Ch. 13
Frequency analysis

Force linear system with input x(t) = A sin ωt .
Here is the output y(t):

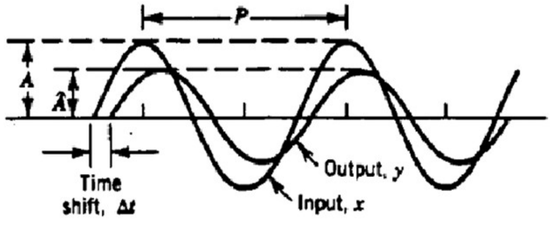
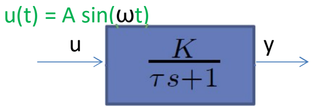


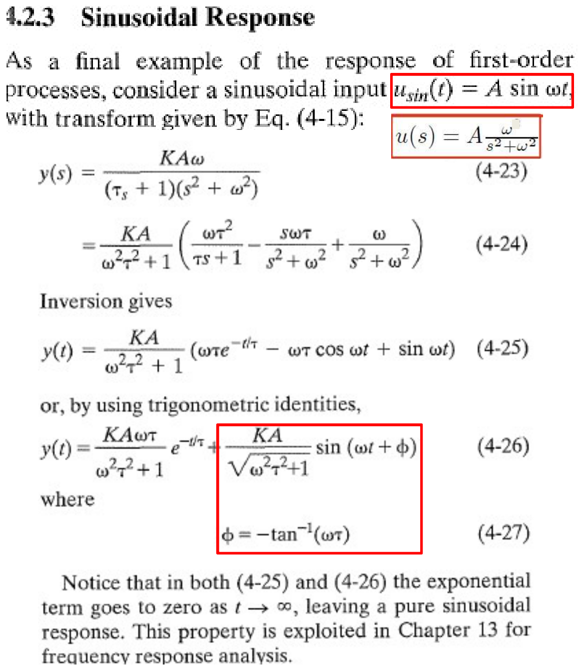
Figure 13.1.

Attenuation and time shift between input and output sine waves ($K = 1$). The phase angle ϕ of the output signal is given by $\phi = -\Delta t/P \times 360$, where Δt is the time (period) shift and P is the period of oscillation.



As t →∞:
y(t) = AR *A*sin(ωt+φ)

General (VERY SIMPLE). Set s=jω in G(s). Then
AR = |G(jω)|
φ = ∠ G(jω)



4.2.3 Sinusoidal Response

As a final example of the response of first-order processes, consider a sinusoidal input $\boxed{u_{sin}(t) = A \sin \omega t}$ with transform given by Eq. (4-15): $\boxed{u(s) = A\frac{\omega}{s^2+\omega^2}}$

$$y(s) = \frac{KA\omega}{(\tau_s + 1)(s^2 + \omega^2)} \tag{4-23}$$

$$= \frac{KA}{\omega^2\tau^2+1}\left(\frac{\omega\tau^2}{\tau s+1} - \frac{s\omega\tau}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2}\right) \tag{4-24}$$

Inversion gives

$$y(t) = \frac{KA}{\omega^2\tau^2 + 1}(\omega\tau e^{-t/\tau} - \omega\tau \cos \omega t + \sin \omega t) \tag{4-25}$$

or, by using trigonometric identities,

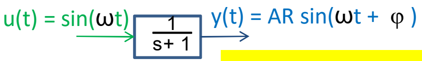
$$y(t) = \frac{KA\omega\tau}{\omega^2\tau^2+1}e^{-t/\tau} + \boxed{\frac{KA}{\sqrt{\omega^2\tau^2+1}}\sin(\omega t + \phi)} \tag{4-26}$$

where

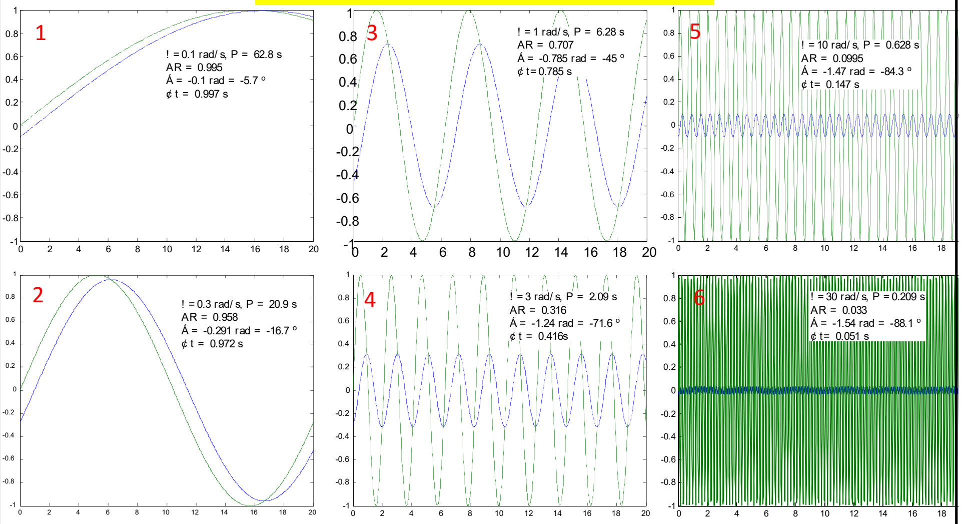
$$\boxed{\phi = -\tan^{-1}(\omega\tau)} \tag{4-27}$$

Notice that in both (4-25) and (4-26) the exponential term goes to zero as $t \to \infty$, leaving a pure sinusoidal response. This property is exploited in Chapter 13 for frequency response analysis.

