Ch 10.4
Stability

Definition of Stability. *An unconstrained linear system is said to be stable if the output response is bounded for all bounded inputs. Otherwise it is said to be unstable.*

NOTE: Stability of a linear system is a SYSTEM property, that is, independent of the input signal (sine or step, etc.) and of where it enters the system (input or disturbance, etc.)

Linear system. Is g(s) stable?

- g(s) = n(s)/d(s). Poles are solutions $s=p_i$ to d(s)=0.

$$g(s) = \frac{n(s)}{d(s)} \text{ where}$$
$$d(s) = a_n s^n + \cdots + a_1 s + a_0 = a_n(s-p_1)(s-p_2)\cdots(s-p_n)$$

Step response. Partial fraction explansion:

$$y(s) = g(s)\frac{1}{s} = \frac{A_0}{s} + \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \cdots$$

Inverse Laplace:

$$y(t) = A_0 + A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots$$

- So time response contains term e^{pt}
  - Example: G(s) = 1/(s-1), p=1
    - Step response is: $y(t) = -1(1-e^t) = e^t - 1$
- Conclusion:

  Stability $\Leftrightarrow$ $Re(p_i)<0$ (all poles have negative real part)

  $\Leftrightarrow$ All poles in left half plane (LHP),

Pole in right half plane (RHP): UNSTABLE

G(s) = n(s)/d(s) where
d(s) = $(s-p_1)(s-p_2)\cdots$

Real pole p: Get term e^{pt}.
For Re(p)>0 (RHP-pole):
Unstable since $e^{pt} \to \infty$ (as $t \to \infty$)

Complex pole pair ($p_{12} = p \pm j\omega$)
Gives oscillations:
$c_1 e^{p1t} + c_2 e^{p2t} = c\, e^{pt} \sin(\omega t + \phi)$
Which are unstable if Re(p)>0

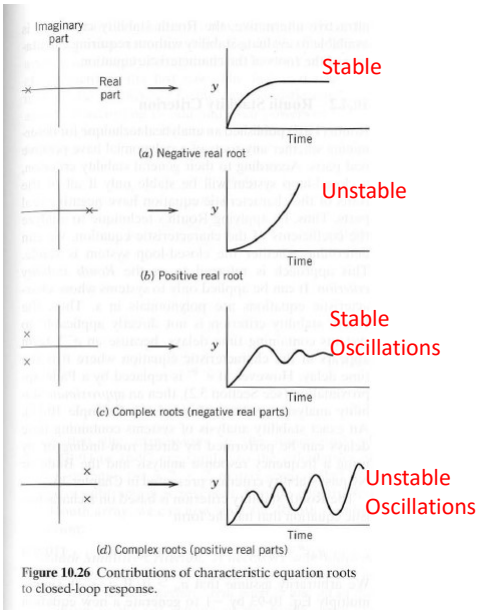
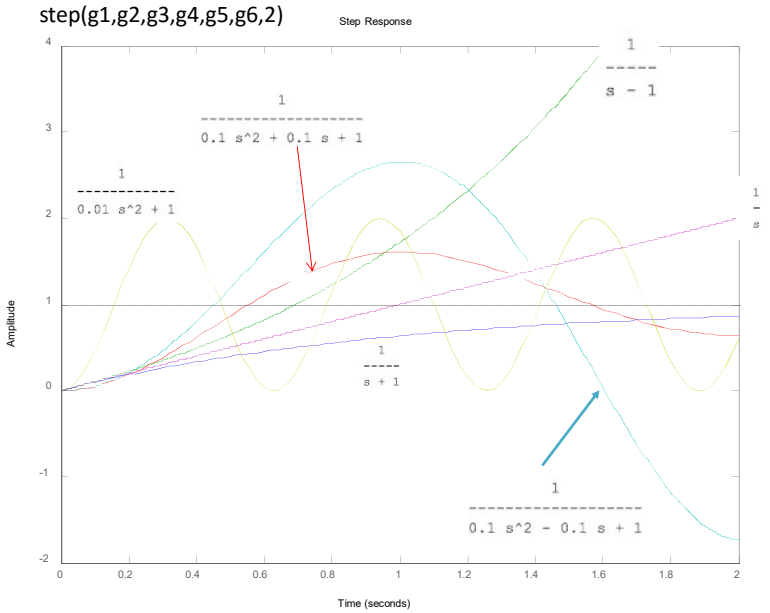


Figure 10.26 Contributions of characteristic equation roots to closed-loop response.



