Process control TKP4140 – Midterm Exam - Proposed Solution

October 2016

- Write your **student number** on **every** page.
- Write your answers in the designated spaces.
- Do **not** separate the sheets.
- If you need extra space, use the extra pages in the end.

Problem 1

1. (7 points) The transfer function $g(s)$ is a first order transfer function with time delay and the output is $y(s) = g(s)u(s)$. The step response $y(t)$ is plotted in Figure 1, and:

$$u(t) = \begin{cases} 0 & \text{for } t < 5 \\ 2 & \text{for } t \geq 5 \end{cases}$$

(a) Sketch $u(t)$ in Figure 1.

(b) Indicate the time delay (θ), time constant (τ), and steady state value (y_∞) in Figure 1.

(c) Write down θ, τ, k, and $g(s)$.

$\theta = 5\ s$ $\quad\quad$ $\tau = 15\ s$ $\quad\quad$ $k = \frac{\Delta y_\infty}{\Delta u} = \frac{3}{2} = 1.5$

$g(s) = \frac{1.5e^{-5s}}{15s+1}$

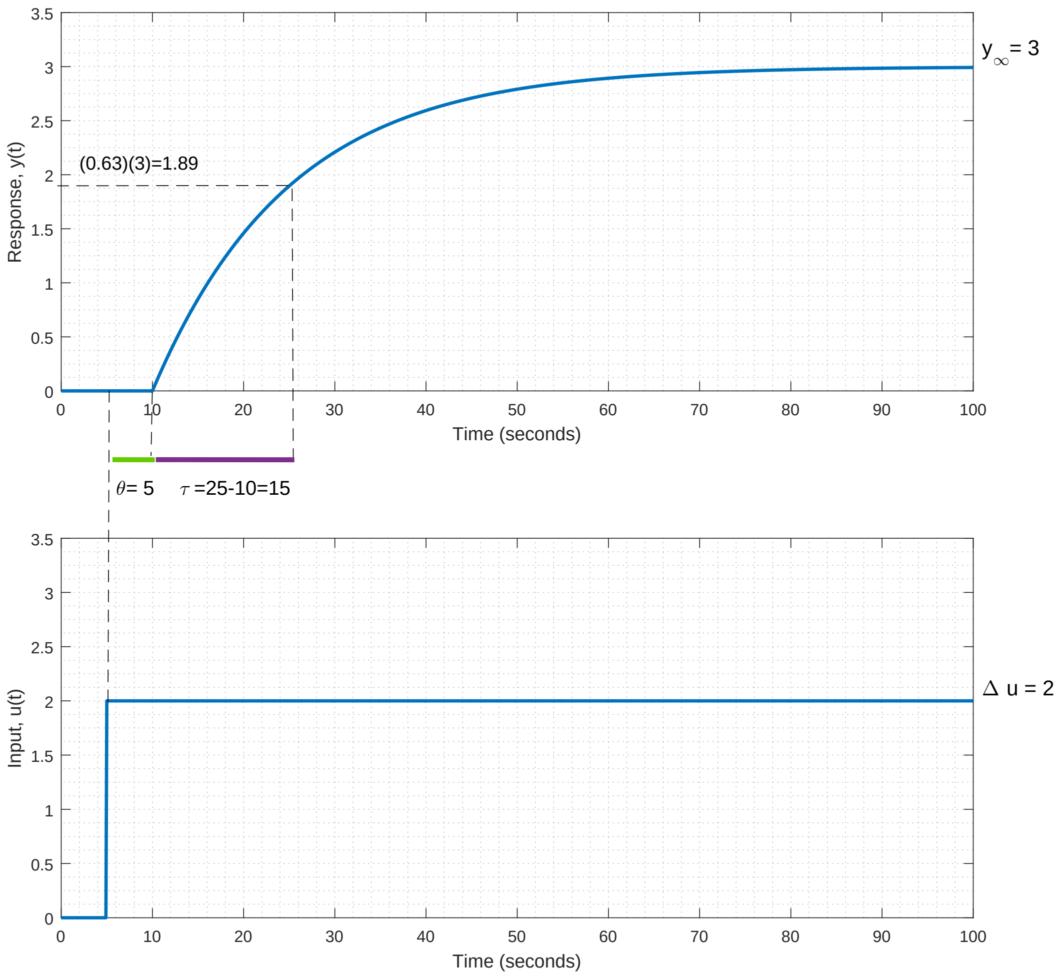

Figure 1: First order transfer function.

