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Parametric Robustness

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Abstract: Gain and time-constant factors shift the inverse process relative to its feedback controller. A new robustness plot cross-graphs these shift factors that take the loop to the stability boundary as a function of frequency.

Keywords: Robustness plot; parameter shift factors; stability region; inverse process; deadtime.

1. INTRODUCTION

Process dynamics may be imperfectly known and may change over time. Production rate, feed composition, energy supply, wear, fouling, and the environment may cause the change. Feedback-loop robustness is an ability to maintain stability in the presence of process change or uncertainty. Robustness limits may be quantified as the smallest combination of probable parameter changes of specific process elements that can bring the nominally stable loop to its stability boundary. Often such a completely parameterized structural dynamic model of the process is either not available or difficult to analyze. Instead generic measures of unstructured uncertainty are commonly applied to nominally stable linearized input-output models of the process and controller.

Phase information in the frequency neighborhood of open-loop unity-gain crossings is critical for predicting feedback-loop stability. Effective delay contributes to phase but not amplitude at these frequencies. The effective delay time includes deadtime plus other high-frequency dynamics. Static nonlinearity also contributes to uncertainty. Zero-frequency gain, as distinct from amplitude in the critical frequency regions, is most easily measured or calculated but may have little impact on stability.

2. TRADITIONAL ROBUSTNESS

The Nyquist plot is a graph of the product of the open-loop process $G\{i\omega\}$ and feedback-controller $C\{i\omega\}$. Its polar amplitude-phase trajectory, as a function of frequency, demonstrates stability by passing to the right of the critical $(1, 180°)$ point, provided no unstable open-loop poles are cancelled by zeros. Traditional robustness measures (Åström & Hägglund, 2005) include gain and phase margins, changes in gain and phase that would bring the loop trajectory to the stability-limit point.

This multiplicative model emphasizes the controlled measurement's response y to process output noise n (multiplied by the sensitivity) or setpoint (and measurement noise) r (multiplied by the complimentary-sensitivity):

$$y = \frac{1}{1+GC}\, n + \frac{1}{1+(GC)^{-1}}\, r \tag{1}$$

Multiplicative compensation (for example stable-pole cancellation with a PID or model-feedback controller) may achieve fast setpoint tracking, provided the controller output does not limit, but may also provide poor load rejection.

Phase depends on frequency and time-constants. Small deadtime changes can cause large phase changes at high frequencies. This is particularly relevant for systems with multiple unity-gain crossings. Multiple crossings may occur if the process has a resonance, a recycle stream, or a lead (a zero), also if the controller has derivative action or has deadtime feedback (e.g. Smith Predictor, Internal Model, or Foxboro's PIDτ (Shinskey 1994, Hansen 2003) controllers).

At the stability limit:

$$1 + (G + \delta G)\, C = 0 \tag{2a}$$

$$\left(\frac{GC}{1+GC}\right)\left(\frac{\delta G}{G}\right) = -1. \tag{2b}$$

Maximizing magnitudes separately over frequency

$$\left|\frac{1}{1+(GC)^{-1}\{\omega\}}\right|_{max} \cdot \left|\frac{\delta G}{G}\{\omega\}\right|_{max} < 1.0 \tag{2c}$$

assures stability. The maximum complementary-sensitivity magnitude is at least 1. This unrestricted-frequency test fails to demonstrate closed-loop robustness whenever the process contains deadtime uncertainty, because the max absolute process-change ratio is at least 2. A specified max abs. process-change ratio less than 1 (restricted to all frequencies where absolute complementary-sensitivity may be 1 or more) leads to an allowable maximum absolute complementary-sensitivity (circle) for use as a Nyquist-plot robustness constraint in a controller-design optimization.

3. PARAMETRIC ROBUSTNESS

Parameterized robustness is based on the sum of the inverse-process $(G\{s\})^{-1}$ and controller $C\{s\}$, thus avoiding pole cancellation issues. This additive model emphasizes the controlled measurement's response y to an unmeasured-load disturbance v at the process input.

$$y = \frac{1}{G^{-1} + C}\ v \tag{3}$$

Additive compensation can achieve good load rejection and setpoint response as shown in Figure 4.

