First-order Transfer Function

First-order scalar system:

$$\frac{dx(t)}{dt} = a\,x(t) + b\,u(t)$$
$$y(t) = c\,x(t) + d\,u(t)$$

Laplace transform:

$$s\,x(s) - x(t=0) = a\,x(s) + b\,u(s)$$
$$(s-a)\,x(s) = x(t=0) + b\,u(s)$$
$$x(s) = (s-a)^{-1}\,(x(t=0) + b\,u(s))$$
$$x(s) = (s-a)^{-1}\,x(t=0) + $$
$$+\,(s-a)^{-1}\,b\,u(s)$$

with zero initial condition $x(0) := 0$ the transfer function results:

$$g(s) = \frac{y(s)}{u(s)} = c\,(s-a)^{-1}\,b + d$$
$$= \frac{n(s)}{d(s)}$$

Fraction of two polynomials in s:
$d(s) = s - a \quad n(s) = ds + bc - da$

Transfer function g(s):
- Effect of forcing system with u(t)
- IMPORTANT!!: g(s) is independent of u(s)!!!
- Fraction of two polynomials

1

General Transfer Matrix

General system with n differential equations in n state variables $x(t)$ (where x, u, y are vectors and A, B, C, D are matrices):

$$\frac{dx(t)}{dt} = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$

Laplace transform with zero intitial condition, $x(0) = 0, u(0) = 0$ (deviation variables):

$$sI\;x(s) = A\,x(s) + B\,u(s)$$
$$(sI - A)\,x(s) = B\,u(s)$$
$$x(s) = (sI - A)^{-1}\;B\,u(s)$$

Get $y(s) = G(s)u(s)$ where transfer matrix is:

$$G(s) = C\;(sI - A)^{-1}\;B + D$$

Here

$$(sI - A)^{-1} = \frac{adj(sI - A)}{det(sI - A)}$$

where $det(SI - A) =$

$$d(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

is a n'th order polynomial in n,

The n roots (generally complex) of the polynomial d(s) are the same as the eigenvalues of the state matrix A, and are known as the «poles» of the system

Poles and zeros

- Transfer functions *G(s)* of linear, time-invariant networks of first-order systems are ratios of two polynomials in *s* (Laplace variable)
  - G(s) = n(s)/d(s)
- Polynomials have roots
  root in denominator, d(s)=0: *G(s)* $\rightarrow \infty$ "pole"
  root in numerator, n(s)=0: *G(s)* $\rightarrow$ 0 "zero"
- Roots & dynamics
  - Zeros are responsible for shape of response
    - Zeros in right half plane (RHP): inverse response
  - Poles roots determine stability and fast or slow dynamics
    - Complex poles (=eigenvalues): Oscillations
    - Poles in right half plane (RHP): Unstable

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Example transfer function

$$g(s) = \frac{4s+2}{5s^2+5.5s+0.5}$$

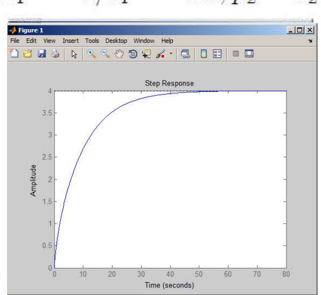
Time constant form:

$$g(s) = k\frac{Ts+1}{(\tau_1 s+1)(\tau_2 s+1)} \text{ with } k=4, T=2, \tau_1=10, \tau_2=1$$

Pole-zero form:

$$g(s) = \frac{4}{5}\frac{s+0.5}{(s+0.1)(s+1)} = k'\frac{s-z}{(s-p_1)(s-p_2)}$$

with $k' = 4/5$,
zero $z = -1/T = -0.5$,
poles (or eigenvalues): $p_1 = \lambda_1 = -1/\tau_1 = -0.1$, $p_2 = \lambda_2 = -1/\tau_2 = -1$



Initial and final values for step response

- Consider response y(t) to step of magnitude M in input
- Transfer function g(s)
- Deviation variables for y(t) and u(t)

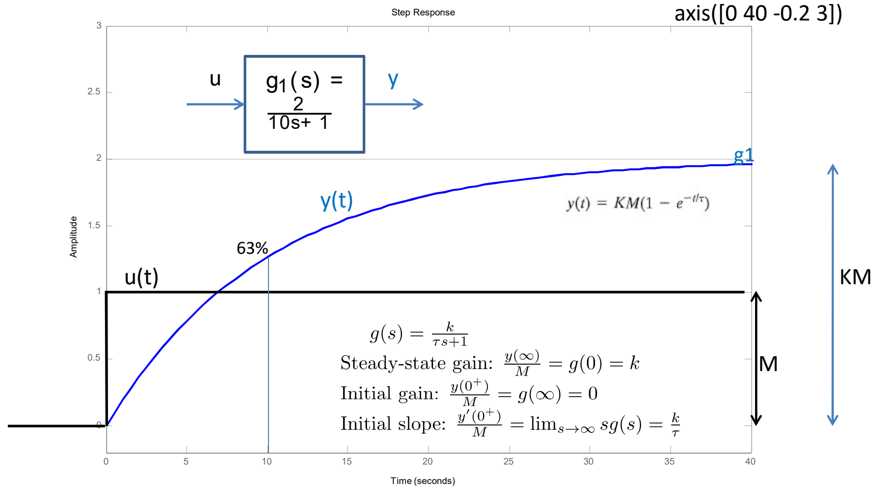
$$\text{Steady-state gain: } \frac{y(\infty)}{M} = g(0)$$
$$\text{Initial gain: } \frac{y(0^+)}{M} = g(\infty)$$
$$\text{Initial slope: } \frac{y'(0^+)}{M} = \lim_{s\to\infty} sg(s)$$