Problem 2

2. (18 points) Consider a heated tank with perfectly controlled level as the one in Figure 2. **Assume**: constant density (ρ), constant heat capacity (C_p), constant volume (V), perfect mixing.

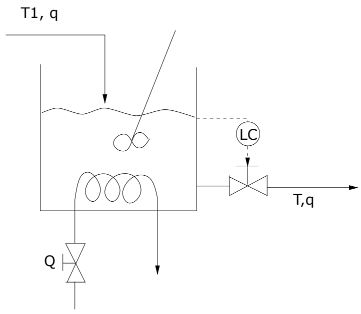

Figure 2: Heated tank

Table 1: Parameters for the heated tank.

V	$=$	10 000	ℓ	$= 10\ m^3$
q	$=$	1.5	m^3/s	
ρ	$=$	1000	kg/m^3	
$C_v \approx C_p$	$=$	4200	J/kgK	

(a) Formulate the dynamic energy balance of the system and write it in the form $\frac{dT}{dt} = \ldots$

$$\rho V C_p \frac{dT}{dt} = \rho q C_p (T_1 - T) + Q$$

$$\frac{dT}{dt} = \frac{q}{V}(T_1 - T) + \frac{1}{\rho V C_p} Q \qquad \rightarrow \text{The units are } [K/s]$$

$\rightarrow$ Note that $\frac{V}{q}$ is the residence time of the tank. If we write the residence time as τ the energy balance can be written as:

$$\frac{dT}{dt} = \frac{1}{\tau}(T_1 - T) + \frac{1}{\rho V C_p} Q$$

(b) Linearize the model and introduce deviation variables.
$f = \frac{dT}{dt}$

$$\frac{dT}{dt} \approx \frac{\partial f}{\partial T_1}(T_1 - T_1^*) + \frac{\partial f}{\partial T}(T - T^*) + \frac{\partial f}{\partial Q}(Q - Q^*) + f^*$$

$$\frac{dT}{dt} - f^* \approx \frac{q}{V}(T_1 - T_1^*) - \frac{q}{V}(T - T^*) + \frac{1}{\rho V C_p}(Q - Q^*)$$

If: $\Delta T = T - T^*$; $\quad \Delta T_1 = T_1 - T_1^*$; $\quad \Delta Q = Q - Q^*$

$$\frac{d\Delta T}{dt} = \frac{q}{V}(\Delta T_1 - \Delta T) + \frac{1}{\rho V C_p}\Delta Q$$

(c) Take the Laplace transform and derive the transfer functions $g_1(s)$ and $g_2(s)$:

$$T(s) = g_1 T_1(s) + g_2(s)Q(s)$$

$$g_1(s) = \frac{1}{\frac{V}{q}s+1} = \frac{1}{\tau_1 s+1} \qquad g_2(s) = \frac{\frac{1}{\rho C_p q}}{\frac{V}{q}s+1} = \frac{k_2}{\tau_1 s+1}$$

$$sT(s) = \frac{q}{V}(T_1(s) - T(s)) + \frac{1}{\rho V C_p}Q(s)$$

$$(s + \frac{q}{V})T(s) = \frac{q}{V}T_1(s) + \frac{1}{\rho V C_p}Q(s)$$

$$T(s) = \frac{\frac{q}{V}}{s+\frac{q}{V}}T_1(s) + \frac{1}{\rho V C_p(s+\frac{q}{V})}Q(s)$$

$$T(s) = \frac{1}{\frac{V}{q}s+1}T_1(s) + \frac{\frac{1}{\rho C_p q}}{\frac{\rho C_p V}{\rho C_p q}s+1}Q(s) = \frac{1}{\tau_1 s+1}T_1(s) + 1 + \frac{k_2}{\tau_1 s+1}Q(s)$$

(d) Considering the parameters in Table 1, what is the value of the steady state gain k_1 and the time constant τ_1 of $g_1(s)$? (write the units)

$$k_1 = 1 \;\; K/K \qquad \tau_1 = \frac{V}{q} = \frac{10m^3}{1.5m^3/s} = \frac{20}{3}\; s = 6.66\; s$$

(e) Considering the parameters in Table 1, what is the value of the steady-state gain k_2 of $g_2(s)$? (write the units)

$$k_2 = \frac{1}{\rho C_p q} = \frac{1}{(1000\frac{kg}{m^3})(4200\frac{J}{kgK})(1.5\frac{m^3}{s})} =$$

$$\frac{1}{6.3\times10^6}\frac{K\ s}{J} = 1.587\times10^{-7}\frac{K}{W} = 1.587\times10^{-4}\frac{K}{kW}$$

(f) Draw the block diagram for the (open loop) process.