SINUSOIDAL RESPONSE OF FIRST-ORDER SYSTEM
k = 1, τ = 1 [s]



6 Plots: Increase ω from 0.1 to 30 rad/s



```
w=0.3; tau=1; t = linspace(0,20,1000);
u = sin(w*t);
AR = 1/sqrt((w*tau)^2+1)
phi = - atan(w*tau), phig=phi*180/pi, dt=-phi/w
y = AR*sin(w*t+phi);
plot(t,y,t,u)
```

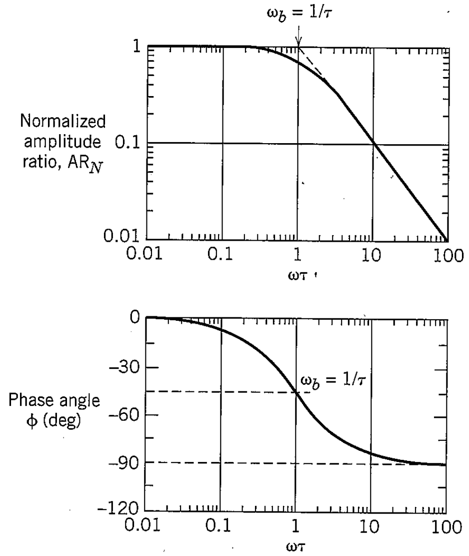
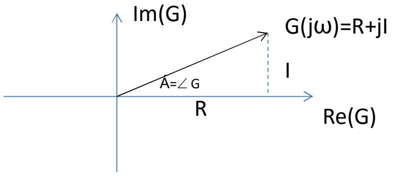


Figure 14.2 Bode diagram for a first-order process

Mathematics. Complex numbers, $j^2=-1$



$$s = j\omega \qquad G(j\omega) = R + Ij$$

$$\left.\begin{array}{l} |G| = AR = \sqrt{R^2 + I^2} \\ \phi = \angle G = \arctan \dfrac{I}{R} \end{array}\right\} \text{Polar form}$$

Polar form:

$G = R + jI = |G|(cos\angle G + jsin\angle G) = |G|e^{j\angle G}$

Note: $e^{j\pi} = -1$

Polar form
Multiply complex numbers:
Multiply magnitudes and add phases

$$G = G_1 \cdot G_2 \cdot G_3$$
$$|G| = |G_1| \cdot |G_2| \cdot |G_3|$$
$$\angle G = \angle G_1 + \angle G_2 + \angle G_3$$

$$Similar - for - ratio:$$
$$G = \frac{G_1}{G_2}$$
$$|G| = |G_1| / |G_2|$$
$$\angle G = \angle G_1 - \angle G_2$$

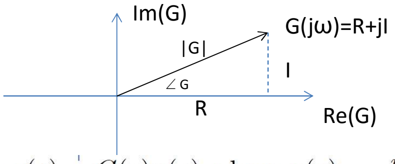
Simple method to find sinusoidal response of system G(s)

1. Input signal to linear system: $u = u_0 \sin(\omega t)$
2. Steady-state ("persistent", t→∞) output signal: $y = y_0 \sin(\omega t + \phi)$
3. What is $AR = y_0/u_0$ and ϕ?

Solution (extremely simple!)
1. Find system transfer function, G(s)
2. Let $s=j\omega$ (imaginary number, $j^2=-1$) and evaluate $G(j\omega) = R + j\,I$ (complex number)
3. Then ("believe it or not!")

$$\boxed{\begin{array}{l} AR = |G(j\omega)| \\ \varphi = \angle G(j\omega) \end{array}}$$

(magnitude of the complex number)
(phase of the complex number)



Proof: $y(s) = G(s)u(s)$ where $u(s) = \frac{u_0\omega}{s^2+\omega^2} = \frac{u_0\omega}{(s-j\omega)(s+j\omega)}$, etc...
(poles of $G(s)$ "die out" as $t \to \infty$)

Example 13.1:

1. $G(s)=\frac{1}{\tau s+1}$

2. $G(j\omega)=\frac{1}{1+\tau j\omega}\cdot\frac{1-\tau j\omega}{1-\tau j\omega}\qquad (j^2=-1)$

$$G(j\omega)=\underbrace{\frac{1}{1+\omega^2\tau^2}}_{\mathbf{R}}-\underbrace{\frac{\omega\tau}{1+\omega^2\tau^2}}_{\mathbf{I}}j$$

3. $|G|=AR=\sqrt{R^2+I^2}=\frac{1}{\sqrt{1+\omega^2\tau^2}}$

$$\phi=\angle G=\arctan\frac{I}{R}=-\arctan(\omega\tau)$$

Gain and phase shift
of sinusoidal response!

SIMPLER: Use G=G1/G2, where G1=1, G2=tau*s+1.
set s=jw. Get |G|=1/|G2|=1/sqrt((w*tau)^2+1), angle(G)=0-angle(G2) = –arctg(w*tau)

$$\boxed{\mathrm{AR}=|G(j\omega)|=\frac{1}{\sqrt{(\omega\tau)^2+1}}}$$

$$\boxed{\phi=\angle G(j\omega)=-\arctan(\omega\tau)}$$

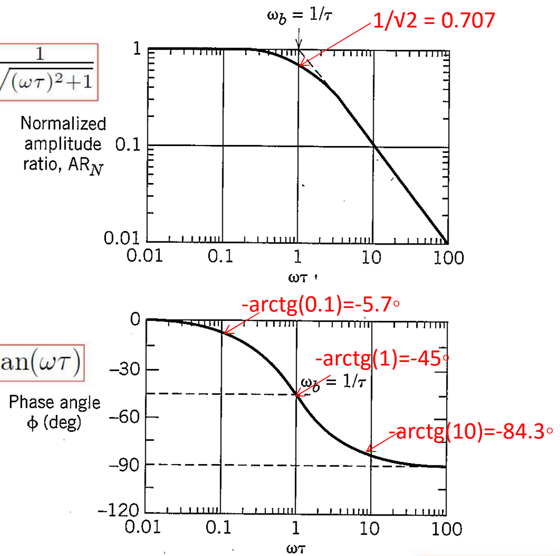


Figure 14.2 Bode diagram for a first-order process $\boxed{G(s)=\frac{1}{\tau s+1}}$

Example 2

$$g(s) = \frac{k(Ts+1)}{(\tau_1 s+1)(\tau_2 s+1)} = \frac{g_1\ g_2}{g_3\ g_4}$$

$$g_1 = k$$
$$g_2 = Ts + 1$$
$$g_3 = \tau_1 s + 1$$
$$g_4 = \tau_2 s + 1$$

Frequency response of time delay

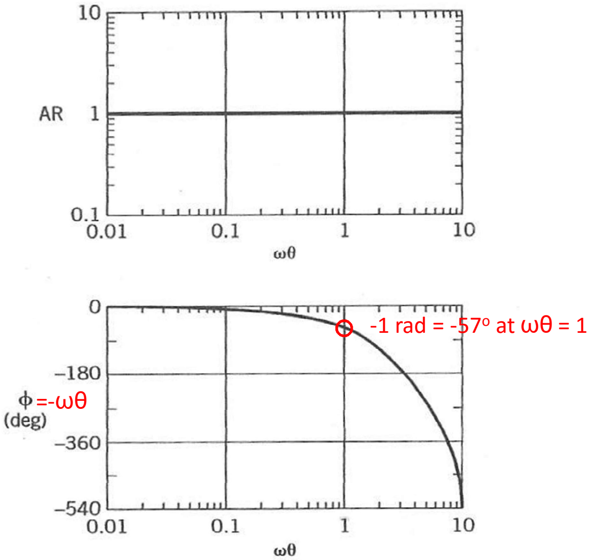


Figure 14.4 Bode diagram for a time delay, $e^{-\theta s}$.