Poles = Eigenvalue of A-matrix

Linear system in deviation variables (state space form)

$dx/dt = A\,x + B\,u$
$y = C\,x + D\,u$

Laplace. Get transfer function from u to y

$G(s) = C\,(sI-A)^{-1}\,B + D = n(s)/d(s)$

From mathematics: $(sI-A)^{-1} = adj(sI-A)/det(sI-A)$, so

$d(s) = det(sI-A)$ = pole polynomial

But $det(sI-A)=0$ is also the formula for finding the eigenvalues of A

Conclusion: Solutions to $d(s)=0$ are the poles which are identical to the eigenvalues of A

$p_i = eig(A)$

Stability of closed-loop systems

- Closed-loop transfer functions, $G_{CL}(s) = n(s)/d(s)$
  - n(s) = direct path
  - d(s) = 1 + loop(s)
    - where loop(s) = $g(s)\,g_m(s)\,c(s)$
- Same d(s) for any input/output!
  - Makes sense because stability is a system property
- Conclusion: Closed-loop poles are given by solutions to "closed-loop characteristic equation"
    - $d(s) = 1 + loop(s) = 0$

How do we test for stability?

T(s) = n(s)/d(s)

Method 1. Compute **poles** (eigenvalues) **p**=$\lambda(A)$ and check if Re(p)<0

- OK numerically, but difficult to find poles p analytically

Method 2. Look at **coefficients a_i** in d(s),

$d(s)= a_0 + a_1 s + \ldots + a_n s^n$

Good for analytical results. Don't need to find poles p

Test 1. All a's must have **same sign*** for stability (necessary condition)

Test 2. Routh array: Necessary and sufficient

Method 3. Closed-loop system. Frequency analysis **(see later)**

- Consider loop transfer function, L = GC
- Bode stability test for stability: |L| < 1 at frequency ω_{180}
- Easy to include time delay

* Necessary and sufficient for 2nd order system

EXAMPLE 2

For which K_c is the closed-loop system stable?

$g(s) = \frac{1}{6s+1}$

$g_m(s) = \frac{-s+1}{s+1}$

$c(s) = K_c$

Method 1. Analytic solution using poles. Much more work*!

Matlab commands:

```
syms s Kc
g=1/(6*s+1)
gm=(-s+1)/(s+1)
clpoles=solve(1+Kc*g*gm==0)
solve(real(clpoles(1))==0)
solve(real(clpoles(2))==0)
```

Solution:

```
g =1/(6*s + 1)
gm =-(s - 1)/(s + 1)
 clpoles =
  Kc/12 + (Kc^2 - 38*Kc + 25)^(1/2)/12 - 7/12
 Kc/12 - (Kc^2 - 38*Kc + 25)^(1/2)/12 - 7/12
 ans =-1.0
 ans =7.0
```

* Almost impossible for systems of order 4 or higher

Method 2. Coefficients

Test 1. Check signs of Char. Eq..:

d(s) = 1 + loop = 1 + Kc*(-s+1)/[(s+1)(6s+1)]=0

6s^2 + (7-Kc)s + (1+Kc) = 0

Stable -> all coefficients positive ->

Kc>-1 (lower limit for positive feedback)

Kc<7 (upper limit because of RHP-zero)

(necessary and sufficient for 2nd order system)

Test 2. ROUTH array: Find location (RHP/LHP) of poles without actually having to find the
"Advanced version of looking for negative sign in d(s)-polynomial"