Proof: Note that $y(s) = g(s)\frac{M}{s}$
Final value theorem: $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sy(s) = \lim_{s\to 0} sg(s)\frac{M}{s} = g(0)M$
Initial value theorem: $\lim_{t\to 0} y(t) = \lim_{s\to\infty} sy(s) = g(\infty)M$
Initial value theorem: $\lim_{t\to 0} y'(t) = \lim_{s\to\infty} s(sy(s)) = \lim_{s\to\infty} sg(s)M$
Initial value theorem: $\lim_{t\to 0} y^{(n)}(t) = \lim_{s\to\infty} s^n(sy(s)) = \lim_{s\to\infty} s^n g(s)M$

Dynamic step response of some systems

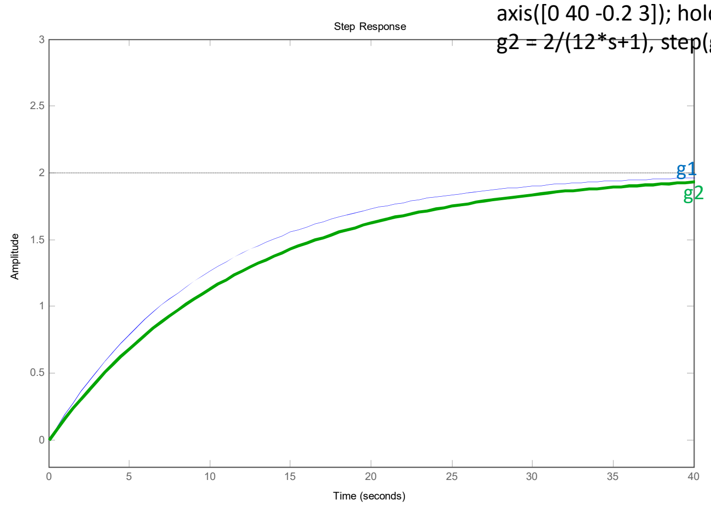
g1: First-order system (k=2, τ=10)

```
s=tf('s')
g1 = 2/(10*s+1)
step(g1,50)
axis([0 40 -0.2 3])
```



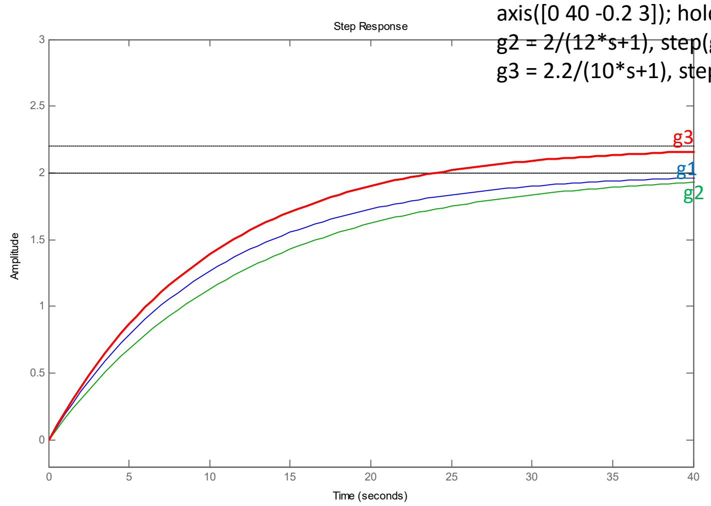
g2: Larger time constant (τ=12)
...gives slower dynamics

```
s=tf('s')
g1 = 2/(10*s+1), step(g1,50)
axis([0 40 -0.2 3]); hold on,
g2 = 2/(12*s+1), step(g2,50)
```



g3: Larger steady-state gain (k=2.2)

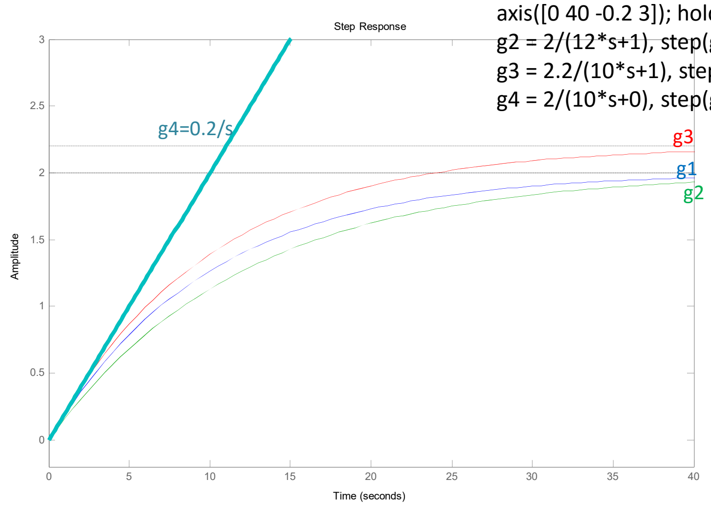
```
s=tf('s')
g1 = 2/(10*s+1), step(g1,50)
axis([0 40 -0.2 3]); hold on,
g2 = 2/(12*s+1), step(g2,50)
g3 = 2.2/(10*s+1), step(g3,50)
```



g4: Integrating system =0.2/s

g1 & g4: Same initial response (slope = 0.2=k/τ)

```
s=tf('s')
g1 = 2/(10*s+1), step(g1,50)
axis([0 40 -0.2 3]); hold on,
g2 = 2/(12*s+1), step(g2,50)
g3 = 2.2/(10*s+1), step(g3,50)
g4 = 2/(10*s+0), step(g4,50)
```