A gain factor k multiplies the process amplitude; a value greater than one shifts the inverse-process amplitude $|kG\{if\omega\}|^{-1}$ downward relative to the controller $|C\{i\omega\}|$ on a Bode plot. A time-constant factor f simultaneously multiplies all process time constants (lag, lead, integral, resonant period, and delay times). Significant time constants are often inversely proportional to production rate. A value of f greater than one shifts the inverse process amplitude and phase to the left, relative to the controller. The robustness plot uses log scales to graph k_{SB} vs. f_{SB}, the values of k and f that bring the nominally stable loop to the stability boundary, as a function of (radian) frequency ω. A trajectory loop further away from the nominal (1, 1) point indicates another root has reached the stability limit.

4. DEADTIME EXAMPLE

Impending instability may not be apparent with a nominal Nyquist trajectory as is demonstrated by an example. Figure 1 is a polar Nyquist plot for the unity-gain pure-delay process $G\{s\} = e^{-\tau s}$ and its "ideal" controller $C\{s\} = (1 - e^{-\tau s})^{-1}$ (Hansen 2000). This model-feedback controller is an extreme example of a PIDτ, a Smith Predictor, or an Internal Model controller. The polar Nyquist trajectory makes an infinite number of clockwise encirclements of the origin and none of the critical (1. 180°) point, keeping this critical point to its left as frequency increases. Thus the loop is stable. The gain margin is 2 (or 6 dB) and the phase margin is ± 60°. The complementary-sensitivity magnitude is 1. Even though the Nyquist plot does not penetrate a max complementary-sensitivity circle, the loop is not robust.

Lack of robustness is shown by shifting the process in the special characteristic equation expressed as:

$$C^{-1}\{i\omega\} + k_{SB} G\{if_{SB}\omega\} = 0 \tag{4}$$

The phase mismatch is $\theta_{SB} \equiv (f_{SB} - 1)\ \tau\omega$. Using algebra and trigonometric identities it can be shown that:

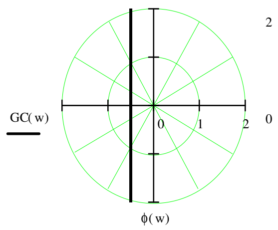
$$\theta_{SB} = \tan^{-1}\left(\frac{\sin \tau\omega}{1 - \cos \tau\omega}\right) = \tan^{-1}\left(\frac{1}{\tan \frac{\tau\omega}{2}}\right) \tag{5a}$$

$$k_{SB} = 2\ \cos \theta_{SB}, \tag{5b}$$

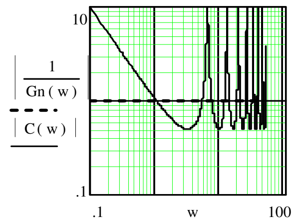
$$f_{SB} = 1 + \frac{\theta_{SB}}{\tau\omega}. \tag{5c}$$

When $\theta_{SB} = 0°$, $k_{SB} = 2$, the gain margin. When $k_{SB} = 1$, $\theta_{SB} = \pm 60°$, the phase margin. However, f_{SB} approaches 1 as frequency increases, indicating a diminishing stability region.

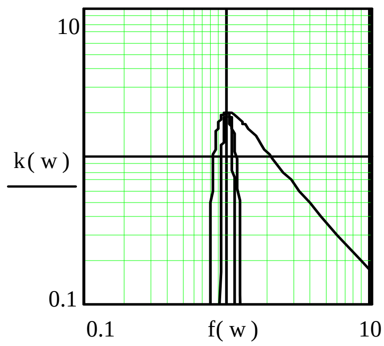
Figure 1c. is the robustness plot, k_{SB} vs. f_{SB}, using logarithmic scales. The robustness trajectory proceeds from the lower right as frequency increases, encircling counterclockwise the nominal (1, 1) point with ever narrowing loops.



(a) Polar Nyquist



(b) Bode Amplitude



(c) Robustness

Fig. 1. "Ideal" Control of Deadtime Process ($w = \tau\omega$)

The parametric change ratios are (a_0 is the inverse gain):

$$\frac{\delta a_0}{a_0} = \frac{1}{k} - 1.0, \tag{6a}$$

$$\frac{\delta\tau}{\tau} = f - 1.0. \tag{6b}$$

5. PID CONTROL

It is convenient to shift a PID (proportional P, Integral I, derivative D) controller:

$$C\{s\} = \frac{1}{P}\ \left(\frac{1}{I\ s} + 1 + Ds\right), \tag{7}$$

relative to the inverse process in order to locate the robustness boundary:

$$\frac{k_{SB}}{P}\left(-i\frac{f_{SB}}{I\bar{\omega}} + 1 + i\frac{D\bar{\omega}}{f_{SB}}\right) = -G^{-1}\{i\bar{\omega}\}, \tag{8}$$

where $\bar{\omega} = f_{SB}\omega$.