1. DERIVATIVE

$$g_1(s) = s$$

Frequency response: $g(j\omega) = j\omega = 0 + j\omega$

$$|g_1(j\omega)| = \omega$$
$$\angle g_1(j\omega) = 90^o = \pi/2 \text{ rad (purely complex at all } \omega)$$

Check:

$$u(t) = u_0 sin(\omega t)$$
$$y(t) = u'(t) = u_0 \omega cos(\omega t) = \omega u_0 sin(\omega t + \pi/2)$$ OK!

2. INTEGRATOR

$$g_2(s) = \frac{1}{s} = \frac{1}{g_1}$$

$$|g_2(j\omega)| = \frac{1}{|g_1|} = \frac{1}{\omega}$$
$$\angle g_2(j\omega) = 0^o - \angle g_1 = -90^o = -\pi/2 \text{ rad}$$

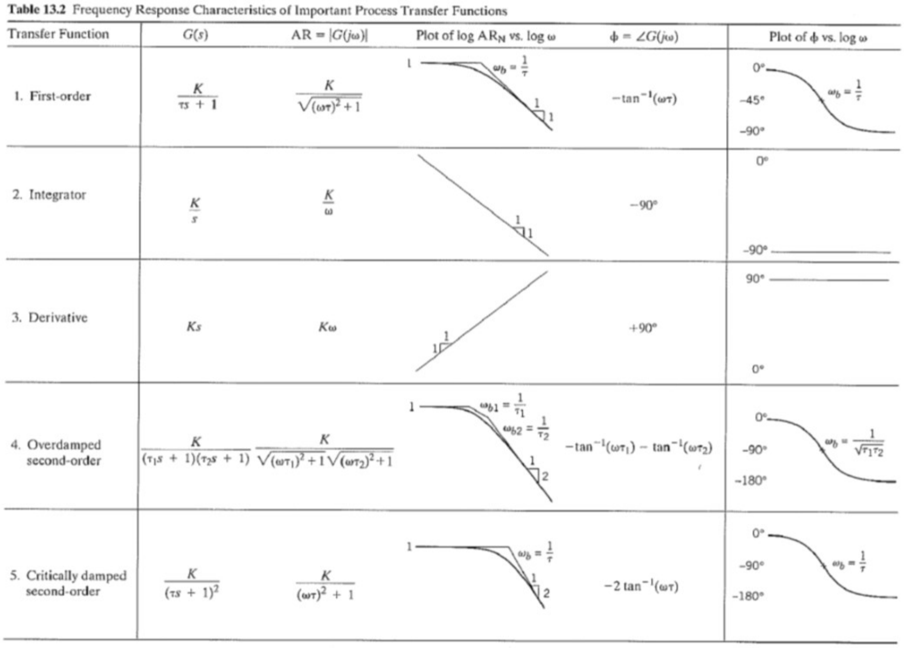
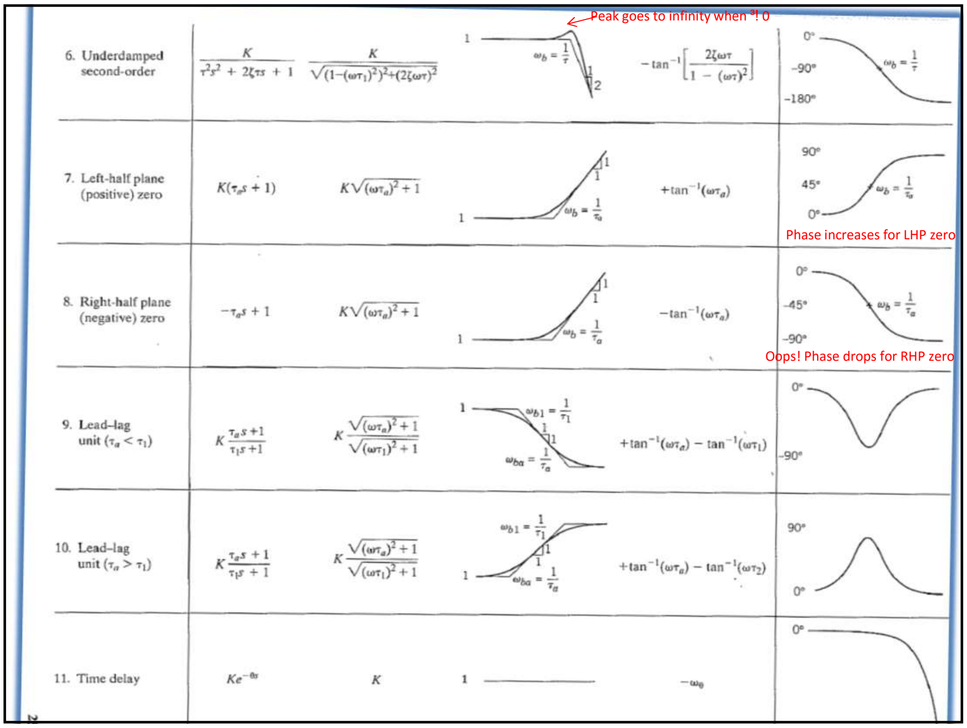


Table 13.2 Frequency Response Characteristics of Important Process Transfer Functions

Transfer Function	$G(s)$	$AR = \|G(j\omega)\|$	Plot of log AR_N vs. log ω	$\phi = \angle G(j\omega)$	Plot of ϕ vs. log ω
1. First-order	$\frac{K}{\tau s + 1}$	$\frac{K}{\sqrt{(\omega\tau)^2 + 1}}$	1 $\omega_b = \frac{1}{\tau}$ 1 1	$-\tan^{-1}(\omega\tau)$	0° $\omega_b = \frac{1}{\tau}$ -45° -90°
2. Integrator	$\frac{K}{s}$	$\frac{K}{\omega}$	1 1	−90°	0° -90°
3. Derivative	Ks	$K\omega$	1 1	+90°	90° 0°
4. Overdamped second-order	$\frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}$	$\frac{K}{\sqrt{(\omega\tau_1)^2+1}\sqrt{(\omega\tau_2)^2+1}}$	1 $\omega_{b1} = \frac{1}{\tau_1}$ $\omega_{b2} = \frac{1}{\tau_2}$ 1 2	$-\tan^{-1}(\omega\tau_1) - \tan^{-1}(\omega\tau_2)$	0° $\omega_b = \frac{1}{\sqrt{\tau_1\tau_2}}$ -90° -180°
5. Critically damped second-order	$\frac{K}{(\tau s + 1)^2}$	$\frac{K}{(\omega\tau)^2 + 1}$	1 $\omega_b = \frac{1}{\tau}$ 1 2	$-2\tan^{-1}(\omega\tau)$	0° -90° $\omega_b = \frac{1}{\tau}$ -180°