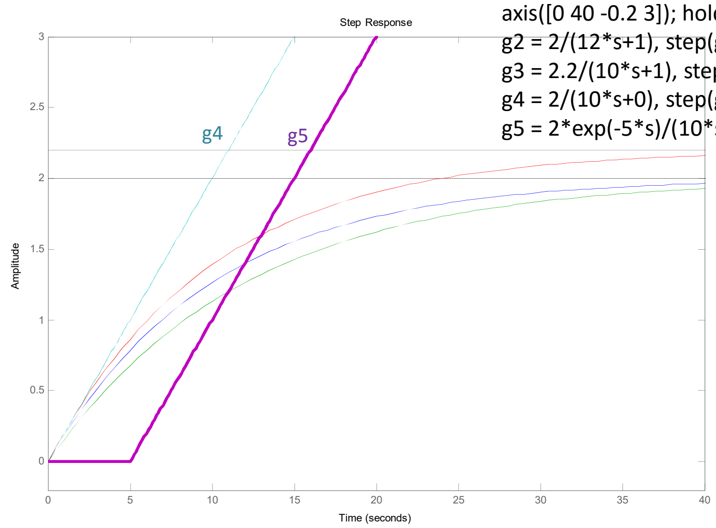


Integrating system, g(s)=k'/s

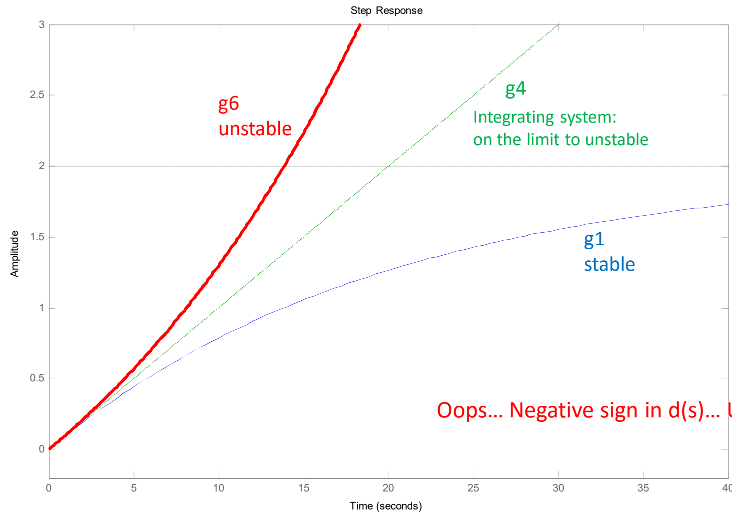
- Special case of first-order system with $\tau=\infty$ and k=∞ but slope k'=k/τ is finite
- g(s)=k/(τ s+1) = k/(τ s) = k'/s
- Step response (u=M): y(t)/M = k't

g5: Integrating system with time delay

```
s=tf('s')
g1 = 2/(10*s+1), step(g1,50)
axis([0 40 -0.2 3]); hold on,
g2 = 2/(12*s+1), step(g2,50)
g3 = 2.2/(10*s+1), step(g3,50)
g4 = 2/(10*s+0), step(g4,50)
g5 = 2*exp(-5*s)/(10*s), step (g5,5
```



g6: Unstable system (e.g., exothermic reactor):
Note: Sign change in denominator d(s)



g1 = $\frac{2}{10\,s + 1}$

g4 = $\frac{2}{10\,s}$

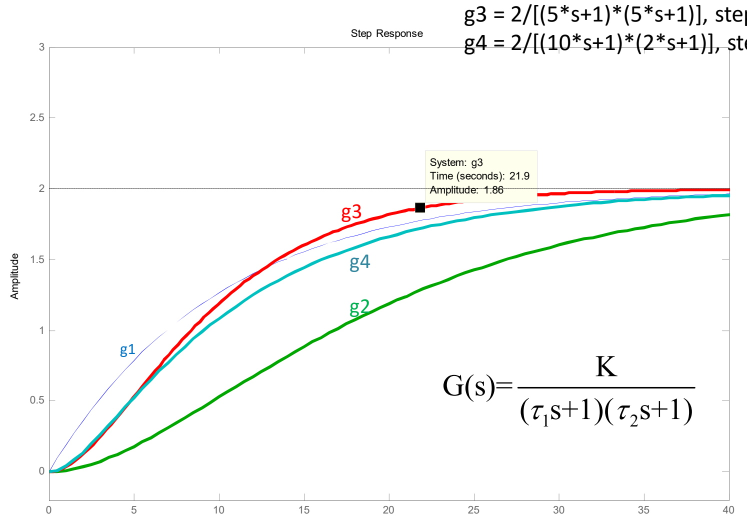
g6 = $\frac{2}{10\,s - 1}$

Oops… Negative sign in d(s)… Unstable!