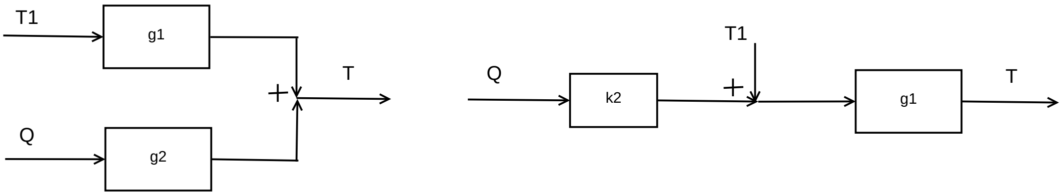

Figure 3: Block diagram for the process. Left: option 1; Right: option 2

Problem 3

3. (30 points) Consider the following block diagram:

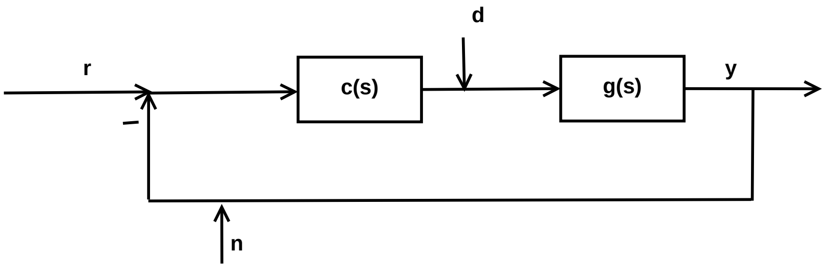

Figure 4: Closed loop block diagram

$$\mathbf{y} = \mathbf{T}(\mathbf{s})\ \mathbf{r} + \mathbf{M}(\mathbf{s})\ \mathbf{d} + \mathbf{N}(\mathbf{s})\ \mathbf{n}$$

(a) Write the transfer functions $T(s)$, $M(s)$ and $N(s)$. Use symbols only (c(s), g(s)).

$T(s) = \frac{c(s)g(s)}{1+c(s)g(s)}$

$M(s) = \frac{g(s)}{1+c(s)g(s)}$

$N(s) = \frac{-c(s)g(s)}{1+c(s)g(s)}$

(b1) Find $T(s)$ and $M(s)$ when: $g(s) = \frac{3}{4s+1}$ and $c(s) = 2$
Write the denominator in the form: $\tau s + 1$

$T(s) = \frac{2(\frac{3}{4s+1})}{1+2(\frac{3}{4s+1})} = \frac{6}{4s+7} = \frac{6/7}{\frac{4}{7}s+1} = \frac{0.857}{0.57\ s+1}$

$M(s) = \frac{\frac{3}{4s+1}}{1+2(\frac{3}{4s+1})} = \frac{3}{4s+7} = \frac{3/7}{\frac{4}{7}s+1} = \frac{0.428}{0.57\ s+1}$

(b2) Find the analytical expression for $y(t)$ when $r(t)$ is a unit step, $d(t) = 0$, and $n(t) = 0$

$y(t) = \frac{6}{7}\left(1 - e^{\frac{-t}{4/7}}\right) = 0.8571\left(1 - e^{\frac{-t}{0.57}}\right)$

The response $y(s)$ is obtained by multiplying $T(s)$ by the unit step $u(s) = 1/s$. Then, $y(s)$ can be rearranged using partial fraction decomposition:

$y(s) = \frac{1}{s}\left(\frac{6/7}{4/7\ s+1}\right) = \frac{A}{s} + \frac{B}{4/7\ s+1} = \frac{6/7}{s} - \frac{24/49}{4/7\ s+1}$

Then, we have to take the inverse Laplace of $y(s)$ to obtain $y(t)$:

$y(t) = \mathcal{L}^{-1}\left\{y(s)\right\} = \frac{6}{7} - \frac{24/49}{4/7}e^{\frac{-t}{4/7}} = \frac{6}{7}\left(1 - e^{\frac{-t}{4/7}}\right)$

A simpler way is to take 6/7 as constant and then use $\mathcal{L}^{-1}\left\{\frac{1}{s(\tau s+1)}\right\} = 1 - e^{-t/\tau}$

(c1) Find $T(s)$ when: $g(s) = \frac{3}{4s+1}$ and $c(s) = 0.5(1 + \frac{1}{s})$
→ Please note that this is not a well-tuned controller.