Equating reals and imaginaries to 0:

$$k_{SB} = -P\ Re\left\{G^{-1}\{i\bar{\omega}\}\right\}, \tag{9a}$$

$$\frac{f_{SB}}{I\bar{\omega}} - \frac{D\bar{\omega}}{f_{SB}} = -\frac{Im}{Re}\left\{G^{-1}\{i\bar{\omega}\}\right\}. \tag{9b}$$

Solving the quadratic:

$$f_{SB} = \bar{\omega}\ \left(\beta + \sqrt{\beta^2 + I\ D}\right), \tag{10}$$

where $\beta \equiv -\frac{I}{2} \frac{Im}{Re} \left\{ G^{-1} \left\{ i\bar{\omega} \right\} \right\}$.

Many industrial processes can be modeled effectively as a quadratic-delay:

$$G^{-1}\{s\} = \left(a_0 + a_1 s + a_2 s^2\right)\ e^{\tau s}. \tag{11}$$

The real and imaginary parts are:

$$Re\left\{G^{-1}\{i\bar{\omega}\}\right\} = \left(a_0 - a_2\bar{\omega}^2\right)\cos\tau\bar{\omega} - a_1\bar{\omega}\ \sin\tau\bar{\omega}, \tag{12a}$$

$$Im\left\{G^{-1}\{i\bar{\omega}\}\right\} = \left(a_0 - a_2\bar{\omega}^2\right)\sin\tau\bar{\omega} + a_1\bar{\omega}\ \cos\tau\bar{\omega}. \tag{12b}$$

Interacting loops and measured-loads may be decoupled using (adapted) additive or multiplicative dynamic feed-forward compensation of the interacting measurements (Hansen 2003). Useful for linearizing are nonlinearly compensated measurements and valves, and a fast cascaded secondary (flow or valve-position) loop.

The f and k factors are also used in an adaptive selftuner (Hansen 2003) to determine how much the process has shifted relative to the controller at a complex closed-loop root, identified from naturally-occurring-response features. The nominally well-tuned PID controller's I and D are then multiplied by f, P is multiplied by k. This nearly restores the original robustness as well as the desired load and setpoint step-response shapes.

6. LAG-DOMINANT PROCESS

A common process type is lag-dominant. Examples include liquid level, gas pressure, temperature, speed, and composition. A limiting example is an integral-delay process ($a_0 = a_2 = 0$). *Algebraic tuning* of a PID for this process gives: $P = 0.938\ \frac{\tau}{a_1}$, $I = 2.7\ \tau$, $D = 0.313\ \tau$.

Algebraic tuning (Hansen 2000) is a performance-based quantitative controller tuning method that approximately minimizes integrated-absolute-error (IAE) in response to a load step with a controlled-variable response shape similar to the Gaussian probability density.

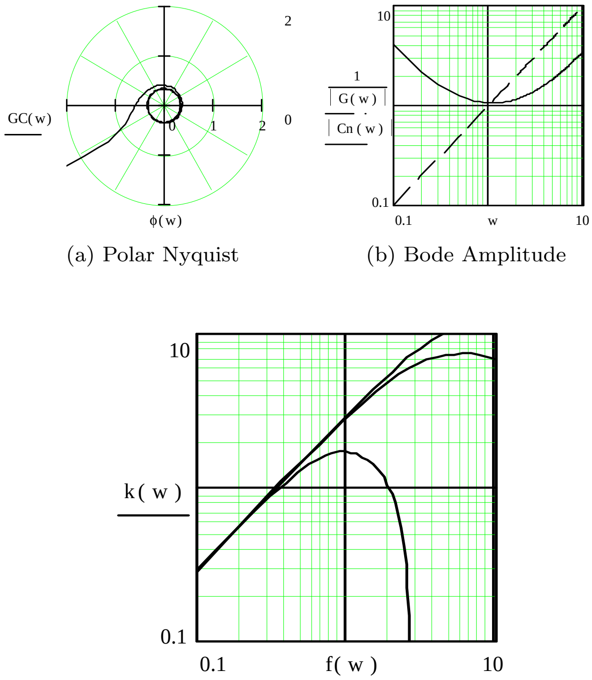
Solving for f_{SB} and k_{SB} closest to 1 at each frequency:

$$k_{SB} = 0.938\ \tau\bar{\omega}\ \ \sin\tau\bar{\omega}, \tag{13a}$$

$$\gamma \equiv \frac{\beta}{\tau} = \frac{0.5\ \cdot\ 2.7}{\tan\tau\bar{\omega}}, \tag{13b}$$

$$f_{SB} = \ \tau\bar{\omega}\ \left(\gamma + \sqrt{\gamma^2 + 2.7\cdot 0.313}\right). \tag{13c}$$