«S-shaped» 2nd order responses

(g2, g3, g4) made from two first-order systems in series

g1 = 2/(10*s+1), step(g1,50)
g2 = 2/[(10*s+1)*(10*s+1)], step(g2,50)
g3 = 2/[(5*s+1)*(5*s+1)], step(g3,50)
g4 = 2/[(10*s+1)*(2*s+1)], step(g4,50)

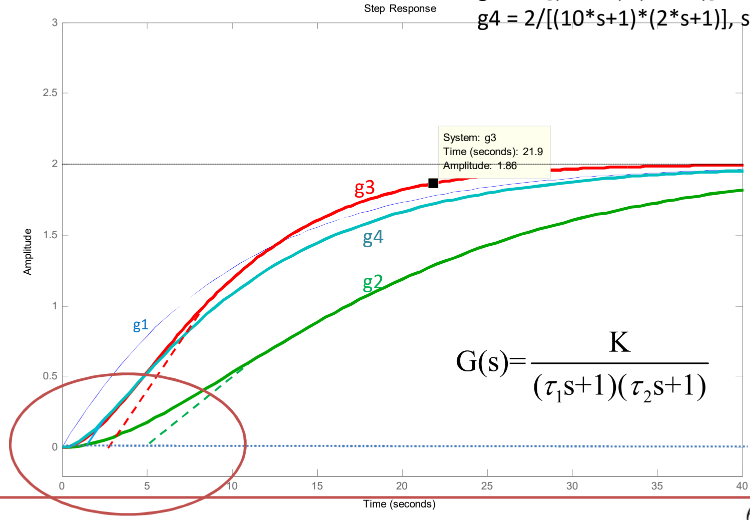


$$G(s)=\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}$$

$$y(t) = KM\left(1 - \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_1 - \tau_2}\right) \qquad (4\text{-}48)$$

2nd order (S-shaped) approximated as first-order plus delay
g1(s)=K*exp(-θs)/(τs+1)

g1 = 2/(10*s+1), step(g1,50)
g2 = 2/[(10*s+1)*(10*s+1)], step(g2,50)
g3 = 2/[(5*s+1)*(5*s+1)], step(g3,50)
g4 = 2/[(10*s+1)*(2*s+1)], step(g4,50)



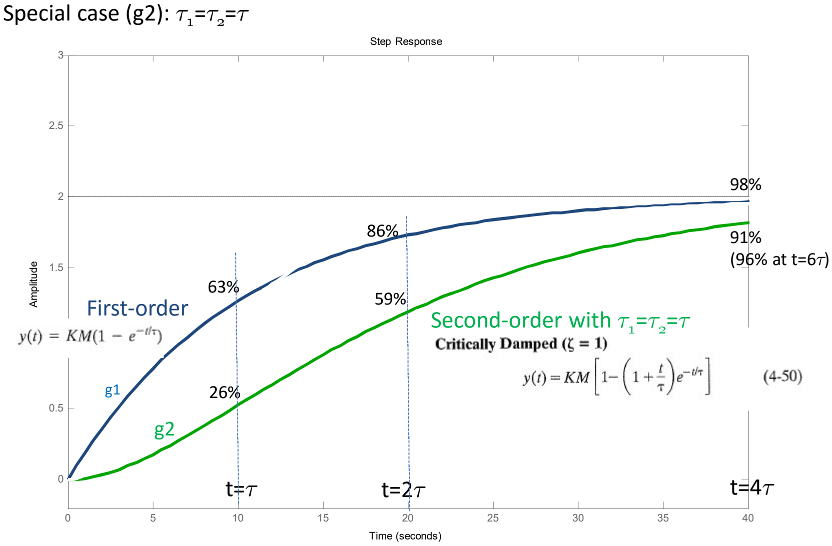
Approximated as first-order with delay, (using half rule):

Half rule: $\theta \approx \tau_2/2$

	$\theta \approx \tau_2/2$
g2	5
g3	2.5
g4	1

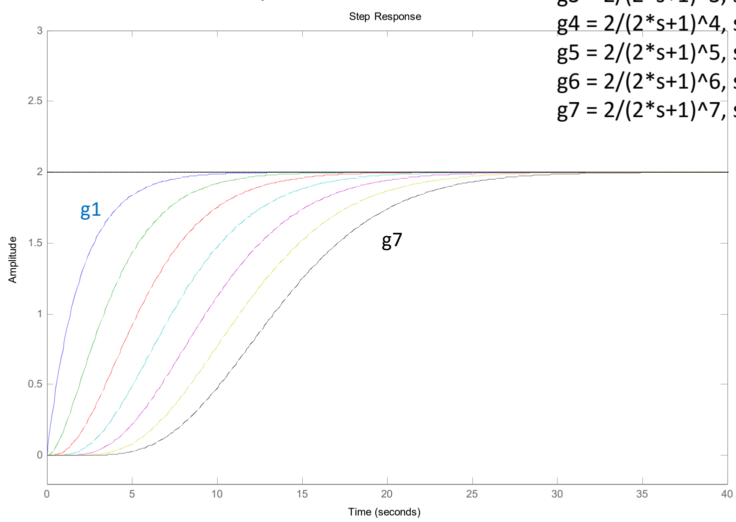
2nd order responses
Special case (g2): $\tau_1=\tau_2=\tau$

g1 = 2/(10*s+1)
g2 = 2/[(10*s+1)*(10*s+1)]



n'th order responses:
From n identical first-order systems in series

g1 = 2/(2*s+1), step(g1,50)
g2 = 2/(2*s+1)^2, step(g2,50)
g3 = 2/(2*s+1)^3, step(g3,50)
g4 = 2/(2*s+1)^4, step(g4,50)
g5 = 2/(2*s+1)^5, step(g5,50)
g6 = 2/(2*s+1)^6, step(g6,50)
g7 = 2/(2*s+1)^7, step(g7,50)