$T(s) = \frac{s+1}{\frac{8}{3}s^2+\frac{5}{3}s+1} = \frac{s+1}{2.66\ s^2+1.66\ s+1}$

This is obtained by substituting $g(s)$ and $c(s)$ in the answer of question (a):

$T(s) = \frac{c(s)g(s)}{1+c(s)g(s)} = \frac{\left(\frac{3}{4s+1}\right)\ 0.5\left(1+\frac{1}{s}\right)}{1+\left(\frac{3}{4s+1}\right)\ 0.5\left(1+\frac{1}{s}\right)} = \frac{1.5(s+1)}{4s^2+2.5\ s+1.5} = \frac{s+1}{2.66\ s^2+1.66\ s+1}$

(c2) Calculate the damping factor and the time constant of T(s).
Hint: the denominator in the second order transfer function is $\tau^2 s^2 + 2\tau\zeta s + 1$

$\zeta = 0.51$ $\tau = \sqrt{8/3} = 1.633$

From the previous answer we know that $\tau^2 = 8/3$, so we can calculate τ from there.

To calculate ζ, we know that:
$2\tau\zeta = 5/3 \approx 1.66$. Then:

$\zeta = \frac{1.66}{(2)(1.633)} = 0.51$

(c3) Compute $y(0)$, $y'(0)$, and $y(\infty)$ for T(s) when a unit step $r(s) = 1/s$ is applied.

$y(0) = 0$ $y'(0) = 0.3759$ $y(\infty) = 1$

The response $y(s)$ is obtained by multiplying $T(s)$ by $r(s)$. In this case, the initial gain, initial slope and steady state gains are calculated as follows:

Initial gain:
$\lim_{t\to 0} y(t) = \lim_{s\to\infty} T(s) = \lim_{s\to\infty} \frac{(s+1)/s^2}{(2.66s^2+1.66s+1)/s^2} = 0$

Initial slope:
$\lim_{t\to 0} y'(t) = \lim_{s\to\infty} sT(s) = \lim_{s\to\infty} \frac{(s^2+s)/s^2}{(2.66s^2+1.66s+1)/s^2} \approx \frac{1}{2.66} = 0.3759$

Steady state gain:
$\lim_{t\to\infty} y(t) = \lim_{s\to 0} T(s) = \lim_{s\to 0} \frac{s+1}{2.66\ s^2+1.66\ s+1} = 1$

(c4) Sketch the response $y(t)$, when $r(t)$ is a unit step, $d = 0$ and $n = 0$.
Hints:

- Consider the answers you gave in (c2) and (c3).
- Note that the period of oscillations is approximately $2\pi\tau$
- The first peak is at $t \approx 4.7s$

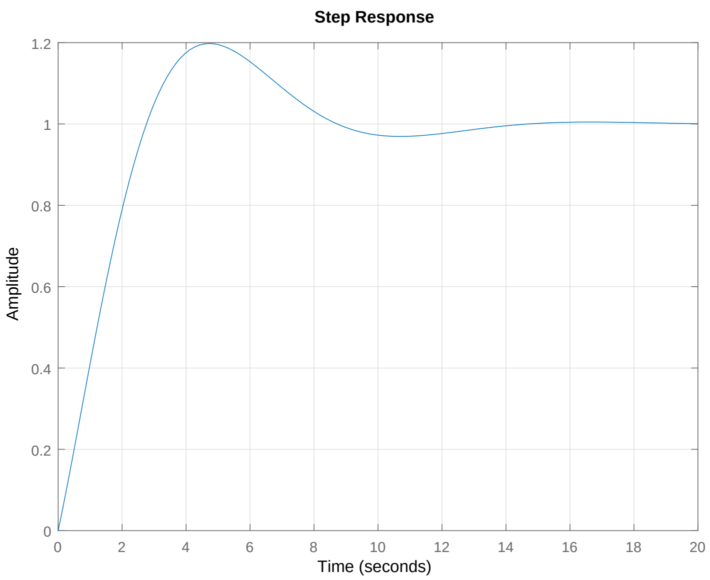

Figure 5: Step response

Problem 4

4. (15 points) Given the transfer function:

$$g(s) = \frac{(-5s+1)e^{-3s}}{(2s+1)^3(7s+1)} \tag{1}$$

(a) Write the first order plus time delay approximation $g_1(s)$ using the half-rule.

