Stability Analysis of PID Controlled Local Model Networks ⋆

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Abstract: This paper addresses closed-loop stability analysis of PID controlled local model networks. The proposed method allows to investigate exponential or asymptotic stability of the closed-loop system. For this purpose a common quadratic Lyapunov function is used as stability criterion. Due to the fact that the Lyapunov approach requires a state-space model a suitable closed-loop state-space system with integration of the controller parameters is introduced. An example demonstrates the effectiveness of the proposed method.

Keywords: Stability analysis, Lyapunov function, PID controllers, Local area networks, Local control

1. INTRODUCTION

For nonlinear systems model-based controller design is a well established approach. For this purpose local model networks (LMN) from the family of multiple-model approaches (e.g. Murray-Smith and Johansen (1997)) offer a versatile structure. These model architectures interpolate between different local models, each valid in a certain operating regime. Each operating regime represents a simple model, e.g. a linear regression model, where the local dynamics are usually defined as transfer functions. Due to the transparency of the LMN structure the incorporation of prior (physical) knowledge is easily possible. When controllers are designed for LMNs closed-loop stability is a key issue. Basically, the global closed-loop may become unstable even if all local closed-loops are stable, Feng (2006). For open-loop and closed-loop systems with a state-space controller common criteria are based on Lyapunov's direct method. These criteria investigate asymptotic or exponential stability, respectively, and result in linear matrix inequalities (LMIs), (e.g. Feng (2006, 2010); Kim and Lee (2000); Precup and Tomescu (2009)). For PID controlled LMNs common criteria investigate bounded input - bounded output (BIBO) stability by using the passivity theorem (e.g. Sio and Lee (1998); Jia et al. (2006)) or the small gain theorem (e.g. Ding et al. (2003); Mohan and Sinha (2008)). The main disadvantage of BIBO stability is that it permits limit cycles and thus it is less meaningful than asymptotic or exponential stability.
The main contribution of this paper is the closed-loop stability analysis of LMNs with PID controllers. For this purpose a novel approach is presented. Due to the fact that both the local dynamics and the PID controller are usually defined as transfer functions and the stability criteria require state-space models, the closed-loop, which consists of LMN and PID controller, is transformed into a suitable state-space model. The used stability criterion is taken from Kim and Lee (2000), adapted for the considered closed-loop system and extended by a decay rate to provide exponential stability as well as asymptotic stability.
This paper is organized as follows: The architecture of local model networks is briefly described in section 2. The PID controller for LMNs is introduced in section 3. In section 4 the transformation as well as the state-space model of the closed-loop control system is described. Section 5 addresses stability and the basic concept of Lyapunov stability as well as the used closed-loop stability criterion. In section 6 the effectiveness of the proposed method is shown by means of two different PID controllers for the same LMN. The paper is concluded by some remarks in section 7.

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2. LOCAL MODEL NETWORKS

The architecture of dynamic local model networks is depicted in Fig. 1. First, an ordered set for the indices of the local models is defined:

$$\mathcal{I} = (i \in \mathbb{N} | 1 \leq i \leq I) \tag{1}$$

where I denotes the number of local linear models. Local model networks have an input vector $\boldsymbol{r}(k)$ with past inputs and outputs according to Fig. 1:

$$\begin{aligned}\boldsymbol{r}^T(k) = [&u(k-1) \ \ldots \ u(k-m) \\ &\hat{y}(k-1) \ \ldots \ \hat{y}(k-n)] \,, \ \boldsymbol{r}^T(k) \in \mathbb{R}^{1\times O}\end{aligned} \tag{2}$$

where m denotes the numerator order and n denotes the system order. All local model outputs

$$\hat{y}_i(k) = \boldsymbol{r}^T(k)\boldsymbol{\theta}_i \tag{3}$$

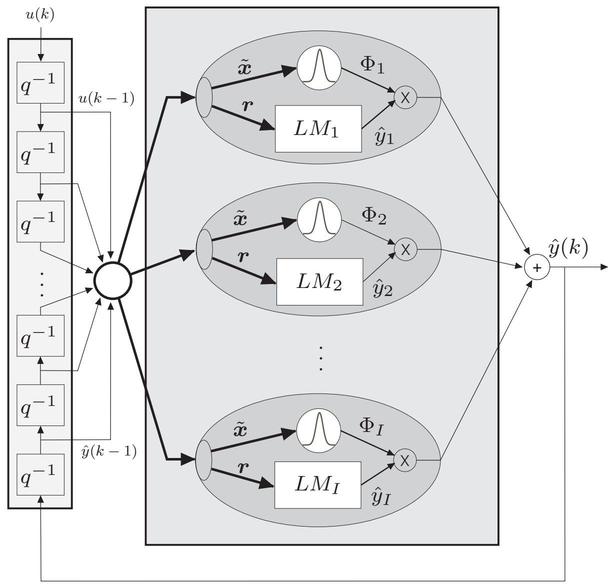

Fig. 1. Architecture of a local model network with external dynamics

with the local parameter vectors

$$\boldsymbol{\theta}_i = \left[b_1^{(i)} \ \dots \ b_m^{(i)} \ a_1^{(i)} \ \dots \ a_n^{(i)} \right]^T, \ \boldsymbol{\theta}_i \in \mathbb{R}^{O\times 1} \qquad (4)$$