6. Underdamped second-order	$\frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$	$\frac{K}{\sqrt{(1-(\omega\tau_1)^2)^2+(2\zeta\omega\tau)^2}}$	$-\tan^{-1}\left[\frac{2\zeta\omega\tau}{1-(\omega\tau)^2}\right]$
7. Left-half plane (positive) zero	$K(\tau_a s + 1)$	$K\sqrt{(\omega\tau_a)^2 + 1}$	$+\tan^{-1}(\omega\tau_a)$
8. Right-half plane (negative) zero	$-\tau_a s + 1$	$K\sqrt{(\omega\tau_a)^2 + 1}$	$-\tan^{-1}(\omega\tau_a)$
9. Lead–lag unit ($\tau_a < \tau_1$)	$K\frac{\tau_a s+1}{\tau_1 s+1}$	$K\frac{\sqrt{(\omega\tau_a)^2+1}}{\sqrt{(\omega\tau_1)^2+1}}$	$+\tan^{-1}(\omega\tau_a) - \tan^{-1}(\omega\tau_1)$
10. Lead–lag unit ($\tau_a > \tau_1$)	$K\frac{\tau_a s + 1}{\tau_1 s + 1}$	$K\frac{\sqrt{(\omega\tau_a)^2+1}}{\sqrt{(\omega\tau_1)^2+1}}$	$+\tan^{-1}(\omega\tau_a) - \tan^{-1}(\omega\tau_2)$
11. Time delay	$Ke^{-\theta s}$	K	$-\omega_\theta$

Peak goes to infinity when ³! 0

Phase increases for LHP zero

Oops! Phase drops for RHP zero

ASYMPTOTES

Frequency response of term (Ts+1): set s=jω.
Asymptotes:

$(j\omega T + 1) \sim 1$ for $\omega T << 1$ (slope n=0, phase=0)
$(j\omega T + 1) \sim j\omega T$ for $\omega T >> 1$ (slope n=1, phase=90°)

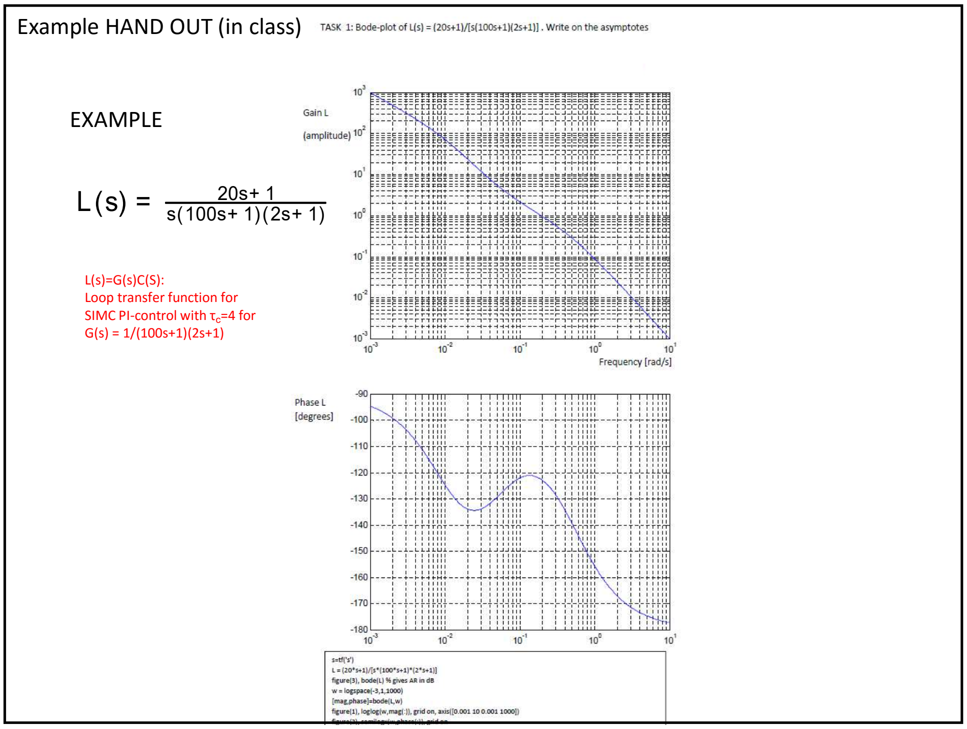
Gain slope n: $|G| \sim \omega^n$

Rule for asymptotic Bode-plot, L = k(Ts+1)/(τs+1)….. :

1. Start with low-frequency asymptote (s→0)
   - (a) If constant (L(0)=k):
     - Gain=k (slope=0)
     - Phase=0°
   - (b) If integrator (L=k'/s):
     - Gain slope= -1 (on log-log plot). Need one fixed point, for example, gain=1 at ω=k'
     - Phase: -90°.
2. Break frequencies (order from large T to small T):

	Change in gain slope	Change in phase
ω=1/T (zero)	+1	+90° (-90° if T negative)
ω=1/**τ** (pole)	-1	-90° (+90° if **τ** negative)

3. Time delay, $e^{-\theta s}$. Gain: no effect, Phase contribution: $-\omega\theta$ [rad] (-1 rad = -57° at $\omega=1/\theta$)



Example HAND OUT (in class)

