p$*, and functions* $X_p(\omega_p)$ *and* $X_g(\omega_g)$ *are defined in (22) and (16) with* $A_p = G(j\omega_p) = M_{A_p}(\omega_p)\, e^{j\varphi_{A_p}(\omega_p)}$ *and* $A_g = G(j\omega_g) = M_{A_g}(\omega_g)\, e^{j\varphi_{A_g}(\omega_g)}$.

b) *for each pair* $(\omega_p,\ \omega_g) \in S_{K_P\omega}$ *compute*

$$\begin{cases} K_D = \dfrac{Y_p\omega_p - Y_g\omega_g}{\omega_p^2 - \omega_g^2} > 0, \\[2ex] K_I = \dfrac{Y_p\omega_p\omega_g^2 - Y_g\omega_g\omega_p^2}{\omega_p^2 - \omega_g^2} > 0. \end{cases} \tag{24}$$

A solution exists only if:
1) K_P *satisfies*

$$0 < K_P < \min\left(\max(X_p(\omega_p)), \max(X_g(\omega_g))\right) \tag{25}$$

2) $S_{K_P\omega}$ *is not empty;* 3) $A_p(\omega_p) \in \mathscr{D}^-_{B_p}$ *and* $A_g(\omega_g) \in \mathscr{D}^-_{B_g}$*;* 4) K_D *and* K_I *in (24) are real and positive.*

Proof: The proof hinges on the fact that $C(s)$ has to be design to move point $A_p = G(j\omega_p)$ to point B_p and point $A_g = G(j\omega_g)$ to point B_g. A solution $C_{K_P}(s, \omega_p, \omega_g)$ exists only if the frequencies ω_p and ω_g satisfy (22) and (16), that is only if they satisfy (23). For each value of K_P satisfying (25), one can find the set $S_{K_P\omega}$ of all the solutions (ω_p, ω_g) of (23).

The solutions of (23) can also be obtained graphically by plotting $X_p(\omega)$ and $X_g(\omega)$ and by finding, for each admissible value of K_P, all the pairs $(\omega_p, \omega_g) \in S_{\omega_p}$ where $X_p(\omega_p)$ and $X_g(\omega_g)$ intersect the horizontal line K_P, see Fig. 6. In the example of Fig. 6 there are three different solutions: $S_{K_P\omega} = \{(\omega_{p1}, \omega_{g1}), (\omega_{p1}, \omega_{g2}), (\omega_{p2}, \omega_{g2})\}$. The solution $(\omega_{p2}, \omega_{g1})$ is not admissible because $\omega_{p2} > \omega_{g1}$. The loop gain frequency responses $H_{11}(s)$, $H_{12}(s)$ and $H_{22}(s)$ of these three solutions on the Nyquist plane are shown in Fig. 8. These solutions are acceptable only if parameters K_D and K_I given in (24) are positive.

The solution of (23) can also be obtained on the Nyquist plane. Given points B_p and B_g and a desired value for $K_P > 0$, the circles $\mathscr{C}^-_{B_pK_P}(j\omega)$ and $\mathscr{C}^-_{B_gK_P}(j\omega)$ can be drawn on the Nyquist plane as the circles having their diameters on the segments defined by points $\{O, \frac{B_p}{K_P}\}$ and $\{O, \frac{B_g}{K_P}\}$, respectively, as described in Property (3). Each pair (ω_p, ω_g) corresponding to the intersections of $G(j\omega)$ with circles $\mathscr{C}^-_{B_{pK_P}}(j\omega)$ and $\mathscr{C}^-_{B_{gK_P}}(j\omega)$ is a possible solution for Design Problem D. If $G(j\omega)$ does not intersect both circles $\mathscr{C}^-_{B_{pK_P}}(j\omega)$ and $\mathscr{C}^-_{B_{gK_P}}(j\omega)$, the chosen value of K_P is not acceptable.