are used subsequently to form the global model output $\hat{y}(k)$ by weighted aggregation, see Fig. 1:

$$\hat{y}(k) = \sum_{\mathcal{I}} \Phi_i(\tilde{\boldsymbol{x}}(k))\hat{y}_i(k), \qquad (5)$$

where the validity functions Φ_i are constrained to form a partition of unity:

$$\sum_{\mathcal{I}} \Phi_i(\tilde{\boldsymbol{x}}(k)) = 1, \ \forall k \in \mathbb{N}^+ \qquad (6)$$

$$0 \le \Phi_i(\tilde{\boldsymbol{x}}(k)) \le 1, \ \forall i \in \mathcal{I}, \ \forall k \in \mathbb{N}^+. \qquad (7)$$

From Fig. 1 it becomes obvious that the input vector $\tilde{\boldsymbol{x}}(k)$ for the validity functions Φ_i, which lies in the so-called partition space, can be chosen differently to the input vector $\boldsymbol{x}$ for the local models. This is an important feature of LMN.

3. PID-CONTROLLER FOR LOCAL MODEL NETWORKS

The discrete-time PID control law is as follows, Ogata (2006):

$$u(k) = K_P \left[e(k) + \frac{T_S}{T_N} \sum_{i=0}^{k-1} e(i) + \frac{T_V}{T_S} \left(e(k) - e(k-1)\right) \right] \qquad (8)$$

where T_S denotes the sampling time. The control error is

$$e(k) = w(k) - \hat{y}(k) \qquad (9)$$

where $w(k)$ denotes the reference signal. Rewriting (8) for $(k-1)$ and subtracting it from (8), one gets the PID control algorithm

$$u(k) = u(k-1) + d_0 e(k) + d_1 e(k-1) + d_2 e(k-2), \qquad (10)$$

with the following coefficients:

$$d_0 = K_P \left[1 + \frac{T_V}{T_S} \right], \qquad (11)$$

$$d_1 = K_P \left[\frac{T_S}{T_N} - \frac{2T_V}{T_S} - 1 \right], \qquad (12)$$

$$d_2 = K_P \frac{T_V}{T_S}. \qquad (13)$$

Eq. (10) can be reformulated by inserting (9) in (10)

$$u(k) = u(k-1) + \underbrace{\underbrace{\boldsymbol{k}_{PID}^T(k)\boldsymbol{w}(k)}_{v(k)} - \boldsymbol{k}_{PID}^T(k)\hat{\boldsymbol{y}}(k)}_{\Delta u(k)} \qquad (14)$$

with

$$\boldsymbol{k}_{PID}^T(k) = [d_2(k) \ d_1(k) \ d_0(k)],$$
$$\hat{\boldsymbol{y}}(k) = \begin{bmatrix} \hat{y}(k-2) \\ \hat{y}(k-1) \\ \hat{y}(k) \end{bmatrix}, \ \boldsymbol{w}(k) = \begin{bmatrix} w(k-2) \\ w(k-1) \\ w(k) \end{bmatrix} \qquad (15)$$

Control design for LMN mostly involves the design of many local controllers, each of which is associated to a particular local model, yielding a so called local controller network (LCN), Hunt and Johansen (1997). In view of the actual nonlinear system which the LMN is supposed to represent the local model partitioning then represents a meaningful scheduling strategy. In this context there is an associated set of PID parameters $\boldsymbol{k}_{PID,i}^T$ for each local model which are aggregated to the global time-varying matrix $\boldsymbol{k}_{PID}^T(k)$ as follows:

$$\boldsymbol{k}_{PID}^T(k) = \sum_{\mathcal{I}} \Phi_i(\tilde{\boldsymbol{x}}(k))\boldsymbol{k}_{PID,i}^T,$$
$$\boldsymbol{k}_{PID,i}^T = \left[d_2^{(i)} \ d_1^{(i)} \ d_0^{(i)} \right], \ \forall i \in \mathcal{I} \qquad (16)$$