Chapter 5 4

General 2nd order system

$$G(s)=\frac{K}{\tau^2 s^2+2\zeta\tau s+1}=\frac{K}{\tau^2(s-\lambda_1)(s-\lambda_2)}$$

Roots (poles, eigenvalues): $\lambda_{1,2}=\frac{-\zeta\pm\sqrt{\zeta^2-1}}{\tau}$

$\zeta<1$ *underdamped (oscillations)*

$\zeta=1$ *critically damped*

$\zeta>1$ *overdamped (no-oscillations)*

Special case: Two first-order in series (overdamped, $\zeta\geq 1$):

$$G(s)=\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}=\frac{K}{\tau_1\tau_2 s^2+(\tau_1+\tau_2)s+1}$$

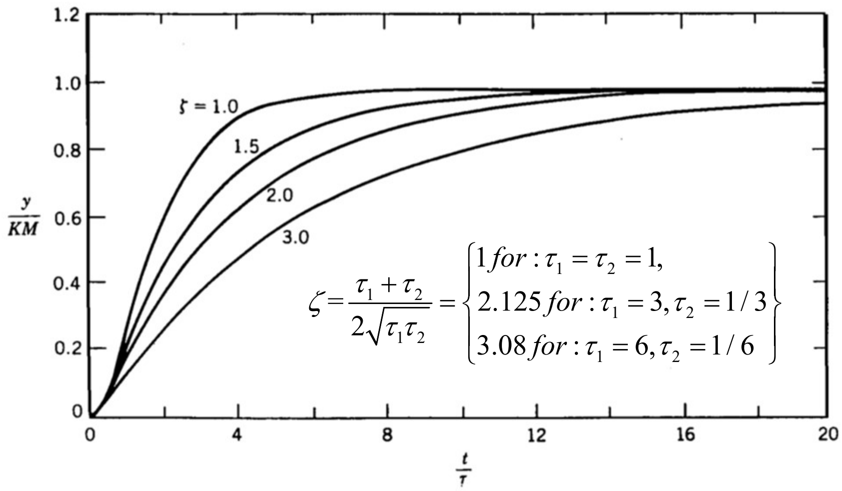
Two-real-poles: $\lambda_1=-1/\tau_1, \lambda_2=-1/\tau_2$

$$\zeta=\frac{\tau_1+\tau_2}{2\sqrt{\tau_1\tau_2}}\geq 1$$

$$\tau=\sqrt{\tau_1\tau_2}$$

Chapter 5

$$G(s)=\frac{K}{(\tau_1 s+1)(\tau_2 s+1)} \qquad y(t) = KM\left(1 - \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_1 - \tau_2}\right) \quad (4\text{-}48)$$



$$\zeta=\frac{\tau_1+\tau_2}{2\sqrt{\tau_1\tau_2}}=\begin{cases}1\, for: \tau_1=\tau_2=1,\\ 2.125\, for: \tau_1=3, \tau_2=1/3\\ 3.08\, for: \tau_1=6, \tau_2=1/6\end{cases}$$

Figure 4.9 Step response of critically-damped and overdamped second-order processes.

$$\tau=\sqrt{\tau_1\tau_2}$$

Underdamped (Oscillating) second-order systems (ζ<1)

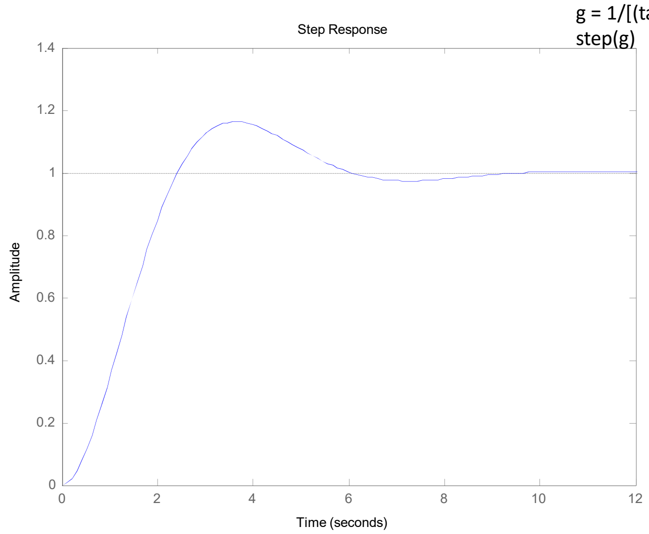
$$G(s)=\frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Corresponds to complex poles