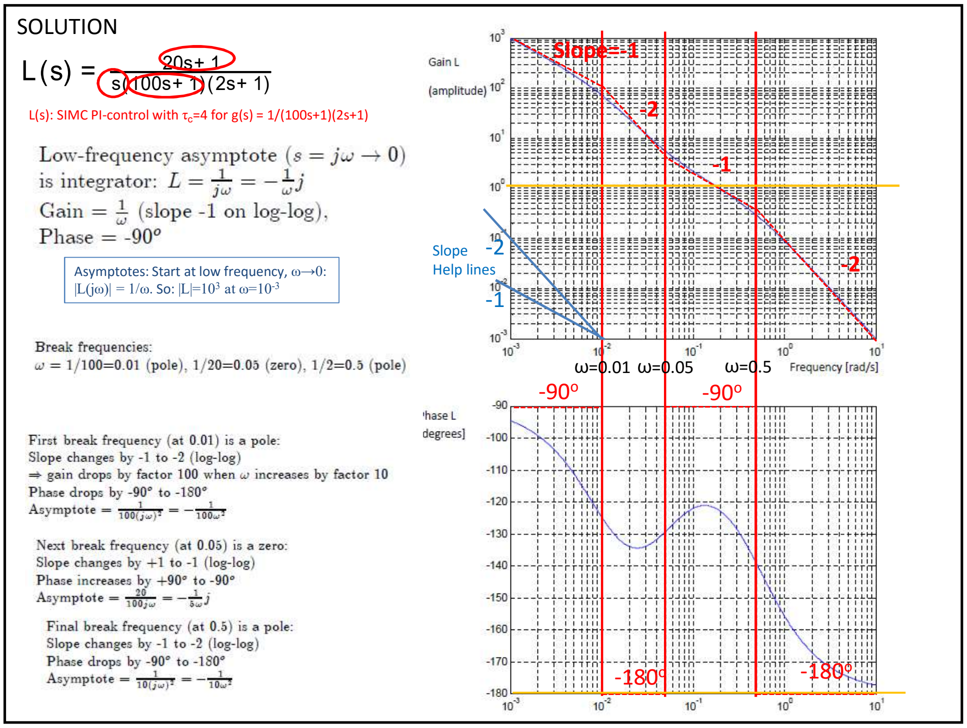
TASK 1: Bode-plot of L(s) = (20s+1)/[s(100s+1)(2s+1)] . Write on the asymptotes

EXAMPLE

$$L(s) = \frac{20s+1}{s(100s+1)(2s+1)}$$

L(s)=G(s)C(S):
Loop transfer function for
SIMC PI-control with τ_c=4 for
G(s) = 1/(100s+1)(2s+1)

```
s=tf('s')
L = (20*s+1)/[s*(100*s+1)*(2*s+1)]
figure(3), bode(L) % gives AR in dB
w = logspace(-3,1,1000)
[mag,phase]=bode(L,w)
figure(1), loglog(w,mag(:)), grid on, axis([0.001 10 0.001 1000])
```



SOLUTION

$$L(s) = \frac{20s+1}{s(100s+1)(2s+1)}$$

L(s): SIMC PI-control with τ_c=4 for g(s) = 1/(100s+1)(2s+1)

Low-frequency asymptote ($s = j\omega \to 0$) is integrator: $L = \frac{1}{j\omega} = -\frac{1}{\omega}j$

Gain = $\frac{1}{\omega}$ (slope -1 on log-log),

Phase = -90^o

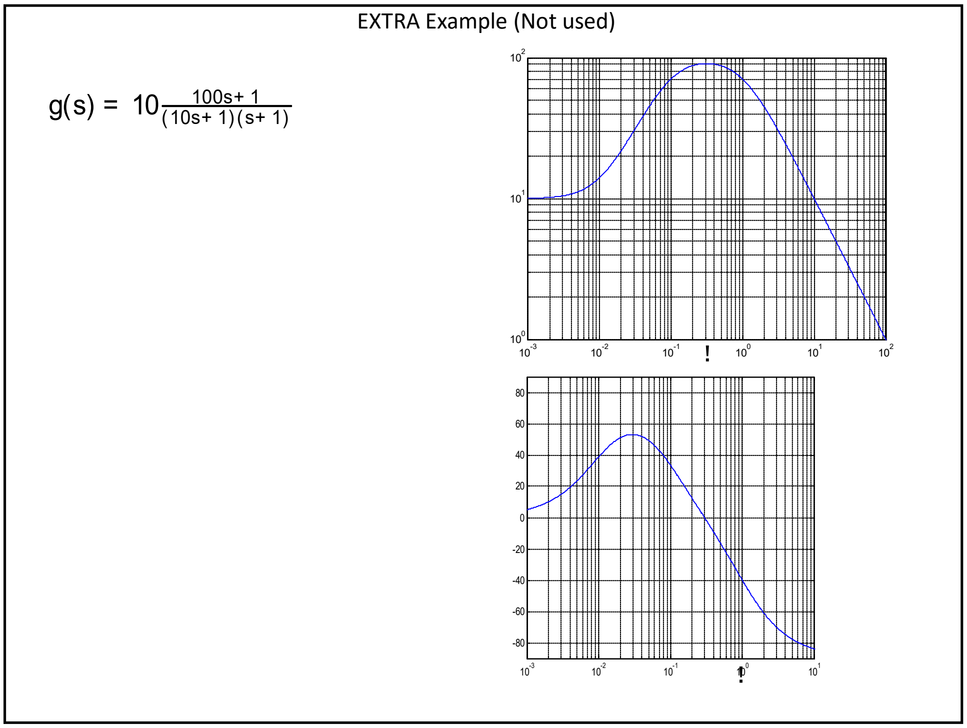
Asymptotes: Start at low frequency, ω→0:
$|L(j\omega)| = 1/\omega$. So: $|L|=10^3$ at $\omega=10^{-3}$

Break frequencies:
$\omega = 1/100=0.01$ (pole), $1/20=0.05$ (zero), $1/2=0.5$ (pole)

First break frequency (at 0.01) is a pole:
Slope changes by -1 to -2 (log-log)
⇒ gain drops by factor 100 when ω increases by factor 10
Phase drops by -90° to -180°
Asymptote = $\frac{1}{100(j\omega)^2} = -\frac{1}{100\omega^2}$