4. CLOSED-LOOP STATE-SPACE MODEL

4.1 Basic Concept

To investigate closed-loop stability analysis by means of Lyapunov's direct method the system has to be transformed into a suitable state-space notation, see Fig. 2. Thus, the LMN as well as the integrator of the PID-controller are transferred into a non-minimal state space system. According to Fig. 2 and (14) the PID control law is divided into three parts:

(1) past input $u(k-1)$ (integrator)
(2) "filtered" reference signal $v(k)$
(3) feedback $\boldsymbol{k}_{PID}^T\hat{\boldsymbol{y}}(k)$

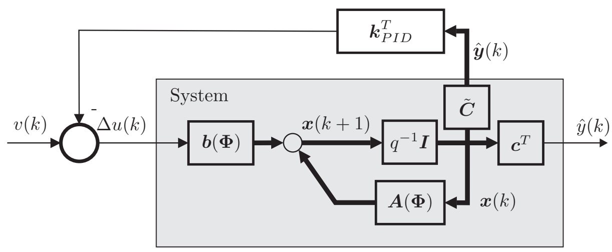

Fig. 2. PID controller in state-space

The system can be generally defined as

$$\boldsymbol{x}(k+1) = \boldsymbol{A}(\boldsymbol{\Phi})\boldsymbol{x}(k) + \boldsymbol{b}(\boldsymbol{\Phi})\Delta u(k) \qquad (17)$$

$$\hat{y}(k) = \boldsymbol{c}^T \boldsymbol{x}(k) \qquad (18)$$

where the state vector contains time shifted in- and outputs as follows

$$\boldsymbol{x}(k) = \begin{pmatrix} u(k-o) \\ \vdots \\ u(k-1) \\ \hline \hat{y}(k-p+1) \\ \vdots \\ \hat{y}(k-1) \\ \hat{y}(k) \end{pmatrix}, \quad \boldsymbol{x} \in \mathbb{R}^{o+p\times 1}. \tag{19}$$

The system matrices are as follows

$$\boldsymbol{A}(\boldsymbol{\Phi}) = \sum_{\mathcal{I}} \Phi_i(\tilde{\boldsymbol{x}}(k))\boldsymbol{A}_i, \quad \boldsymbol{A} \in \mathbb{R}^{o+p\times o+p} \tag{20}$$

$$\boldsymbol{b}(\boldsymbol{\Phi}) = \sum_{\mathcal{I}} \Phi_i(\tilde{\boldsymbol{x}}(k))\boldsymbol{b}_i, \quad \boldsymbol{b} \in \mathbb{R}^{o+p\times 1} \tag{21}$$

$$\boldsymbol{c}^T = [0 \ \cdots \ 0 \ 1], \quad \boldsymbol{c}^T \in \mathbb{R}^{1\times o+p} \tag{22}$$

with

$$o = \max(m-1, 1) \tag{23}$$

$$p = \max(n, 3). \tag{24}$$

4.2 Construction of the system matrices

Rewriting (3) for $(k+1)$:

$$\begin{aligned} \hat{y}_i(k+1) =& u(k)b_1^{(i)} + u(k-1)b_2^{(i)} + \\ &+ \ldots + u(k-m+1)b_m^{(i)} + \\ &+ \hat{y}(k)a_1^{(i)} + \ldots + \hat{y}(k-n+1)a_n^{(i)} \end{aligned} \tag{25}$$

and inserting $u(k)$ of (14) in (25) the outputs of the local models are as follows:

$$\begin{aligned} \hat{y}_i(k+1) =& u(k-1)(b_1^{(i)} + b_2^{(i)}) + \Delta u(k)b_1^{(i)} + \\ &+ \ldots + u(k-m+1)b_m^{(i)} + \\ &+ \hat{y}a_1^{(i)} + \ldots + \hat{y}(k-n+1)a_n^{(i)} \end{aligned} \tag{26}$$

The feedback loop of the input $u(k-1)$ is integrated in the system matrix to provide a feedback matrix with only three entries:

$$\boldsymbol{A}_i = \hat{\boldsymbol{A}}_i + \boldsymbol{b}_i \cdot [\boldsymbol{0}_{1\times o-1} \ 1 \ \boldsymbol{0}_{1\times p}], \ \boldsymbol{A}_i \in \mathbb{R}^{o+p\times o+p} \tag{27}$$

with $\hat{\boldsymbol{A}}_i$ according to (28) (see next page). The local input matrices $\boldsymbol{b}_i$ are as follows:

$$\boldsymbol{b}_i = \begin{bmatrix} \boldsymbol{0}_{o-1\times 1} \\ 1 \\ \hline \boldsymbol{0}_{p-1\times 1} \\ b_1^{(i)} \end{bmatrix}, \quad \boldsymbol{b}_i \in \mathbb{R}^{o+p\times 1} \tag{29}$$

Remark 1. The notation of $\boldsymbol{A}_i$ according to (27) and $\boldsymbol{b}_i$ according to (29) provides that (26) and the last row of (17) are equal.