Process systems:
Oscillations are usually caused by (too) aggressive control

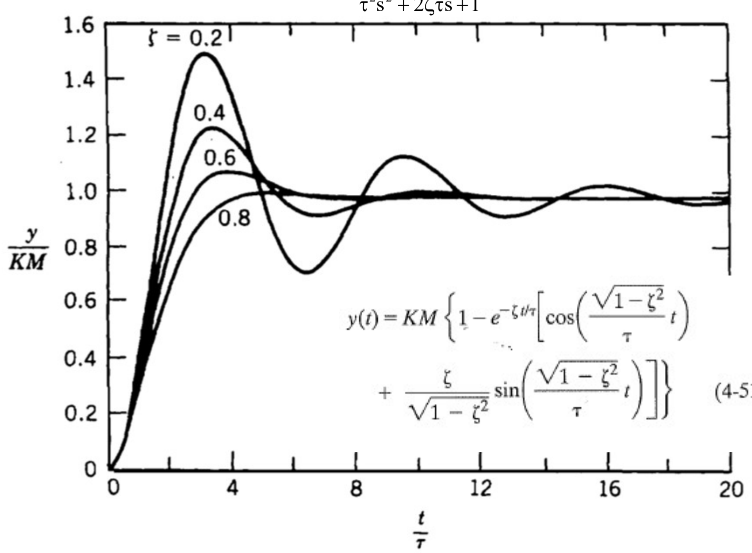
Example 1: P-control of second-order process, $k/(\tau_1 s+1)(\tau_2 s+1)$
- Oscillates (ζ<1) if $K_c k$ is large (see exercise)

Example 2: PI-control of integrating process, k'/s
- Need control to stabilize
- Oscillates (ζ<1) if $K_c k'$ is small (see derivation SIMC PID-rules)




```
s=tf('s')
zeta=0.5, tau=1
g = 1/[(tau*s)^2 + 2*tau*zeta*s + 1]
step(g)

g =
          1
     ----------
     s^2 + s + 1

>> pole(g)

ans =

  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
```

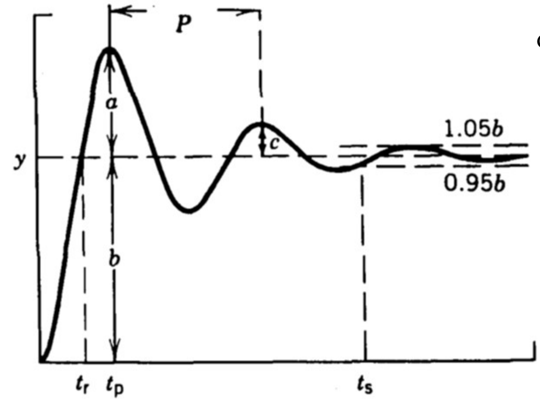

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$



$$y(t) = KM\left\{1 - e^{-\zeta t/\tau}\left[\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)\right]\right\} \qquad (4\text{-}51)$$

Figure 5.8. Step response of underdamped second-order processes.

Chapter 5



$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$|\zeta|<1$

Time to first peak: $t_p = \pi\tau/\sqrt{1-\zeta^2}$ (4-52)

Overshoot: $OS = \exp\left(-\pi\zeta/\sqrt{1-\zeta^2}\right)$ (4-53)
=a/b

Decay ratio: $DR = (OS)^2 = \exp(-2\pi\zeta/\sqrt{1-\zeta^2})$ (4-54)
=c/a

Period: $P = \frac{2\pi\tau}{\sqrt{1-\zeta^2}}$ (4-55)

Small ζ

$t_p=\pi\tau$

$OS=\exp(-\pi\zeta)$

$P=2\pi\tau$

Zeros

Zeros

- g(s) = n(s)/d(s)
- Zeros: roots of numerator polynomial, n(s)=0
- Example, g_1(s)= (3s+1)/(10s+1)(s+1). Zero: s=-1/3
- Problem for control if n(s) has coefficient with different signs (positive zeros in the right half plane (RHP). Give inverse response.
  - Example, g_2(s)= (-3s+1)/(10s+1)(s+1). Zero: s=1/3

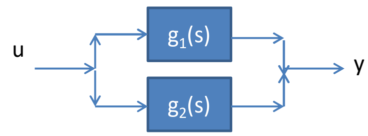
↑

Oops… Negative sign in n(s)… Inverse response!