$g_1(s) = \frac{e^{-13\ s}}{8\ s+1}$

$g_1(s)$ can be written as: $g_1(s) = \frac{(-5s+1)e^{-3s}}{(7s+1)(2s+1)(2s+1)(2s+1)}$ Then:

$\theta \approx 3+5+2+2+2/2 = 13$
$\tau_1 = 7+2/2 = 8$

(b) Give the SIMC PI settings, using $\tau_c = \theta$, where θ is the effective delay.

$K_c = 0.3077$ $\tau_I = 8$

$K_c = \frac{1}{k}\left(\frac{\tau_1}{\tau_c+\theta}\right) = \frac{1}{1}\left(\frac{8}{13+13}\right) = 0.3077$

$\tau_I = min\{\tau_1,\ 4(\tau_c+\theta)\} = min\{8,\ 4(13+13)\} = 8$

(c) Write the second order plus time delay approximation $g_2(s)$ using the half-rule.

$g_2(s) = \frac{e^{-11\ s}}{(7\ s+1)(3\ s+1)}$

$\theta = 3+5+2+2/2 = 11$

$\tau_2 = 2+2/2 = 3$

(d) Give the SIMC PID settings (cascade PID), using $\tau_c = \theta$, where θ is the effective delay.

$K_c = 0.318$ $\tau_I = 7$ $\tau_D = 3$

$K_c = \frac{1}{k}\left(\frac{\tau_1}{\tau_c+\theta}\right) = \frac{1}{1}\left(\frac{7}{11+11}\right) = 0.318$
$\tau_I = min\{\tau_1,\ 4(\tau_c+\theta)\} = min\{7,\ 4(11+11)\} = 7$
$\tau_D = \tau_2$

(e) Would you recommend a PI or a PID controller? Explain briefly.
The second order model (and derivative action) should be used when $\tau_2 > \theta$, meaning that it is a dominant second-order process. In this case, $\tau_2 = 3$ and $\theta = 11$, so, it is not recommended to use a PID. → **Use PI controller**.

Problem 5

5. (5 points) Given the transfer function:

$$g(s) = \frac{20e^{-3s}}{s(2s+1)(s+1)} \quad (2)$$

(a) Write the first order plus time delay approximation $g_1(s)$ using the half-rule.

$g_1(s) = \frac{20}{s}e^{-5\ s}$

$\frac{1}{s} \approx \frac{1}{Ts+1}$ with $T \to \infty$

So, for the time constant: $\infty + 2/2 = \infty$

For the time delay: $\theta = 3 + 1 + 2/2 = 5$

(b) Give the SIMC PI settings. Use $\tau_c = \theta$, where θ is the effective delay.

$Kc = 1/200$ $\tau_I = 40$

We have an integrating process with $k' = 20$ and $\theta = 5$

$K_c = \frac{1}{k'}\left(\frac{1}{\tau_c+\theta}\right) = \frac{1}{20}\left(\frac{1}{5+5}\right) = \frac{1}{200}$

$\tau_I = 4(\tau_c + \theta) = 4(5 + 5) = 40$

Problem 6

6. (5 points) Indicate whether the following statements are true of false.

	True	False
The PID SIMC rule gives tunings for PID in ideal form.		X
Increasing the integral term τ_I in a PID controller increases the effect of integral action.		X
After tuning a PI controller using the SIMC rules with $\tau_c = \theta$, you realized that the closed loop response is faster than what you would like it to be. In order to slow down the response, you should decrease τ_c.		X
A closed loop with P-control always has a steady-state offset of $\frac{1}{1+K_cK}$, where K_c is the controller gain and K is the steady state process gain.	X	
Windup is caused by the integral part of the PI controller.	X	