Example 3. $g(s) = \frac{1}{(5s+1)(2s+1)}$; $g_m = \frac{1}{s+1}$, P-control

10.4.2 Routh Stability Criterion

Routh (1905) published an analytical technique for determining whether any roots of a polynomial have positive real parts. According to their general stability criterion, a closed-loop system will be stable only if all of the roots of the characteristic equation have negative real parts. Thus, by applying Routh's technique to analyze the coefficients of the characteristic equation, we can determine whether the closed-loop system is stable. This approach is referred to as the *Routh stability criterion*. It can be applied only to systems whose characteristic equations are polynomials in s. Thus, the Routh stability criterion is not directly applicable to systems containing time delays, because an $e^{-\theta s}$ term appears in the characteristic equation where θ is the time delay. However, if $e^{-\theta s}$ is replaced by a Padé approximation (see Section 5.2), then an *approximate* stability analysis can be performed (cf. Example 10.11). An exact stability analysis of systems containing time delays can be performed by direct root-finding or by using a frequency response analysis and the Bode or Nyquist stability criterion presented in Chapter 13.

The Routh stability criterion is based on a characteristic equation that has the form

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0 \qquad (10\text{-}93)$$

We arbitrarily assume that $a_n > 0$. If $a_n < 0$, simply multiply Eq. 10-93 by -1 to generate a new equation that satisfies this condition. A *necessary* (but not sufficient) condition for stability is that all of the coefficients ($a_0, a_1, \ldots, a_n$) in the characteristic equation be positive. If any coefficient is negative or zero, then at least one root of the characteristic equation lies to the right of, or on, the imaginary axis, and the system is unstable. If all of the coefficients are positive, we next construct the following Routh array:

Row				
1	a_n	a_{n-2}	a_{n-4}	$\cdots$
2	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
3	b_1	b_2	b_3	$\cdots$
4	c_1	c_2	$\cdots$	
$\vdots$	$\vdots$	$\vdots$		
$n+1$	z_1			

The Routh array has $n+1$ rows, where n is the order of the characteristic equation, Eq. 10-93. The Routh array has a roughly triangular structure with only a single element in the last row. The first two rows are merely the coefficients in the characteristic equation, arranged according to odd and even powers of s. The elements in the remaining rows are calculated from the formulas

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \qquad (10\text{-}94)$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} \qquad (10\text{-}95)$$

$$\vdots$$

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1} \qquad (10\text{-}96)$$

$$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1} \qquad (10\text{-}97)$$

$$\vdots$$

Note that the expressions in the numerators of Eqs. 10-94 to 10-97 are similar to the calculation of a determinant for a 2×2 matrix except that the order of subtraction is reversed. Having constructed the Routh array, we can now state the Routh stability criterion:

Routh Stability Criterion. *A necessary and sufficient condition for all roots of the characteristic equation in Eq. 10-93 to have negative real parts is that all of the elements in the left column of the Routh array are positive.*

Next we present three examples that show how the Routh stability criterion can be applied.

EXAMPLE 10.9

Determine the stability of a system that has the characteristic equation

$$s^4 + 5s^3 + 3s^2 + 1 = 0 \qquad (10\text{-}98)$$

SOLUTION

Because the s term is missing, its coefficient is zero. Thus, the system is unstable. Recall that a necessary condition for stability is that all of the coefficients in the characteristic equation must be positive.

EXAMPLE 10.10

Find the values of controller gain K_c that make the feedback control system of Example 10.4 stable.

SOLUTION

From Eq. 10-76, the characteristic equation is

$$10s^3 + 17s^2 + 8s + 1 + K_c = 0 \qquad (10\text{-}99)$$

All coefficients are positive provided that $1 + K_c > 0$ or $K_c > -1$. The Routh array is

10	8	
17	$1 + K_c$	
b_1	b_2	
c_1		

To have a stable system, each element in the left column of the Routh array must be positive. Element b_1 will be positive if $K_c < 12.6$. Similarly, c_1 will be positive if $K_c > -1$. Thus, we conclude that the system will be stable if

$$-1 < K_c < 12.6 \qquad (10\text{-}100)$$

This example illustrates that stability limits for controller parameters can be derived analytically using the Routh array; in other words, it is not necessary to specify a numerical value for K_c before performing the stability analysis.