4.3 Feedback Loop

The input vector $\hat{\boldsymbol{y}}(k)$ of the feedback matrix is calculated as follows:

$$\hat{\boldsymbol{y}}(k) = \tilde{\boldsymbol{C}}\boldsymbol{x}(k) \tag{30}$$

with

$$\tilde{\boldsymbol{C}} = \begin{bmatrix} \boldsymbol{0}_{3\times o+p-3} & \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \end{bmatrix}, \ \tilde{\boldsymbol{C}} \in \mathbb{R}^{3\times o+p}$$

5. STABILITY CONCEPTS FOR LOCAL MODEL NETWORKS

5.1 Basic Notations

For nonlinear systems a number of refined stability concepts, such as marginal stability, asymptotic stability and exponential stability are available, Slotine and Li (1991):

Definition 2. An equilibrium state $\boldsymbol{x}^e$ is *marginally stable* (Fig. 3, dashed line) if for every neighborhood $U > 0$ of $\boldsymbol{x}^e$ there is a neighborhood $T > 0$, $T \subseteq U$ of $\boldsymbol{x}^e$ such that every solution $\boldsymbol{x}(k)$ starting within $T(\boldsymbol{x}(0) \in T)$ remains within U for all $k > 0$. Otherwise the equilibrium point $\boldsymbol{x}^e$ is *unstable* (Fig. 3, dotted line).
Note that $\boldsymbol{x}(k)$ need not approach $\boldsymbol{x}^e$.

Definition 3. An equilibrium state $\boldsymbol{x}^e$ is *asymptotically stable* (Fig. 3, dash-dotted line) if it is marginal stable and additionally T can be chosen so that $\|\boldsymbol{x}(k) - \boldsymbol{x}^e\| \to 0$ as $k \to \infty$ for all $\boldsymbol{x}(0) \in T$.

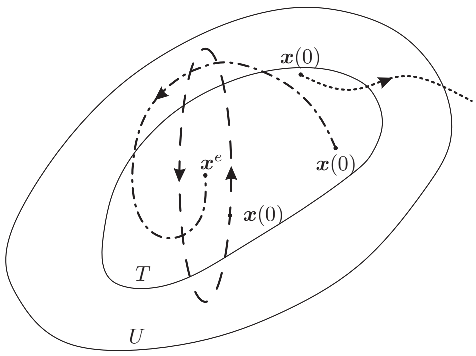

Fig. 3. Stability definitions

The following definition of exponential stability was adopted from Bernal and Husek (2005):

Definition 4. A discrete-time system is said to be *globally exponentially stable* if there exist positive constants α, $0 < \alpha < 1$ and $\beta > 0$, such that

$$\|\boldsymbol{x}(k)\| \leq \alpha^k \beta \|\boldsymbol{x}(0)\|, \ \forall k \in \mathbb{N}^+ \tag{31}$$

The number α is known as the decay rate.

5.2 Lyapunov Stability

The global stability of a PID controlled LMN can be proved by Lyapunov's direct method. This general approach is based on a state space formulation of the system, such as (17), and one has to find a suitable Lyapunov function $V(\boldsymbol{x}): \ \mathbb{R}^{o+p} \to \mathbb{R}$, Feng (2010).

Definition 5. A Lyapunov function basically has to satisfy four properties to provide *asymptotic* stability of a discrete-time system:

i) $V(\boldsymbol{x}(k) = 0) = 0$
ii) $V(\boldsymbol{x}(k)) > 0$ for $\boldsymbol{x}(k) \neq 0$
iii) $V(\boldsymbol{x}(k))$ approaches infinity as $\|\boldsymbol{x}(k)\| \to \infty$
iv) $\Delta V(\boldsymbol{x}(k)) = V(k+1) - V(k) < 0, \ \forall k \in \mathbb{N}^+$.

Lemma 6. For *exponential* stability, the Lyapunov function must satisfy i)-iii) of Definition 5 and decrease strictly

$$\hat{\boldsymbol{A}}_i = \left[\begin{array}{cc|c} \boldsymbol{0}_{o-1\times 1} & \boldsymbol{I}_{o-1\times o-1} & \\ \multicolumn{2}{c|}{\boldsymbol{0}_{1\times o}} & \boldsymbol{0}_{o\times p} \\ \hline \multicolumn{2}{c|}{\boldsymbol{0}_{p-1\times o}} & \boldsymbol{0}_{p-1\times 1} \quad \boldsymbol{I}_{p-1\times p-1} \\ \multicolumn{2}{c|}{b_{o+1}^{(i)} \cdots b_s^{(i)} \cdots b_2^{(i)}} & a_p^{(i)} \cdots a_q^{(i)} \cdots a_1^{(i)} \end{array}\right], \quad \begin{array}{c} \hat{\boldsymbol{A}}_i \in \mathbb{R}^{o+p\times o+p}, \\ \text{if } o=1: \; b_{o+1}^{(i)} = 0, \\ \forall\, q > n: \; a_q^{(i)} = 0, \\ \forall i \in \mathcal{I} \end{array} \tag{28}$$