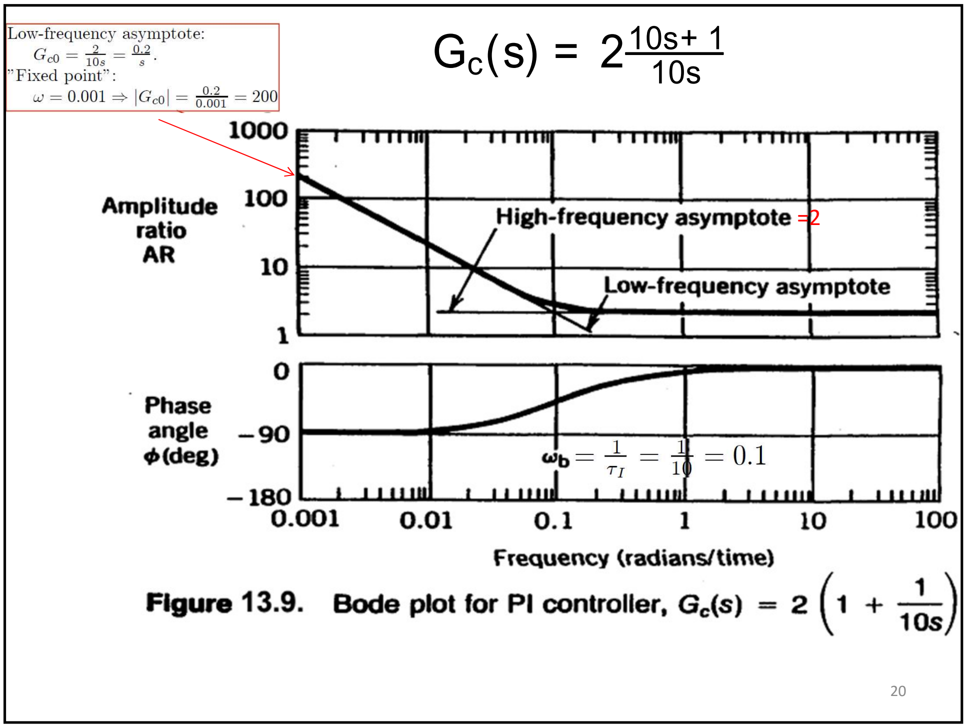
Next break frequency (at 0.05) is a zero:
Slope changes by +1 to -1 (log-log)
Phase increases by +90° to -90°
Asymptote = $\frac{20}{100j\omega} = -\frac{1}{5\omega}j$

Final break frequency (at 0.5) is a pole:
Slope changes by -1 to -2 (log-log)
Phase drops by -90° to -180°
Asymptote = $\frac{1}{10(j\omega)^2} = -\frac{1}{10\omega^2}$



EXTRA Example (Not used)

$$g(s) = 10\frac{100s+1}{(10s+1)(s+1)}$$



Low-frequency asymptote:

$G_{c0} = \frac{2}{10s} = \frac{0.2}{s}$.

"Fixed point":

$\omega = 0.001 \Rightarrow |G_{c0}| = \frac{0.2}{0.001} = 200$

$$G_c(s) = 2\frac{10s+1}{10s}$$

$\omega_b = \frac{1}{\tau_I} = \frac{1}{10} = 0.1$

Figure 13.9. Bode plot for PI controller, $G_c(s) = 2\left(1 + \frac{1}{10s}\right)$

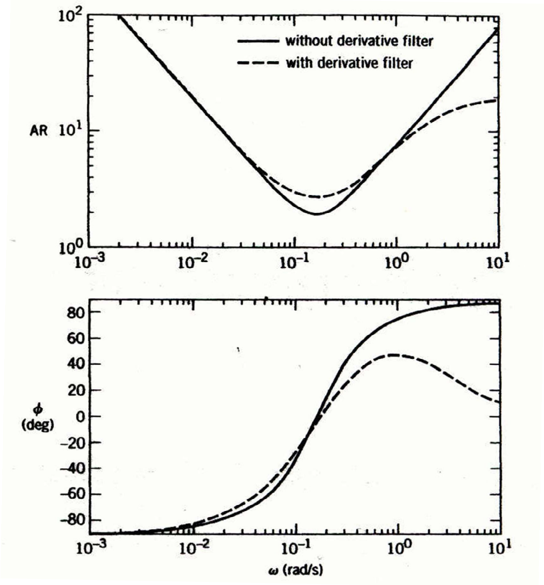


Figure 14.6 Bode plots of ideal parallel PID controller and series PID controller with derivative filter ($\alpha = 0.1$).

Ideal parallel:

$$G_c(s) = 2\left(1 + \frac{1}{10s} + 4s\right)$$

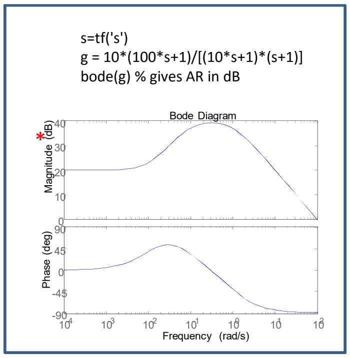
Series with Derivative Filter:

$$G_c(s) = 2\left(\frac{10s+1}{10s}\right)\left(\frac{4s+1}{0.4s+1}\right)$$

Electrical engineers (and Matlab) use decibel for gain

- $|G|$ [dB] = $20 \log_{10}|G|$

|G|	|G| [dB]
0.1	-20 dB
1	0 dB
2	6 dB
10	20 dB
100	40 dB
1000	60 dB



```
s=tf('s')
g = 10*(100*s+1)/[(10*s+1)*(s+1)]
bode(g) % gives AR in dB
```

*To change magnitude from dB to abs: Right click + properties + units

Other way: $|G| = 10^{|G|(\text{dB})/20}$

GM=2 is same as GM = 6dB

CLOSED-LOOP STABILITY

- $L = gcg_m$ = loop transfer function with negative feedback
- Bode's stability condition: $|L(\omega_{180})|<1|$
  - Limitations
    - Open-loop stable (L(s) stable)
    - Phase of L crosses -180° only once

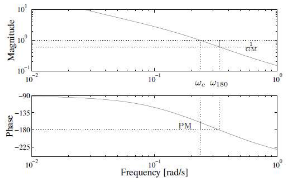
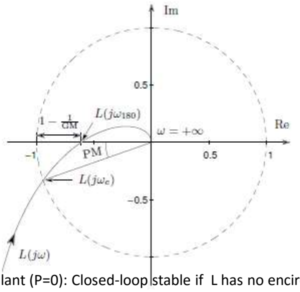


Figure 2.12: Typical Bode plot of $L(j\omega)$ with PM and GM indicated

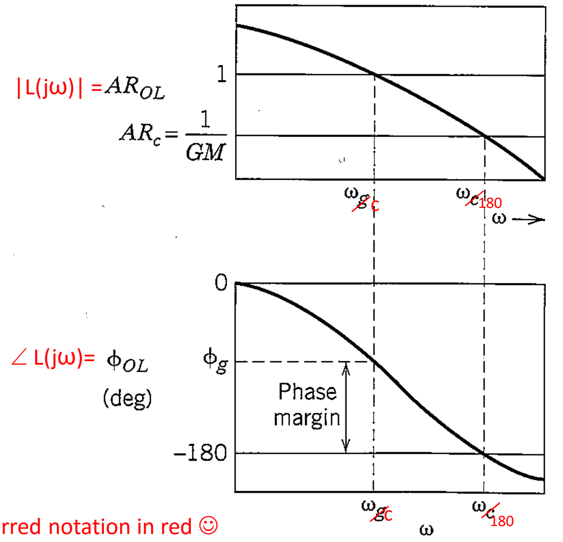
- More general: Nyquist stability condition:

Locus of L(jω) should encircle the (-1)-point P times in the anti-clockwise direction (where P = no. of unstable poles in L).



