UTILITY FUNCTIONS FOR BAYESIAN DESIGN OF TESTS FOR FAULT DETECTION AND ISOLATION

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Abstract

Fault severity assessment is a challenging task especially in the presence of system uncertainty. Methods based on design of experiments can be leveraged for active Fault Detection and Isolation (FDI), but these methods find optimal FDI tests around a neighborhood of an anticipated set of values for system uncertainty and the fault(s) severity. To address this issue of locally optimal designs, Bayesian experimental design is applied for active FDI with robust test designs. In this work, we present the formulation and performance of Frequentist and Bayesian D- and D_s-optimal FDI designs in a 3-tank system.

Keywords

Bayesian Experimental Design, Optimal Experimental Design, Fault Detection and Isolation

Introduction

Fault Detection and Isolation (FDI) has been studied extensively due to the implications of uncertainty and system disturbance on the system state assessment and the corresponding impact on system maintenance cost. Active FDI approaches solve an optimal design problem that improves the information extracted from the FDI test. However, these approaches are often local as they design FDI tests for a predetermined set of fault scenarios at fixed uncertainty expectation. Palmer et al. (2018) calculated optimal FDI tests for assessing heat exchanger fouling severity using a function of the Fisher Information Matrix as the objective for FDI optimality. Their steady state FDI test designs were shown to be functions of heat exchanger fouling severity, which imposes a requirement for good anticipation of the fault severity or the execution of several parallel tests. Bayesian optimal experimental design (BOED) can overcome this issue by considering the entire range (distribution) of anticipated uncertainty and fault severity. BOED can, thus, calculate a unique FDI design for all instantiations of uncertainty and fault severity. Successful examples of BOED include that by Atkinson and Bogacka (1997) for D- and D_s-optimal designs to determine the order of a chemical reaction with uncertain reaction rate and the work by Huan and Marzouk (2013), who used the prior to posterior Kullback-Leibler divergence for parameter inference in a combustion kinetics problem. This work explores several of these criteria for BOED for FDI in the benchmark 3-tank system.

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Mathematical Formulation

Bayesian Experimental Design

The Bayesian approach to experimental design is based on Bayes' theorem where the posterior expectation of parameters $\boldsymbol{\theta}$ given a design $\boldsymbol{d}$ and observations $\boldsymbol{y}$, can be found from Eq. (1):

$$p(\boldsymbol{\theta}|\boldsymbol{y},\boldsymbol{d}) = \frac{p(\boldsymbol{y}|\boldsymbol{\theta},\boldsymbol{d})p(\boldsymbol{\theta}|\boldsymbol{d})}{p(\boldsymbol{y}|\boldsymbol{d})}, \tag{1}$$

where $p(\boldsymbol{\theta}|\boldsymbol{y},\boldsymbol{d})$ is the posterior density of the parameters $\boldsymbol{\theta}$, $p(\boldsymbol{\theta}|\boldsymbol{d})$ is the prior density of the parameters for the design $\boldsymbol{d}$, and $p(\boldsymbol{y}|\boldsymbol{\theta},\boldsymbol{d})$ is the likelihood of the observation $\boldsymbol{y}$. The denominator $p(\boldsymbol{y}|\boldsymbol{d})$ is the evidence for any value of $\boldsymbol{\theta}$, $p(\boldsymbol{y}|\boldsymbol{d}) = \int_{\Theta} p(\boldsymbol{y}|\boldsymbol{\theta},\boldsymbol{d})p(\boldsymbol{\theta}|\boldsymbol{d})\ d\boldsymbol{\theta}$. Lindley

(1972) suggested that the expected utility $U(\boldsymbol{d})$ of Eq. (2) can be used as the objective for experimental design:

$$U(\boldsymbol{d}^*) = \max_{\boldsymbol{d}\in\boldsymbol{D}} \int_{\boldsymbol{Y}} \int_{\boldsymbol{\Theta}} u(\boldsymbol{d},\boldsymbol{y},\boldsymbol{\theta}) p(\boldsymbol{\theta}|\boldsymbol{y},\boldsymbol{d}) p(\boldsymbol{y}|\boldsymbol{d}) \, d\boldsymbol{\theta} d\boldsymbol{y}, \quad (2)$$

where $u(\boldsymbol{d},\boldsymbol{y},\boldsymbol{\theta})$ is a utility function. Applying Bayes' theorem to Eq. (2) the expected utility becomes a function of the observations $\boldsymbol{y}$:

$$U(\boldsymbol{d}^*) = \max_{\boldsymbol{d}\in\boldsymbol{D}} \int_{\boldsymbol{Y}} \int_{\boldsymbol{\Theta}} u(\boldsymbol{d},\boldsymbol{y},\boldsymbol{\theta}) p(\boldsymbol{y}|\boldsymbol{\theta},\boldsymbol{d}) p(\boldsymbol{\theta}) \, d\boldsymbol{\theta} d\boldsymbol{y}. \quad (3)$$

The choice of utility function in Eq. (3) is decisive and should reflect the information gain from a test at the design $\boldsymbol{d}$. The Fisher Information of a system can be used as a utility function and a connection to alphabetical design criteria can be made.