monotonically over time k with a decay rate α according to Definition 4:

$$V(k+1) - \alpha^2 V(k) \leq 0, \; \forall k \in \mathbb{N}^+ \tag{32}$$

The proof of Lemma 6 is given in Bernal and Husek (2005).

When LMN are considered it is common to restrict the search to the class of quadratic Lyapunov functions

$$V(k) = \boldsymbol{x}^T(k)\boldsymbol{P}\boldsymbol{x}(k) > 0, \quad \boldsymbol{P} \succ 0 \tag{33}$$

where $\boldsymbol{P}$ is a positive definite matrix, Feng (2006).
The following theorem provides a statement on exponential stability of a PID controlled LMN using a common quadratic Lyapunov function:

Theorem 7. The equilibrium of the PID controlled dynamic LMN (5) is asymptotically ($\alpha = 1$) or exponentially ($0 < \alpha < 1$) stable via the control law (14) in the large if there exist symmetric matrices $\boldsymbol{Q}$ and $\boldsymbol{Y}_{ij}$ and a decay rate α such that

$$\boldsymbol{Q} \succ 0 \tag{34}$$

$$\begin{bmatrix} \alpha^2\boldsymbol{Q} - \boldsymbol{Y}_{ii} & \boldsymbol{Q}\left(\boldsymbol{A}_i^T - \tilde{\boldsymbol{C}}^T\boldsymbol{k}_{PID,i}\boldsymbol{b}_i^T\right) \\ \left(\boldsymbol{A}_i - \boldsymbol{b}_i\boldsymbol{k}_{PID,i}^T\tilde{\boldsymbol{C}}\right)\boldsymbol{Q} & \boldsymbol{Q} \end{bmatrix} \succ 0 \tag{35}$$

for (36) see next page

$$\tilde{\boldsymbol{Y}} = \begin{pmatrix} \boldsymbol{Y}_{11} & \boldsymbol{Y}_{12} & \cdots & \boldsymbol{Y}_{1I} \\ \boldsymbol{Y}_{12} & \boldsymbol{Y}_{22} & \cdots & \boldsymbol{Y}_{2I} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{Y}_{1I} & \boldsymbol{Y}_{2I} & \cdots & \boldsymbol{Y}_{II} \end{pmatrix} \succ 0 \tag{37}$$
$$\forall i \in \mathcal{I}, \; \forall i < j \leq I$$

Proof. The proof of Theorem 7 is based on the ideas given in Tanaka et al. (1998); Kim and Lee (2000).
From inserting $\Delta u(k)$ with $v(k) = 0$ of (14), (20) and (21) in (17) follows

$$\boldsymbol{x}(k+1) = \sum_{i,j\in\mathcal{I}} \Phi_i(k)\Phi_j(k)\left\{\boldsymbol{A}_i - \boldsymbol{b}_i\boldsymbol{k}_{PID,j}^T\tilde{\boldsymbol{C}}\right\}\boldsymbol{x}(k) \tag{38}$$

$$\begin{aligned} &= \sum_{i\in\mathcal{I}} \Phi_i^2(k)\boldsymbol{\Lambda}_{ii}\boldsymbol{x}(k) \\ &\quad + 2\sum_{i<j\leq I} \Phi_i(k)\Phi_j(k)\boldsymbol{\Lambda}_{ij}\boldsymbol{x}(k) \end{aligned} \tag{39}$$

with

$$\boldsymbol{\Lambda}_{ii} = \boldsymbol{G}_{ii}, \; \boldsymbol{\Lambda}_{ij} = \frac{\boldsymbol{G}_{ij} + \boldsymbol{G}_{ji}}{2}, \; \boldsymbol{G}_{ij} = \boldsymbol{A}_i - \boldsymbol{b}_i\boldsymbol{k}_{PID,j}^T\tilde{\boldsymbol{C}}. \tag{40}$$