Stable plant (P=0): Closed-loop stable if L has no encirclements of -1 (=Bode's stability condition)

- Example 1. P-control of delay process. For what Kc is system stable?
- Example 2. I-control of delay process. For what KI is system stable? & compare with SIMC for delay process



Sigurd’s preferred notation in red ☺

Figure 14.12 Gain and phase margins on a Bode plot

Time delay margin, $\Delta\theta$= PM[rad]/ω_c

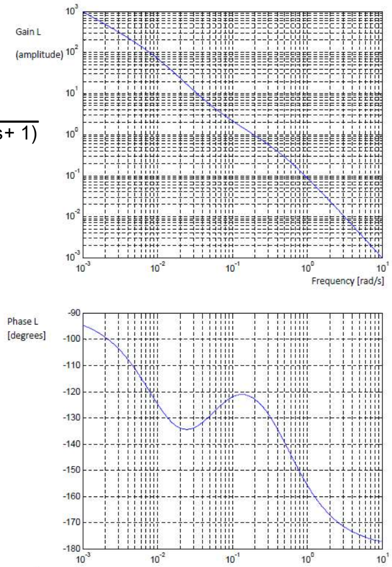
HAND OUT (in class)

TASK 1: Bode-plot of L(s) = (20s+1)/[s(100s+1)(2s+1)] . Write on the asymptotes
TASK 2: What is GM and PM?
TASK 2: How does the plot change if we add a delay of 2 time units (e^{-2*s})
TASK 3: What is now GM and PM? How much extra time delay can we allow?

EXAMPLE 3

$$L(s) = \frac{20s+1}{s(100s+1)(2s+1)}$$

L(s): SIMC PI-control with τ_c=4 for g(s) = 1/(100s+1)(2s+1)

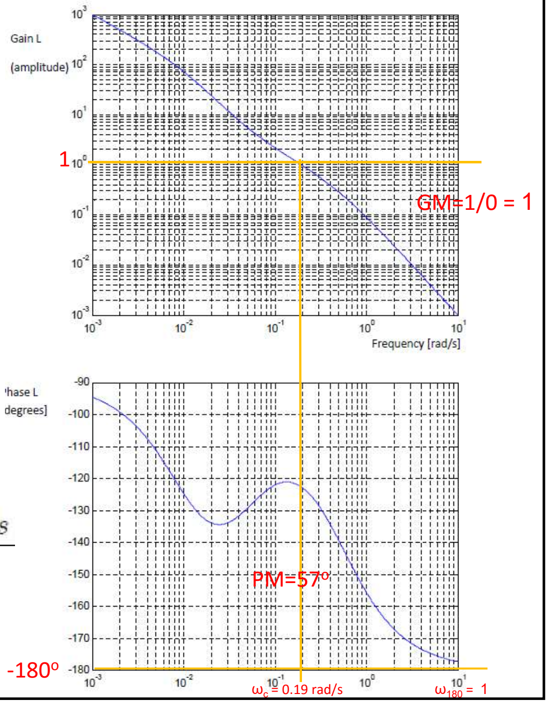


```
s=tf('s')
L = (20*s+1)/[s*(100*s+1)*(2*s+1)]
figure(3), bode(L) % gives AR in dB
w = logspace(-3,1,1000)
[mag,phase]=bode(L,w)
figure(1), loglog(w,mag(:)), grid on, axis([0.001 10 0.001 1000])
figure(2), semilogx(w,phase(:)), grid on
```

SOLUTION

$$L(s) = \frac{20s+1}{s(100s+1)(2s+1)}$$

L(s): SIMC PI-control with τ_c=4 for g(s) = 1/(100s+1)(2s+1)



GM=1/0 = 1

PM=57°

ω_c = 0.19 rad/s

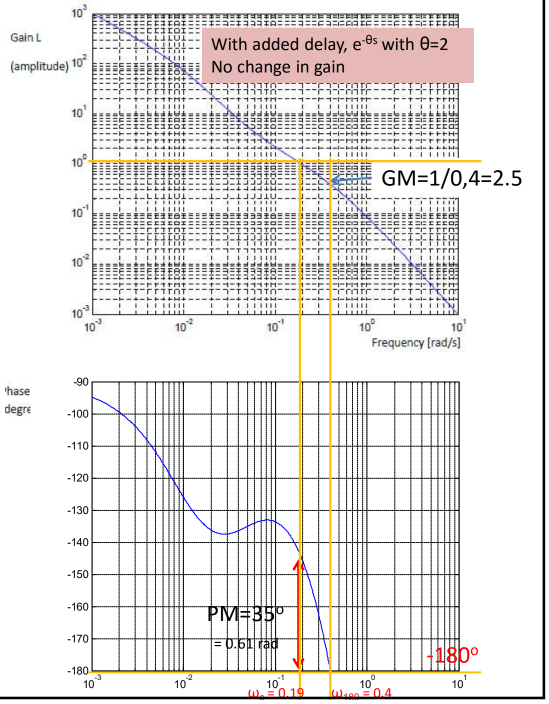
ω_{180} = 1

Time delay margin

$$\Delta\theta = \frac{PM[rad]}{\omega_c[rad/s]} = \frac{1}{0.19} = 5.2s$$

SOLUTION: ADD 2 UNITS OF DELAY

$$L = \frac{20s+1}{s(100s+1)(2s+1)}e^{-2s}$$



With added delay, $e^{-\theta s}$ with θ=2
No change in gain

GM=1/0,4=2.5

PM=35°

= 0.61 rad

-180°

ω_c = 0.19

ω_{180} = 0.4

New time delay margin

$$\Delta\theta = \frac{PM[rad]}{\omega_c[rad/s]} = \frac{0.61}{0.19} = 3.2s$$

Example. PI-control of integrating process with delay

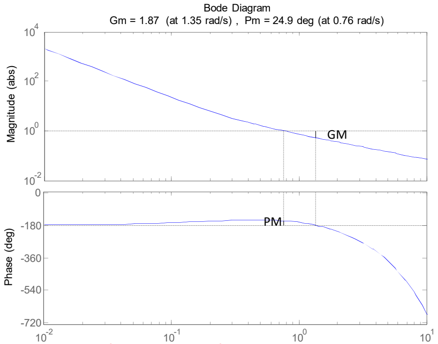
- $g(s) = k' e^{-\theta s}/s$
  - Derive: $P_u = 4\theta$ and $K_u = (\pi/2)/(k'\theta)$
- PI-controller, $c(s) = K_c\,(1+1/(\tau_I s))$