Frequentist D- and D_s-optimal Designs

The Fisher Information Matrix (FIM) is a common information metric, calculated through sensitivity analysis of the observations w.r.t. system parameters:

$$\mathcal{I}(\boldsymbol{\theta},\boldsymbol{d}) = \sum_{r=1}^{N_y}\sum_{s=1}^{N_y} \mathrm{Q}_r^T \sigma_{rs} \mathrm{Q}_s, \quad (4)$$

with σ_{rs} the $[r,s]$ element of the inverse of the variance-covariance matrix, N_y the number of elements in the observation array $\boldsymbol{y}$ and Q_r, Q_s are the sensitivity matrices of the r, s responses w.r.t. parameters θ (Franceschini and Macchietto, 2008):

$$Q_r = \left[\frac{\partial y_r(\boldsymbol{\theta},\boldsymbol{d})}{\partial\theta_i \partial\theta_j}\right], \ with\ i,j = 1,2,...,k,$$

where k is the number of the parameters. The D-optimal experimental design in the Frequentist's context maximizes the logarithmic determinant of the FIM:

$$\phi_{F,DOpt}(\boldsymbol{d}^*) = \max_{\phi\in\Phi} \log\det\left[\mathcal{I}(\boldsymbol{\theta},\boldsymbol{d})\right].$$

If only a part of the parameter vector is of interest, we can split the parameter vector such as $\boldsymbol{\theta} = [\boldsymbol{\theta}_1\ \boldsymbol{\theta}_2]$, with $\boldsymbol{\theta}_1$ the parameters of interest and $\boldsymbol{\theta}_2$ the nuisance parameters. The Frequentist D_s-optimal experimental design can, then, be found from Eq. (5):

$$\phi_{F,DsOpt}(\boldsymbol{d}^*) = \max_{\phi\in\Phi} \log\left\{\frac{\det[\mathcal{I}(\boldsymbol{\theta},\boldsymbol{d})]}{\det[\mathcal{I}(\boldsymbol{\theta}_2,\boldsymbol{d})]}\right\}. \quad (5)$$

Bayesian D and D_s-optimal Designs

Bayesian Fisher Information is equal to the second derivative of the logarithmic likelihood of the observations $p(\boldsymbol{y}|\boldsymbol{\theta},\boldsymbol{d})$ w.r.t. the parameters $\boldsymbol{\theta}$ (Berger, 1985):

$$\mathcal{I}(\boldsymbol{\theta},\boldsymbol{d}) = -E_\theta\left[\frac{\partial^2 \log p(\boldsymbol{y}|\boldsymbol{\theta},\boldsymbol{d})}{\partial\boldsymbol{\theta}^2}\right]. \quad (6)$$

Assuming the observations $\boldsymbol{y}$ follow a Gaussian distribution model, Eq. (6) reduces to Eq. (4). Using the logarithmic determinant of FIM as the utility in Eq. (3), with Monte Carlo simulation for the estimation of the integral yields the Bayesian D-optimal design criterion:

$$\phi_{B,DOpt}(\boldsymbol{d}^*) = \max_{\boldsymbol{d}\in\boldsymbol{D}} \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} \log\det\left[\mathcal{I}(\boldsymbol{\theta}^{(i)},\boldsymbol{d})\right].$$

with $\theta^{(i)}$ the i^{th} draw from the prior $p(\boldsymbol{\theta})$ and n_{MC} the number of Monte Carlo samples. Similarly, the Bayesian D_s-optimal design criterion can be derived using as utility function the term in brackets in Eq. (5):

$$\phi_{B,DsOpt}(\boldsymbol{d}^*) = \max_{\boldsymbol{d}\in\boldsymbol{D}} \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} \log\left[\frac{\det[\mathcal{I}(\boldsymbol{\theta}^{(i)},\boldsymbol{d})]}{\det[\mathcal{I}(\boldsymbol{\theta}_2^{(i)},\boldsymbol{d})]}\right].$$