Problem 7

7. (20 points) Figure 6 depicts the responses of the following transfer functions to a step input:

$$g_1 = \frac{1}{7s+1} - \frac{1.1}{3s+1} \tag{3}$$

$$g_2 = \frac{1}{3s+1} - \frac{1.1}{7s+1} \tag{4}$$

$$g_3 = \frac{(10s+1)}{(7s+1)(1.5s+1)^2} \tag{5}$$

$$g_4 = \frac{1}{s^2+0.4s+1} \tag{6}$$

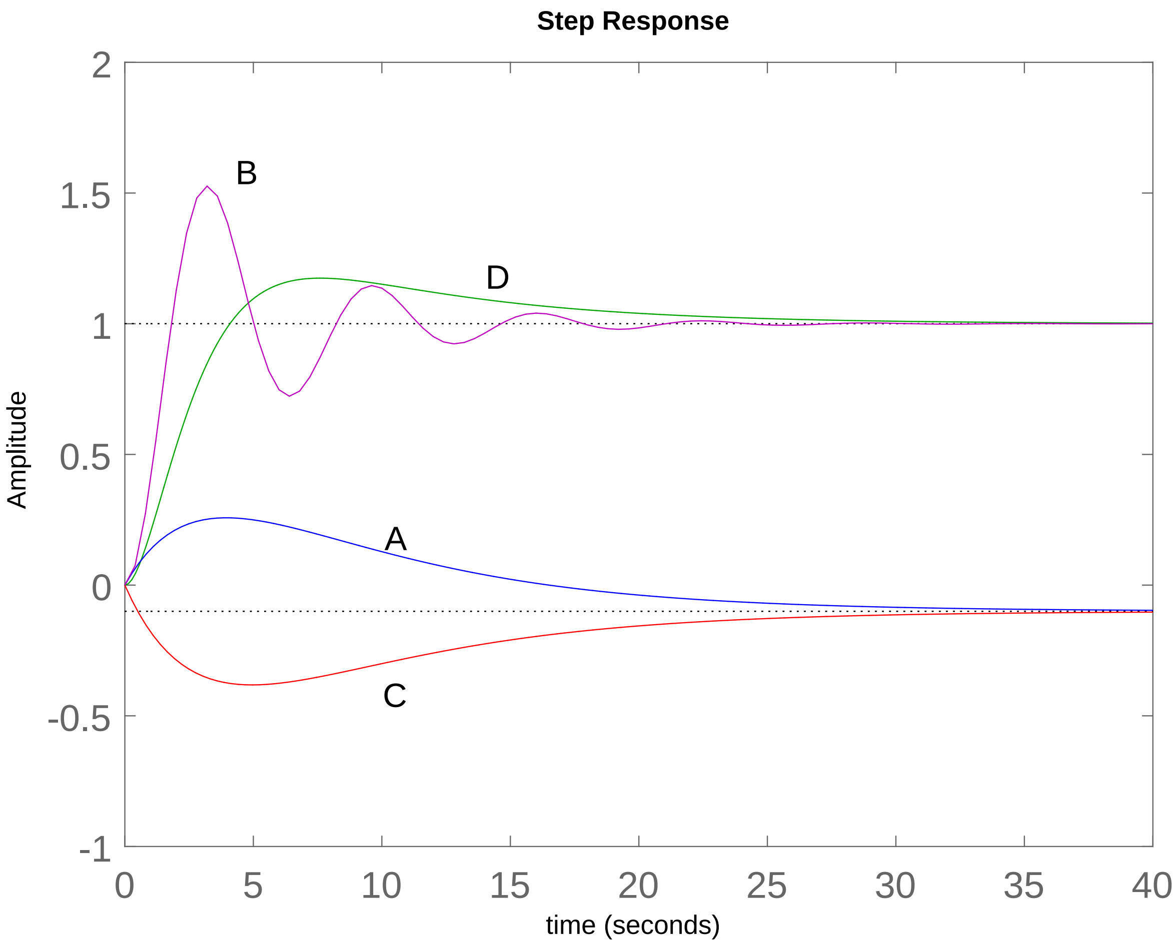

Figure 6: Step response of $g_i(s)$

Fill in the missing values in Table 2:

- Poles and zeros of $g_i(s)$
- Steady state gain (SS gain), initial gain and initial slope **when a unit step input** $u(s) = 1/s$ **is applied at** $t = 0$**.**
- As conclusion, identify the step responses in Figure 6 (A,B,C, or D).