Inserting the closed-loop transfer function (38) and the Lyapunov function candidate (33) in (32) and using the abbreviations of (40) results in:

$$\begin{aligned} &V(k+1) - \alpha^2 V(k) = \\ &= \sum_{i,j,l,m\in\mathcal{I}} \Phi_i(k)\Phi_j(k)\Phi_l(k)\Phi_m(k) \\ &\quad \cdot \boldsymbol{x}^T(k)\left[\boldsymbol{G}_{ij}^T\boldsymbol{P}\boldsymbol{G}_{kl} - \alpha^2\boldsymbol{P}\right]\boldsymbol{x}(k) \\ &= \frac{1}{4}\sum_{i,j,l,m\in\mathcal{I}} \Phi_i(k)\Phi_j(k)\Phi_l(k)\Phi_m(k) \\ &\quad \cdot \boldsymbol{x}^T(k)\left[(\boldsymbol{G}_{ij} + \boldsymbol{G}_{\boldsymbol{ji}})^T\boldsymbol{P}(\boldsymbol{G}_{lm} + \boldsymbol{G}_{ml}) - 4\alpha^2\boldsymbol{P}\right]\boldsymbol{x}(k) \\ &\leq \frac{1}{4}\sum_{i,j\in\mathcal{I}} \Phi_i(k)\Phi_j(k) \\ &\quad \cdot \boldsymbol{x}^T(k)\left[(\boldsymbol{G}_{ij} + \boldsymbol{G}_{\boldsymbol{ji}})^T\boldsymbol{P}(\boldsymbol{G}_{ij} + \boldsymbol{G}_{ji}) - 4\alpha^2\boldsymbol{P}\right]\boldsymbol{x}(k) \\ &= \sum_{i,j\in\mathcal{I}} \Phi_i(k)\Phi_j(k)\boldsymbol{x}^T(k)\left[\boldsymbol{\Lambda}_{ij}^T\boldsymbol{P}\boldsymbol{\Lambda}_{ij} - \alpha^2\boldsymbol{P}\right]\boldsymbol{x}(k) \\ &\leq -\sum_{\mathcal{I}} \Phi_i^2(k)\boldsymbol{x}^T(k)\boldsymbol{X}_{ii}\boldsymbol{x}(k) \\ &\quad - 2\sum_{i<j\leq I} \Phi_i(k)\Phi_j(k)\boldsymbol{x}^T(k)\boldsymbol{X}_{ij}\boldsymbol{x}(k) \\ &= -\begin{pmatrix} \Phi_1(k)\boldsymbol{x}(k) \\ \Phi_2(k)\boldsymbol{x}(k) \\ \vdots \\ \Phi_I(k)\boldsymbol{x}(k) \end{pmatrix}^T \begin{pmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{12} & \dots & \boldsymbol{X}_{1I} \\ \boldsymbol{X}_{12} & \boldsymbol{X}_{22} & \dots & \boldsymbol{X}_{2I} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{X}_{1I} & \boldsymbol{X}_{2I} & \dots & \boldsymbol{X}_{II} \end{pmatrix} \begin{pmatrix} \Phi_1(k)\boldsymbol{x}(k) \\ \Phi_2(k)\boldsymbol{x}(k) \\ \vdots \\ \Phi_I(k)\boldsymbol{x}(k) \end{pmatrix} \\ &= -\boldsymbol{x}(k)^T\boldsymbol{\Phi}^T(k)\tilde{\boldsymbol{X}}\boldsymbol{\Phi}(k)\boldsymbol{x}(k) < 0 \end{aligned} \tag{41}$$

The introduction of the $\boldsymbol{X}_{ij}$ matrices allows that not all controller-plant combinations ($\boldsymbol{G_{ij}}, \; i \neq j$) need to be stable. Thus, it relaxes the conservatism of the proposed approach.
From (41) follows:

$$\begin{aligned} &\sum_{i\in\mathcal{I}} \Phi_i^2(k)\boldsymbol{x}^T(k)\left[\boldsymbol{\Lambda}_{ii}^T\boldsymbol{P}\boldsymbol{\Lambda}_{ii} - \alpha^2\boldsymbol{P} + \boldsymbol{X}_{ii}\right]\boldsymbol{x}(k) \\ &+ 2\sum_{i<j\leq I} \Phi_i(k)\Phi_j(k) \\ &\cdot \boldsymbol{x}^T(k)\left[\boldsymbol{\Lambda}_{ij}^T\boldsymbol{P}\boldsymbol{\Lambda}_{ij} - \alpha^2\boldsymbol{P} + \boldsymbol{X}_{ij}\right]\boldsymbol{x}(k) < 0 \end{aligned} \tag{42}$$

$$\boldsymbol{x}(k)^T\boldsymbol{\Phi}^T(k)\tilde{\boldsymbol{X}}\boldsymbol{\Phi}(k)\boldsymbol{x}(k) > 0 \tag{43}$$

Eq. (35) and (36) of Theorem 7 follow from premultiplying and postmultiplying (42) with $\boldsymbol{Q}$ where

$$\boldsymbol{Q} = \boldsymbol{P}^{-1}, \; \boldsymbol{Y}_{ij} = \boldsymbol{Q}\boldsymbol{X}_{ij}\boldsymbol{Q}, \tag{44}$$

inserting (40) and applying the Schur complement (Boyd et al. (1994)). From (43) and (44) directly follows (37) of Theorem 7.