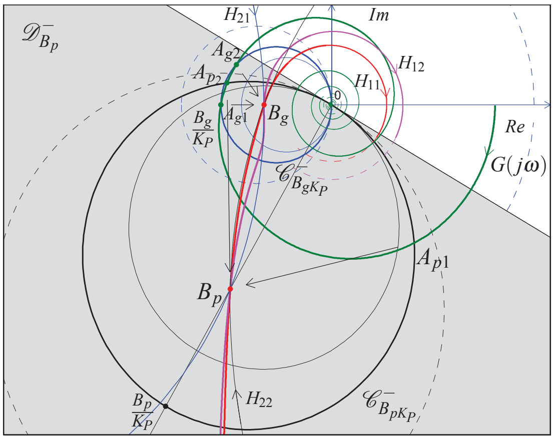

Fig. 8. Graphical solution of Design Problem *D*.

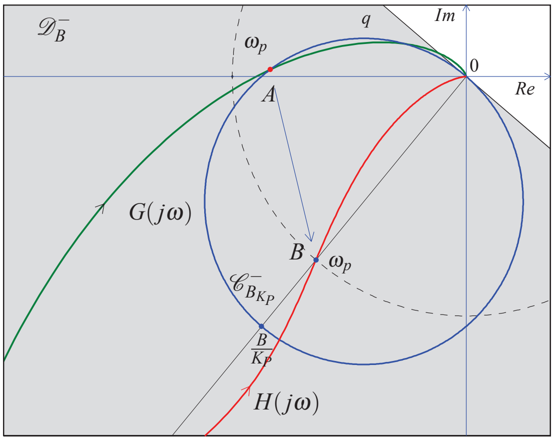

Fig. 9. Graphical solution of Design problem A .

5. NUMERICAL EXAMPLES

Design Problem A: Given the following type-1 plant

$$G(s) = \frac{28(s+1)}{s(s+1.5)^2(s+3)},$$

design a PID compensator $C(s)$ in order to achieve the acceleration constant $K_a = 2$, the phase margin $\phi_m = 50^o$ and the gain crossover frequency $\omega_p = 2.5$.

Solution: The integral constant K_I is determined by steady-state requirement as

$$K_a = \lim_{s\to 0} s^2 C(s)\, G(s) = \frac{28K_I}{1.5^2 \cdot 3} = 2,$$

that leads to $K_I = 0.482$. The point $A = G(j\omega_p) = 0.84e^{j178^o}$ belongs to the admissible domain $\mathscr{D}^-_B$ defined by $B = e^{-j230^o}$, see Fig. 9. From inversion formulae (9) it follows that $X = 0.734$ and $Y = 0.939$. Finally relations (11) lead to $K_P = 0.734$ and $K_D = 0.433$. The designed compensator (1) is

$$C(s) = 0.734 + 0.433s + \frac{0.482}{s}.$$

The corresponding loop gain transfer function $H(j\omega) = C(j\omega)G(j\omega)$ is plotted in red in Fig. 9. The improvement of the closed-loop step response is shown in Fig. 10

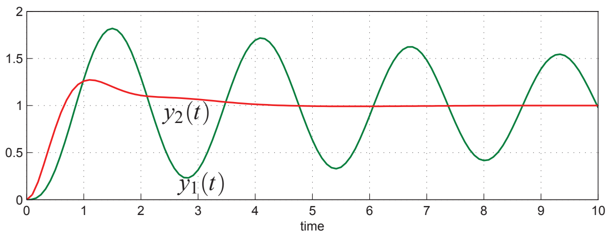

Fig. 10. Design problem A: closed-loop step responses without the controller $y_1(t)$ and with the controller $y_1(t)$.

Design Problem B: Given the system proposed in Wang et al. (1999)

$$G(s) = \frac{1}{0.12s^2 + 1.33s + 1.24} e^{-2s}, \tag{26}$$

design a PID compensator to meet the following design specifications: phase margin $\phi_m = 60°$, gain margin $G_m = 3$ and gain crossover frequency $\omega_p = 0.3325$.