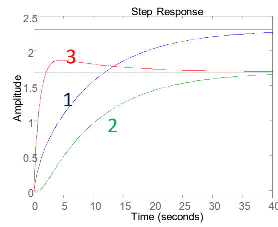
Zeros

Zeros

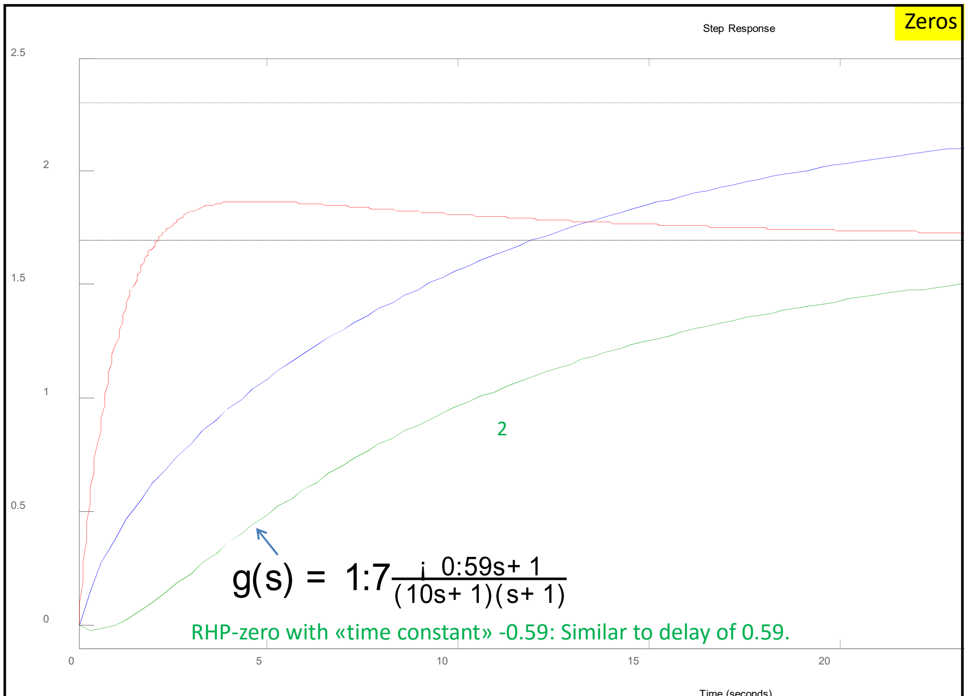
- Zeros are common in practise
- Occur when there are several «paths» to the output.



- Example 1. $g_1(s) = \frac{2}{10s+1}$; $g_2(s) = \frac{0.3}{s+1}$
  $g(s) = g_1 + g_2 = \frac{2(s+1) + 0.3(10s+1)}{(10s+1)(s+1)} = 2.3\frac{2.17s+1}{(10s+1)(s+1)}$
  All coefficients positive: LHP zero
- Example 2 $g_1(s) = \frac{2}{10s+1}$; $g_2(s) = -\frac{0.3}{s+1}$
  $g(s) = g_1 + g_2 = \frac{2(s+1) - 0.3(10s+1)}{(10s+1)(s+1)} = 1.7\frac{-0.59s+1}{(10s+1)(s+1)}$
  Sign change: RHP zero ⇒ Inverse response
- Example 3 $g_1(s) = -\frac{0.3}{10s+1}$; $g_2(s) = \frac{2}{s+1}$
  $g(s) = g_1 + g_2 = \frac{2(s+1) - 0.3(10s+1)}{(10s+1)(s+1)} = 1.7\frac{11.3s+1}{(10s+1)(s+1)}$
  Note; Overshoot since 11.3>10



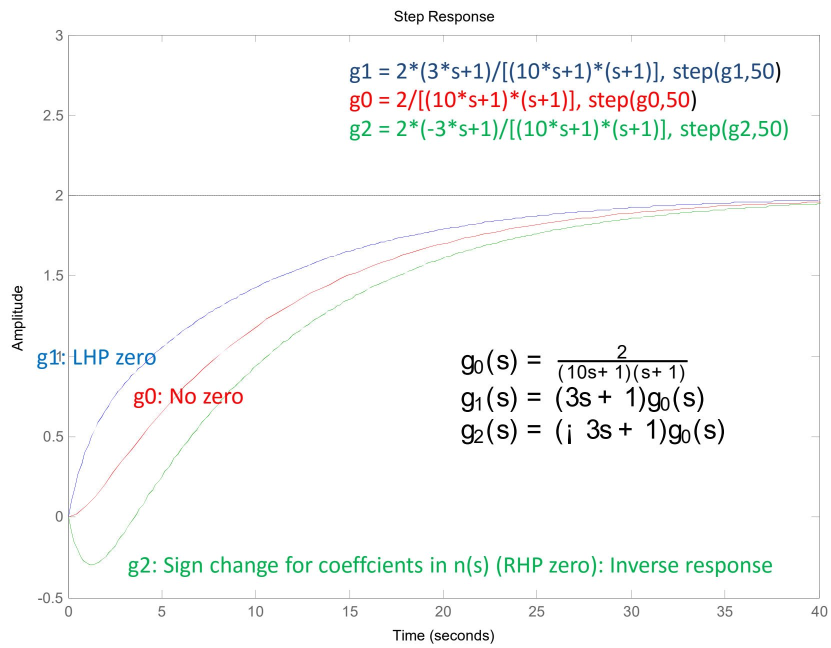
Zeros



$$g(s) = 1.7\frac{-0.59s+1}{(10s+1)(s+1)}$$

RHP-zero with «time constant» -0.59: Similar to delay of 0.59.