6. EXAMPLE

A stable second order Wiener model is considered. It consists of a dynamic linear block with a normalized transfer function $G_L(z) = V(z)/U(z)$ in cascade with a static nonlinearity $f(v)$ at the output with v as the intermediate variable at the output of the linear block.

$$\begin{bmatrix} \alpha^2 \boldsymbol{Q} - \boldsymbol{Y}_{ij} & \frac{1}{2}\boldsymbol{Q}\left[\boldsymbol{A}_i^T + \boldsymbol{A}_j^T - \tilde{\boldsymbol{C}}^T\left(\boldsymbol{k}_{PID,j}\boldsymbol{b}_i^T + \boldsymbol{k}_{PID,i}\boldsymbol{b}_j^T\right)\right] \\ \frac{1}{2}\left[\boldsymbol{A}_i + \boldsymbol{A}_j - \left(\boldsymbol{b}_i\boldsymbol{k}_{PID,j}^T + \boldsymbol{b}_j\boldsymbol{k}_{PID,i}^T\right)\tilde{\boldsymbol{C}}\right]\boldsymbol{Q} & \boldsymbol{Q} \end{bmatrix} \succeq 0, \qquad (36)$$

For the present simulation results $G_L(z)$ and $f(v)$ where chosen as

$$G_L(z) = \frac{0.0187z^{-1} + 0.0175z^{-2}}{1 - 1.64z^{-1} + 0.6929z^{-2}} \qquad (45)$$

$$y(k) = f(v(k)) = \arctan(v(k)). \qquad (46)$$

The structure of Wiener systems enable a simple representation of nonlinear systems. The nonlinearity $f(v)$ has full impact on the output and stability analysis can become challenging, in particular when the nonlinearity has a saturation character like in the present example, Kozek and Jovanovic (2002).
In this example the input $u(k)$ is bounded to the interval $[-3,\ 3]$.
A local model network comprising six local models was generated by the algorithm presented in Hametner and Jakubek (2007), where the local models are constructed using an axis oblique decomposition of the partition space. Fig. 4 shows the identification data as well as a contour plot of the validity functions where its input vector is as follows:

$$\tilde{\boldsymbol{x}}(k) = [u(k-1)\ \hat{y}(k-1)]. \qquad (47)$$

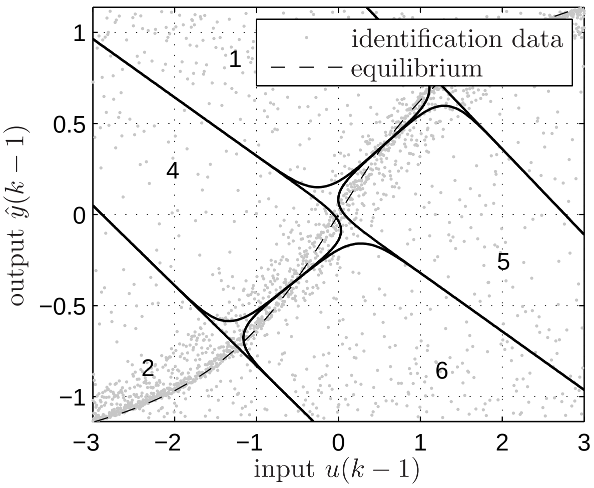

Fig. 4. Contour plot of the validity functions and identification data sequence

The partitioning strategy in Hametner and Jakubek (2007) uses statistical methods to avoid overfitting by local models generated with only few observations.
Fig. 5 illustrates the outputs of the the Wiener model $y(k)$ and the local model network $\hat{y}(k)$ for the same input sequence $u(k)$. Thus, the good approximation capability of the local model network is illustrated.

In order to demonstrate the effectiveness of the proposed method two different PID controllers are compared. The parameter sets of the two controllers are given in Table 1 and Table 2, respectively.