Figure 2c is the robustness plot for this loop. The trajectory proceeds from the lower right as frequency increases, encircling counter-clockwise the (1, 1) point. Another root reaches the stability limit on each trajectory loop. High-frequency loops are not closer to the (1, 1) point in this example. The gain margin for this loop is 1.70 (or 4.61 dB). The phase margin is 28.2°. The max abs. complementary-sensitivity is 2.1, allowing $\left|\frac{\delta G}{G}\right| = 0.48$. The Bode amplitude plot Figure 2b uses $a_1 = 1$ time unit. The nominal setpoint- and load-step responses are shown in Figure 4a.



(a) Polar Nyquist (b) Bode Amplitude

(c) Robustness

Fig. 2. PID Control of an Integral-Delay Process ($w = \tau\bar{\omega}$)

When $f = 1.0$, it appears that a process gain shift $k \ll 1.0$ can be tolerated. However, the Bode amplitude crossing would tend toward the integrating portion of the controller's amplitude trajectory where the net slope approaches -2, resulting in nearly 180° of open-loop phase lag and lightly-damped oscillatory closed-loop behavior.

Parametric change ratios are:

$$\frac{\delta a_1}{a_1} = \frac{f}{k} - 1.0, \tag{14a}$$

$$\frac{\delta\tau}{\tau} = f - 1.0 \tag{14b}$$

When both f and k are inversely proportional to production rate, a dominant-lag control loop, tuned at minimum production rate, will remain well damped, but not optimally tuned, as production rate increases.

7. QUADRATIC-DOMINANT PROCESS

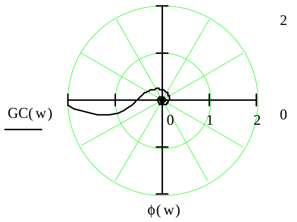
A less common process type is quadratic or resonance dominant. Examples include control of a mass's position or two lags with small effective delay. A limiting example is a two-integral-delay process ($a_0 = a_1 = 0$). *Algebraic tuning* of a PID yields: $P = 3.75\ \frac{\tau^2}{a_2}$, $I = 5.5\ \tau$, $D = 2.5\ \tau$.

Solving for f_{SB} and k_{SB}:

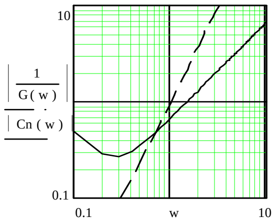
$$k_{SB} = 3.75\ (\tau\bar{\omega})^2 \cos\tau\bar{\omega}, \tag{15a}$$

$$\gamma \equiv \frac{\beta}{\tau} = -0.5\cdot 5.5\ \ \tan\tau\bar{\omega}, \tag{15b}$$

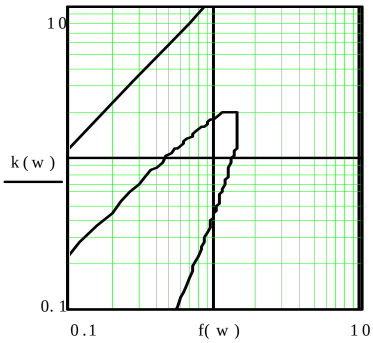
$$f_{SB} = \tau\bar{\omega}\ \left(\gamma + \sqrt{\gamma^2 + 5.5\cdot 2.5}\right). \tag{15c}$$



(a) Polar Nyquist

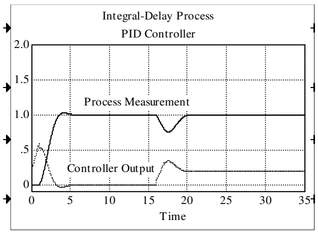


(b) Bode Amplitude

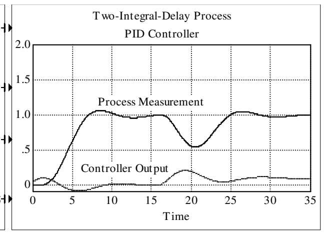


(c) Robustness

Fig. 3. PID Control of a Two-Integral-Delay ($w = \tau\bar{\omega}$)