	K_c	τ_I
Ziegler-Nichols	$0.45K_u = 0.707/(k'\,\theta)$	$P_u/1.2=3.33\theta$
SIMC ($\tau_c=\theta$)	$0.5/(k'\,\theta)$	8θ

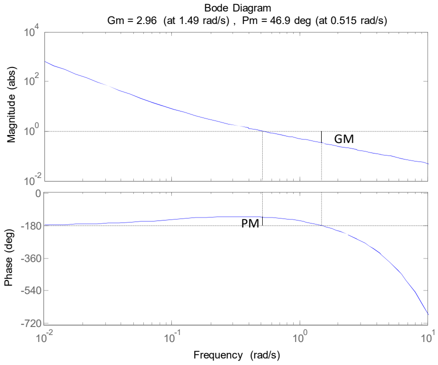
Task: Compare Bode-plot (L=gc), robustness and simulations (use k'=1, θ=1).

$$g(s) = \frac{e^{-s}}{s}, \quad c(s) = K_c \frac{\tau_I s+1}{\tau_I s}$$

Ziegler-Nichols PI



SIMC-PI



SIMC is a lot more robust:

	GM	PM	Delay margin, Δθ
Ziegler-Nichols	1.87	24.9º	0.57 s
SIMC (τ_c=θ)	2.97	46.9º	1.88 s

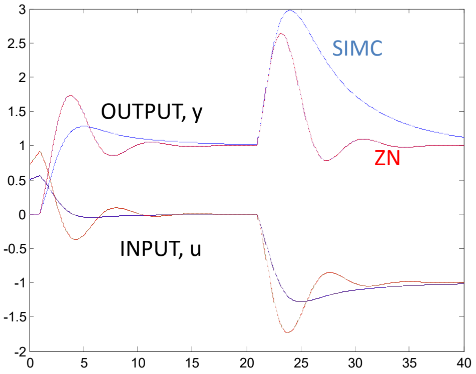
```
s=tf('s')
g = exp(-s)/s
Kc=0.707, taui=3.33
c = Kc*(1+1/(taui*s))
L1 = g*c
figure(1), margin(L1) % Bode-plot with margins
% To change magnitude from dB to abs: Right click + properties + units
Kc=0.5, taui=8
c = Kc*(1+1/(taui*s))
L2 = g*c
figure(2), margin(L2)
```

$\Delta\theta$= PM[rad]/ω_c

ZN: Δθ = 24.9*(3.14/180)/0.76 = 0.572s

SIMC: Δθ = 46.9*(3.14/180)/0.515 = 1.882s

Closed-loop response: PI-control of $g(s) = \frac{e^{-s}}{s}$



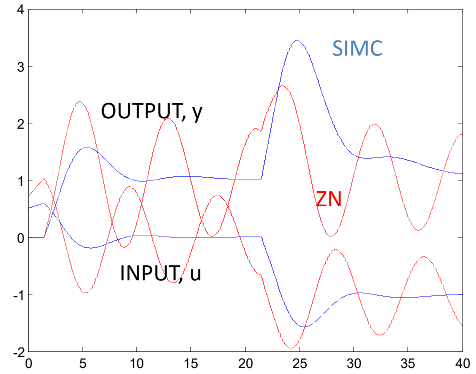
t=0: setpoint change, t=20: input disturbance

```
Simulink file: tunepid4
s=tf('s')
g = exp(-s)/s
Kc=0.707, taui=3.33, taud=0 % ZN
sim('tunepid4')
plot(Tid,y,'red',Tid,u,'red')
Kc=0.5, taui=8, taud=0 % SIMC
sim('tunepid4')
hold
plot(Tid,y,'blue',Tid,u,'blue')
hold off
```

Conclusion: Ziegler-Nichols (ZN) responds faster to the input disturbance, but is much less robust.

- ZN goes unstable if we increase delay from 1s to 1.57s.
- SIMC goes unstable if we increase delay from 1s to 2.88s.

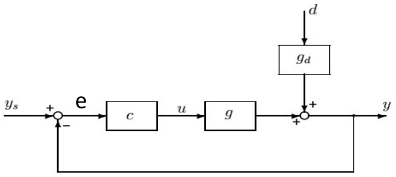
INCREASE DELAY: $g(s) = \frac{e^{-1.5s}}{s}$



t=0: setpoint change, t=20: input disturbance

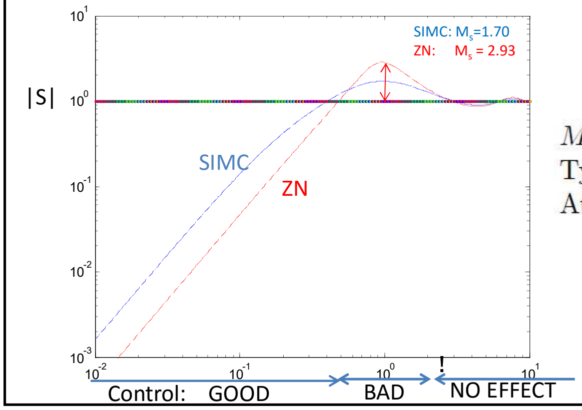
ZN is almost unstable when the delay is increased from 1s to 1.5s.
SIMC does not change very much

Closed-loop frequency response



No control ($c = 0$): $e_{OL} = y_s - y = y_s - g_d d$
With control: $e = y_s - y = Sy_s - Sg_d d = Se_{OL}$
$S = \frac{1}{1+L}$ - sensitivity function = effect of control
$L = gc$ - loop transfer function

Low ω where $|S| < 1$: Control reduces error
Intermediate ω where $|S| > 1$: Control increases error
High ω where $S = 1$ $(L \to 0)$: Control has no effect



M_s = peak of $|S|$
Typical requirement: $M_s < 2$
At stability limit: $M_s \to \infty$

```
w = logspace(-2,1,1000);
[mag1,phase]=bode(1/(1+L1),w);
[mag2,phase]=bode(1/(1+L2),w);
figure(1), loglog(w,mag1(:),'red',w,mag2(:),'blue',w,1,'-.')
axis([0.01,10,0.001,10])
```

Example Ziegler Nichols

Task: Find ZN-settings for integrating+ delay process

- First need to find Pu and Ku