Hints:
Initial slope of response to unit step input: $\lim_{t\to 0} y'(t) = \lim_{s\to\infty} sg(s)$
$j = \sqrt{-1}$

Table 2: Problem 7; SS: steady state; TF: transfer function

TF	Poles	Zeros	SS gain	Initial gain	Initial slope	Conclusion
g_1	$p_1 = -1/3$ $p_2 = -1/7$	$z_1 = -1/47$	-0.1	0	-0.2238	C
g_2	$p_1 = -1/3$ $p_2 = -1/7$	$z_1 = 1/37$	-0.1	0	0.1762	A
g_3	$p_1 = -1/7$ $p_2 = p_3 = -2/3$	$z_1 = -1/10$	1	0	0	D
g_4	$p_{1,2} = -0.2 \pm 0.9798j$	-	1	0	0	B

Calculation of poles and zeros:

$\mathbf{g_1(s)} = \frac{3s+1-7.7s-1.1}{(7s+1)(3s+1)} = \frac{-0.1(47s+1)}{(7s+1)(3s+1)}$ Then, the poles and zeros can be easily calculated.

$\mathbf{g_2(s)} = \frac{7s+1-3.3s-1.1}{(7s+1)(3s+1)} = \frac{-0.1(-37s+1)}{(7s+1)(3s+1)}$ Then, the poles and zeros can be easily calculated.

For $\mathbf{g_3(s)}$ poles and zeros can be easily calculated.

For $\mathbf{g_4(s)}$ there are no zeros and the poles can be calculated by solving the quadratic equation:
$p_{1,2} = -0.4 \pm \frac{\sqrt{0.4^2-4(1)(1)}}{2} = -0.2 \pm \frac{\sqrt{-3.84}}{2} = -0.2 \pm 0.9798j$

For the steady state gain, initial gain, and initial slope, we have to consider that the response to a unit step is $y(s) = g(s)u(s)$ where $u(s) = 1/s$

Calculation of the steady state gain:

Steady state gain of the response when a unit step is applied:

$\lim_{t\to\infty} y(t) = \lim_{s\to 0} g(s)$

All the limits can be taken directly from the transfer functions as given.

Calculation of the initial gain:

Initial gain of the response when a unit step is applied:

$\lim_{t\to 0} y(t) = \lim_{s\to\infty} g(s)$

For $g_1(s)$: $\lim_{t\to 0} y_1(t) = \lim_{s\to\infty} g_1(s) = \lim_{s\to\infty} \frac{(-4.7s-0.1)/s^2}{(21s^2+10s+1)/s^2} = 0$

The numerator becomes zero: $\lim_{s\to\infty}(-4.7/s - 0.1/s^2) = 0$

Similarly for $g_2(s)$ and $g_4(s)$, you divide by s^2 so that the limit is defined when the limit is taken.

For $g_3(s)$: $\lim_{t\to 0} y_3(t) = \lim_{s\to\infty} g_3(s) = \lim_{s\to\infty} \frac{(10s+1)/s^3}{((7s+1)(1.5s+1)^2)/s^3} = 0$

Calculation of the initial slope:

Initial slope of the response when a unit step is applied:

$\lim_{t\to 0} y'(t) = \lim_{s\to\infty} sg(s)$

For $g_1(s)$: $\lim_{t\to 0} y_1'(t) = \lim_{s\to\infty} sg_1(s) = \lim_{s\to\infty} \frac{(-4.7s^2-0.1s)/s^2}{(21s^2+10s+1)/s^2} \approx \frac{-4.7}{21} = -0.2238$

For $g_2(s)$: $\lim_{t\to 0} y_2'(t) = \lim_{s\to\infty} sg_2(s) = \lim_{s\to\infty} \frac{(3.7s^2-0.1s)/s^2}{(21s^2+10s+1)/s^2} \approx \frac{3.7}{21} = 0.1762$

For $g_3(s)$: $\lim_{t\to 0} y_3'(t) = \lim_{s\to\infty} sg_3(s) = \lim_{s\to\infty} \frac{(10s^2+s)/s^3}{((7s+1)(1.5s+1)^2)/s^3} = 0$

For $g_4(s)$: $\lim_{t\to 0} y_4'(t) = \lim_{s\to\infty} sg_4(s) = \lim_{s\to\infty} \frac{1/s^2}{(s^2+0.4s+1)/s^2} = 0$