(a)



(b)

Fig. 4. Setpoint and Load Step Responses

Figure 3c is the robustness plot for this loop. The trajectory proceeds from the lower right as frequency increases, encircling counter-clockwise the (1, 1) point. High-frequency trajectory loops are further away from the (1, 1) point. The smallest shift to the stability boundary is $f = 1.41$.

This conditionally stable loop has slightly oscillatory load and setpoint step-response shapes shown in Figure 4b. The gain margin is 1.85 (or 5.34 dB). However the phase margin is only 16.0°. The Bode amplitude plot Figure 3b uses $a_2 = 1$.

Parametric change ratios are:

$$\frac{\delta a_2}{a_2} = \frac{f^2}{k} - 1.0, \tag{16a}$$

$$\frac{\delta\tau}{\tau} = f - 1.0. \tag{16b}$$

If $f^2 \leq k \leq 1.0$ the loop would remain well damped as the process speeds up. The max absolute complementary-sensitivity is 3.6, yielding an allowable $\left|\frac{\delta G}{G}\right| = 0.28$ compared with $\frac{\delta\tau}{\tau} = 0.41$, $\frac{\delta a_2}{a_2} = 1.0$ allowed by the robustness plot.

8. CONCLUSIONS

Feedback control-loop robustness is an ability to maintain stability in the presence of process change or uncertainty. Unmodeled high-frequency dynamics may be represented as effective delay, contributing phase shift at the frequencies critical for stability. Static nonlinearity may be represented as gain uncertainty.

The Nyquist plot represents the nominal process and controller as a trajectory and the stability limit as a point. Linear closed-loop stability can be proven with a Nyquist plot of the open-loop gain – phase trajectory as a function of frequency provided unstable poles in the process G or controller C are not cancelled in the GC product. For the three examples, the max abs. complementary-sensitivity occurs at the unity-gain crossing. There, $\left|\frac{\delta G}{G}\right| = 2\ sin\frac{|\theta|}{2}$ where θ is the phase margin. One example shows that a stable Nyquist plot may not reveal poor robustness when there are multiple unity-gain crossings.

Gain and time-constants shifts of the inverse process relative to the controller are visualized on a Bode plot, when $\left|G^{-1}\right|$ and $|C|$ are plotted separately. At the intersection point the open-loop amplitude $|GC|$ is one. If the intersection point occurs in the well-tuned controller's integral region, the process is dominant effective delay, dominant lag for a proportional region crossing, and dominant quadratic for a derivative region crossing. A process parameter significantly affecting robustness is a_0 for a dominant effective-delay process, a_1 for a dominant lag, and a_2 for a dominant quadratic. As in the examples, the other significant parameter is the effective delay which combines phase contributions of higher frequency roots and deadtime. Derivative action is usually avoided for a dominant effective delay process to avoid multiple Bode crossings.

Shifts less than 1.0 that maintain the same relative slope at a single Bode crossing will have less phase shift contributed by effective delay and therefore provide well-damped but degraded unmeasured-load rejection.

The robustness plot represents the nominal process and controller as a point and the stability limit as a trajectory or boundary. Pole cancellation does not occur when the characteristic equation is expressed as $G^{-1} + C = 0$. Shifts to the stability boundary locate the stability region on the robustness plot even when there can be multiple Bode plot crossings. A stable region of the robustness plot is on the left side of all (counter-clockwise) trajectory loops as frequency ω increases. Non-significant process parameters do not affect the stability boundary near the nominal (1, 1) point.

The robustness trajectory is relatively easy to generate for a PID control loop by solving a quadratic at each frequency point. A more complicated controller may require iterative solutions or the shifting of a simple (two-parameter) inverse-process model relative to the controller. When using computer software, special care may be required to avoid trigonometric or polar angle jumps.

When the controller is designed for superior unmeasured-load rejection, robustness is compromised significantly by dominant process lags (or integrals) and by a controller using deadtime. To maintain both good load rejection and robustness over a range of production rates, it may be necessary to gain schedule or frequently (adaptively) retune the controller.

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