Complete Routh array
b1 = (17*8-10*(1+Kc)) / 17
b2 = 0
c1 = (1+Kc)
Stability: Elements in first column > 0.
Conclusion:
b1>0 -> Kc<12.6
c1>0 -> Kc>-1

UNSTABLE | STABLE | UNSTABLE K_c

-1 12.6

How do the closed-loop poles depend on the controller gain K_c?

Example 4

c(s) = K_c

$g(s) = \frac{4}{(s+1)(s+2)(s+3)} = \frac{4=6}{(s+1)(0.5s+1)(0.33s+1)}$

10.5 ROOT LOCUS DIAGRAMS

In the previous section we have seen that the roots of the characteristic equation play a crucial role in determining system stability and the nature of the closed-loop responses. In the design and analysis of control systems, it is instructive to know how the roots of the characteristic equation change when a particular system parameter such as a controller gain changes. A root locus diagram provides a convenient graphical display of this information, as indicated in the following example.

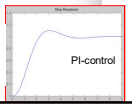
EXAMPLE 10.13

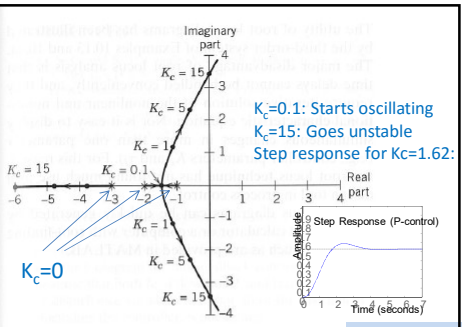
Consider a feedback control system that has the open-loop transfer function,

$$G_{OL}(s) = \frac{4K_c}{(s+1)(s+2)(s+3)} \qquad (10\text{-}108)$$

Plot the root locus diagram for $0 \le K_c \le 20$.

Comment:
Half rule: $g(s) = \frac{4/6}{(s+1)(0.5s+1)(0.33s+1)} \approx \frac{0.67e^{-0.58s}}{1.25s+1}$
SIMC with $\tau_c = \theta = 0.58 : K_c = \frac{0.5}{k}\frac{\tau_1}{\theta} = 1.62, \tau_I = 1.25$
BUT here we use only P-control





K_c=0.1: Starts oscillating
K_c=15: Goes unstable
Step response for Kc=1.62:

Figure 10.27 Root locus diagram for third-order system.

```
s=tf('s')
Kc=1.62
g = 4/[(s+1)*(s+2)*(s+3)]
loop = Kc*g
T = loop/(1+loop)
step(T)
sisotool(g) % root locus
```

SOLUTION

The characteristic equation is $1 + G_{OL} = 0$ or

$$(s+1)(s+2)(s+3) + 4K_c = 0 \qquad (10\text{-}109)$$

The root locus diagram in Fig. 10.27 shows how the three roots of this characteristic equation vary with K_c. When $K_c = 0$, the roots are merely the poles of the open-loop transfer function, -1, -2, and -3. These are designated by an $\times$ symbol in Fig. 10.27. As K_c increases, the root at -3 decreases monotonically. The other two roots converge and then form a complex conjugate pair when $K_c = 0.1$. When $K_c = K_{cm} = 15$, the complex roots cross the imaginary axis and enter the unstable region. This illustrates why the substitution of $s = j\omega$ (Section 10.3) determines the unstable controller gain. Thus, the root locus diagram indicates that the closed-loop system is unstable for $K_c > 15$. It also indicates that the closed-loop response will be nonoscillatory for $K_c < 0.1$.