Solution. The modulus and the phase of point B_p are $M_{B_p} = 1$, $\varphi_{B_p} = -120°$. The value of K_P obtained by (14) is $K_P = X_p = 0.6107$, the frequencies obtained solving (16) are $\omega_{g1} = 1.1052$ and $\omega_{g2} = 1.257$, see Fig. 6. The corresponding points on $G(j\omega)$ are $A_{g1} = 0.546\, e^{j180°}$ and $A_{g2} = 0.506\, e^{j158.1°}$. From (14) and (15), the regulators satisfying the Design Problems B are

$$C_1(s) = \frac{0.3449s^2 + 0.6107s + 0.4212}{s}, \tag{27}$$

$$C_2(s) = \frac{0.4706s^2 + 0.6107s + 0.4351}{s}. \tag{28}$$

The corresponding loop gain transfer functions $H_{11}(j\omega)$ (red line) and $H_{12}(j\omega)$ (magenta line) are plotted in Fig. 5.

Design Problem C: Given the plant (26), design a PID compensator to meet design specifications on the phase margin $\phi_m = 60°$, the gain margin $G_m = 3$ and the phase crossover frequency $\omega_g = 1.1052$.

Solution. The modulus and the phase of point B_g are $M_{B_g} = 0.333$ and $\varphi_{B_g} = 180°$. The value of K_P obtained by (20) is $K_P = X_g = 0.6107$. The frequencies obtained solving (22) are $\omega_{p1} = 0.332$ and $\omega_{p2} = 1.18$, see Fig. 6. The corresponding points on $G(j\omega)$ are $A_{p1} = 0.767\, e^{-j57.9°}$ and $A_{p2} = 0.525\, e^{j168.7°}$. Since $\omega_{p2} > \omega_g$, the corresponding PID is not acceptable. The unique admissible solution is the PID (27) obtained substituting ω_{p1} in (20) and (21) . The corresponding loop gain transfer function $H_{11}(j\omega)$ is plotted in red in Fig. 7.

Design Problem D: Given the plant (26), design a PID compensator to meet design specifications on the phase margin $\phi_m = 60°$and the gain margin $G_m = 3$.

Solution. The design specifications define the points $B_p = e^{-j120°}$, $B_g = 0.333e^{j180°}$ and $K_P = X_p = X_g = 0.6107$. The four solutions $(\omega_{pi}, \omega_{gj})$ of equation (23) can be graphically determined as shown in Fig. 6. The two acceptable regulators (27), (28) are obtained for $(\omega_{p1} = 0.332, \omega_{g1} = 1.1052)$ and $(\omega_{p1} = 0.332, \omega_{g2} = 1.257)$. The solution determined by $(\omega_{p2} = 1.18, \omega_{g1} = 1.1052)$ is not acceptable because $\omega_{p2} > \omega_{g1}$. The solution determined by $(\omega_{p2} = 1.18, \omega_{g2} = 1.257)$ is not acceptable because the PID parameters K_D and K_I are negative and the controlled system is unstable. The obtained loop gain transfer functions are plotted in Fig. 8.

6. CONCLUSION

The presented graphical and analytical methods for the design of PID controllers are based on the use of PID inversion formulae, which allow to achieve given design specifications on the gain margin, the phase margin and the gain or phase crossover frequency. One of the main advantages of the graphical solution over other graphical approaches, such as Yeung (2000), is that it can be directly determined in the complex plane by finding the intersections of the frequency response of the plant with circles, that can be easily determined from the design specifications. The simplicity of the method and its graphical interpretation on the Nyquist plane seem to be useful both for educational and industrial purposes. It is well known that in the process of learning the graphical representation of a numerical solution makes it easier to understand and easier to recall. The drawing and the comparison of the same function plotted in different diagrams, i.e. the loop gain frequency response on the Bode, the Nyquist and the Nichols diagrams, have a great educational value. They emphasize some properties hidden in a single representation and lead to a deeper knowledge of the concepts. In this paper some properties of PID regulators on the Nyquist diagrams are pointed out. These have been used to exactly solve the considered design problem to obtain the system robustness.

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