Theorem 7 fails to prove stability of the considered local model network in connection with controller A. In Fig. 6 the oscillatory behavior shows a poor controller

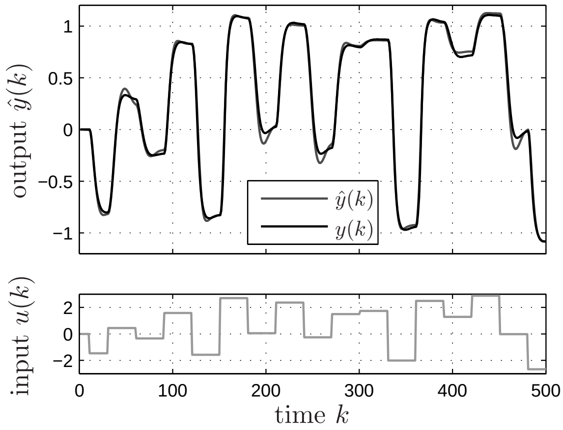

Fig. 5. Comparison of the open-loop behavior of the process and the local model network

Table 1. Parameters of controller A

Model #	K_p	T_N	T_V
1	9.481	79.817	1.039
2	5.778	4.043	0.649
3	5.519	4.029	0.649
4	2.878	7.418	0.722
5	2.934	7.603	0.726
6	9.240	79.762	0.986

Table 2. Parameters of controller B

Model #	K_p	T_N	T_V
1	0.664	4.538	1.908
2	0.073	1.638	18.588
3	0.092	2.046	14.834
4	0.830	4.973	1.947
5	0.821	4.883	1.947
6	0.637	4.393	2.011

performance and the strongly different damping indicates that a stability proof will be difficult. Nevertheless, the closed-loop with controller A may be stable although the stability proof fails because Lyapunov stability criteria are sufficient rather than necessary for stability. The closed-loop stability of the local model network controlled by controller B can be proven by Theorem 7. The controller performance in the time domain looks good as well, see Fig. 6. The closed loop performance also depends on the approximation capability of the local model network. This becomes visible when the local controller network is applied to the actual plant rather than to the local model network, see Fig. 7.

7. CONCLUSION AND OUTLOOK

In this article a method to investigate exponential stability of PID controlled local model networks was proposed. First, a suitable state-space model was introduced to offer the possibility to investigate closed-loop stability with a Lyapunov based approach. Second, a decay rate was intro-

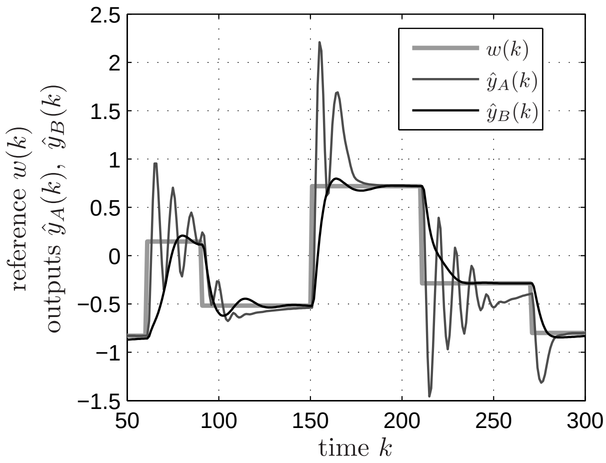

Fig. 6. Comparison of the closed loop performance of the local model network with two PID controllers
Controller A: no stability assertion possible
Controller B: exponentially stable, $\alpha = 0.98$

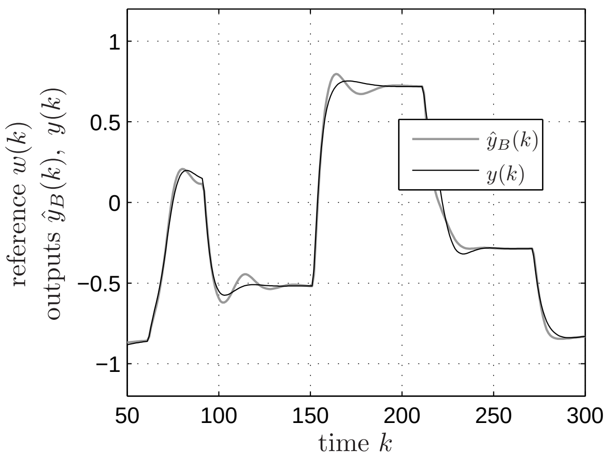

Fig. 7. Comparison of the local model network output $\hat{y}(k)$ and the output of the original process $y(k)$ controlled by the globally stable controller

duced and implemented into a common stability criterion. Further, the used stability criterion was adapted for the considered state-space model. A simulation example highlights the effectiveness of the proposed method.

The proposed stability Theorem 7 may be usable for controller design as well. For this task, the main issue is that state of the art LMI solver cannot simultaneously determine the Lyapunov function *and* the PID controller parameters (bilinear matrix inequalities). Thus, methods to solve BMI such as Genetic Algorithms (GA), Particle Swarm Optimization (PSO) and iterative LMI methods (iLMI) may be suitable for controller design.

The introduced state-space architecture offers a versatile closed-loop description of PID controlled local model networks. Thus, it is possible to relax the conservatism by applying a more advanced Lyapunov criterion (e.g. Fuzzy Lyapunov approach).

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