Comparison of Optimal Designs

The suitability of the different utility functions for BOED are compared here in terms of the Hellinger distance, $\mathcal{H}$. $\mathcal{H}$ is calculated with results from Monte Carlo simulations that produce the distributions of each observation for all fault scenarios, n_c. Then, the mean $\mathcal{H}$ values for all the fault scenarios including the no-fault case, from all observations are calculated as:

$$\mathcal{H} = \frac{1}{C_1} \sum_{i=1}^{(n_c-1)} \sum_{j=i+1}^{n_c} \sum_{l=1}^{n_y} H(P_i(y_l) \parallel P_j(y_l)),$$

$$C_1 = \frac{n_c!}{2!\cdot(n_c-2)!} + n_y.$$

Results of 3-Tank System Case Study

We apply BOED with different utility functions in a variation of the 3-tank system of Palmer and Bollas (2019). In the updated system shown in Figure (1), the FDI test design problem searches for the optimal opening of the valves, u_1 and u_2, for FDI of two tank holes, r_1 and r_2, and an actuator fault, a, in the 3-way valve. We assume that the uncertain parameters, flow coefficients c_{1-3}, follow normal distributions, while the fault parameters are uniform distributed from the no-fault value to a maximum value that cannot be exceeded.

Figure 2 presents the contour plots of the solution plane, $U(\boldsymbol{d^*})$, for the entire design space $\boldsymbol{D}$. BOED finds the optimal solution in the whole parameter space $\boldsymbol{\Theta}$ in contradiction to Frequentist approach, which finds different optimal designs for each set of different anticipated values of $\boldsymbol{\theta}$. The calculated values of $\mathcal{H}$ in-

Table 1. Frequentist D- and D_s-optimal designs and the Hellinger distance for different values of the parameter array, θ. The BOED used $\theta = [0.6, 0.002, 0.002, 1, 1, 0.8]$

θ	$D - Opt$		$D_s - Opt$		$Hellinger\ Distance$	
$[a, r_1, r_2, c_1, c_2, c_3]$	u_1	u_2	u_1	u_2	$D - Opt$	$D_s - Opt$
$[1, 0, 0, 1, 1, 0.8]$	0.2520	0.05	0.3622	0.05	0.84282	0.85793
$[0.8, 0, 0, 1, 1, 0.8]$	0.3071	0.05	0.4357	0.05	0.85167	0.86285
$[1, 0.002, 0, 1, 1, 0.8]$	0.1786	0.05	0.2520	0.05	0.82727	0.84282
$[1, 0, 0.002, 1, 1, 0.8]$	0.2888	0.05	0.3990	0.05	0.84891	0.86086
$[0.8, 0.002, 0, 1, 1, 0.8]$	0.2153	0.05	0.3071	0.05	0.83517	0.85167
Bayesian OED	0.2750	0.05	0.5000	0.05	0.84841	0.85917

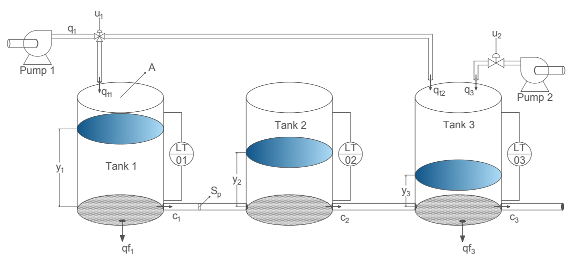
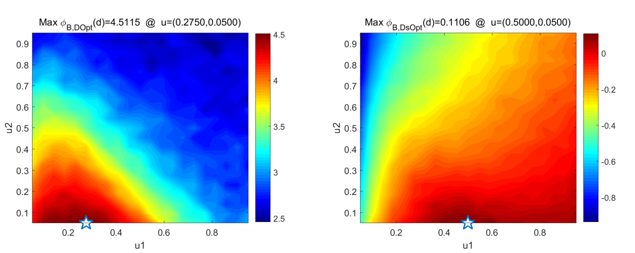


Figure 1. Modified 3-tank system for case study.

dicate that the D_s-optimal criterion results in a better design for FDI than D-optimal, which is in agreement with the literature. Moreover, BOED results in greater $\mathcal{H}$ distances, even compared to the average $\mathcal{H}$ distances of the Frequentist solution for D-optimality or D_s-optimality. The results of Table 1 support the hypothesis that BOED is a robust method for FDI, when prior information for the faults is uniform-distributed.



(a) Bayesian D-Optimal *(b) Bayesian D_s-Optimal*

Figure 2. D- and D_s- optimality criteria performance.

Conclusions

Frequentist and Bayesian D- and D_s- optimal design criteria were presented and applied for the purpose of active FDI in a variation of the benchmark 3-tank system. The FDI designs were compared in terms of Hellinger distance of the sensed outputs for each fault scenario. The Bayesian D_s-optimality criterion was shown to be superior in terms of detection and isolation of the faults using output measurements in which the evidence of system uncertainty is significant. Current work focuses on the use of Shannon Entropy as a less computationally intensive criterion for FDI test design.

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