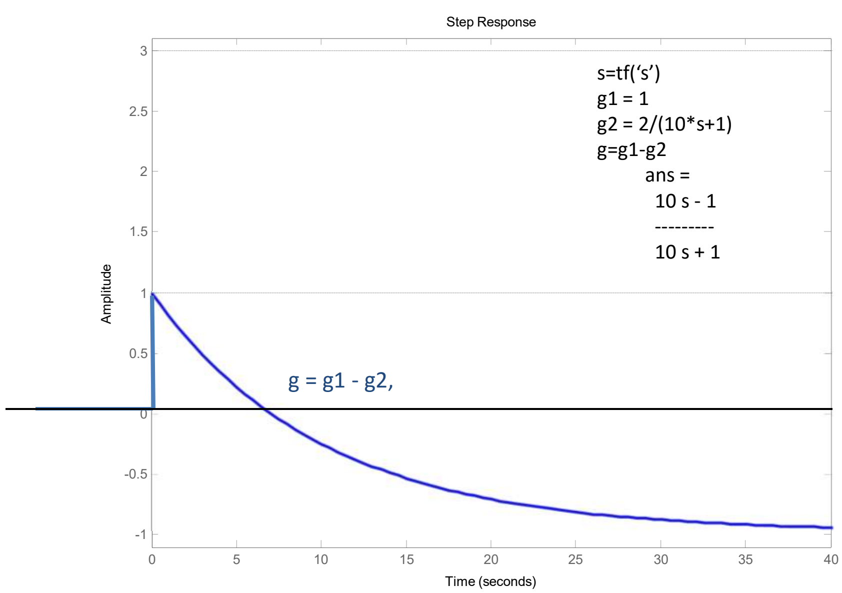
Zeros



Zeros

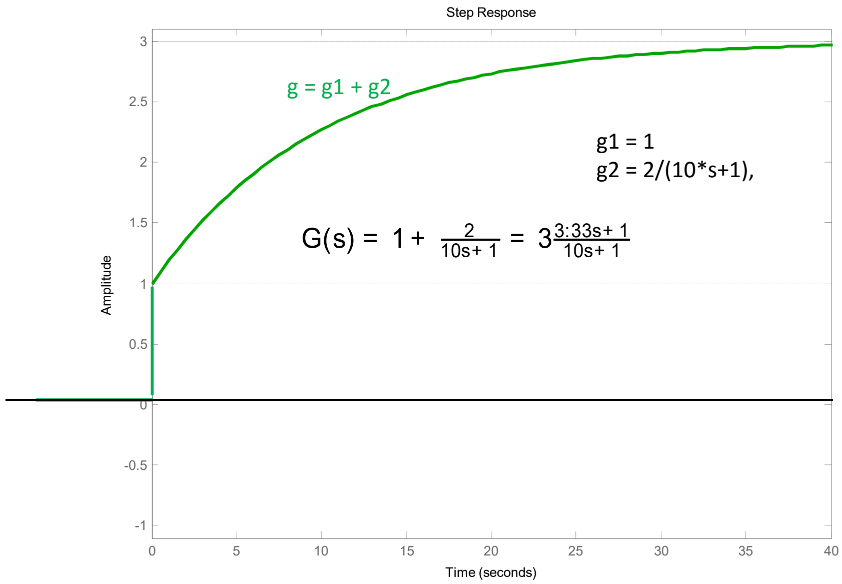
Simple Inverse response example (**RHP zero: «competing effects where slow wins»).**

Physical example electric heater: Increase hot water flow when Q is constant. $u = q_h$, $y = T$ (see below)



Zeros

Example LHP zero: Note no overshoot here (since T=3.33 < τ=10)



Zeros

$$G(s) = \frac{K(\tau_a s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} \qquad (5\text{-}14)$$

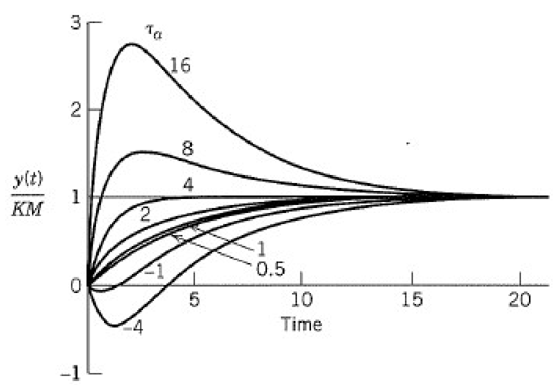


Figure 5.3 Step response of an overdamped second-order system (Eq. 5-14) for different values of τ_a ($\tau_1 = 4$, $\tau_2 = 1$).

Zeros

$$G(s) = \frac{k_1}{\tau_1 s + 1} + \frac{k_2}{\tau_2 s + 1} = k\frac{\tau_a s + 1}{(\tau_1 s + 1)(\tau_2 s + 1)} = k\frac{s - z}{\tau_a(\tau_1 s + 1)(\tau_2 s + 1)}$$

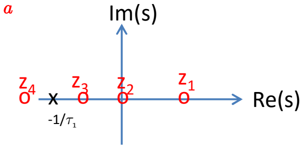
k_1 : "slow" effect ($\tau_1 > \tau_2$)

$k = k_1 + k_2$

$\tau_a = \frac{k_1\tau_2 + k_2\tau_1}{k_1 + k_2}$

$z = -1/\tau_a$

Zero at $z=-1/\tau_a$



z_1: RHP-zero (τ_a <0)
- **Inverse response = Competing effects (k's oppsite signs) where slow wins ($|k_1|>|k_2|$)**

z_2: Zero at origin (z_2=0, $\tau_a \to \infty$)
- **Competing effect with same magnitude (k_1=-k_2)**
- **Steady-state gain is zero**

z_3: LHP-zero close to RHP ($\tau_a > \tau_1$, approx)
- **Overshoot = Competing effects where fast wins**

z_4: LHP-zero far from RHP ($\tau_a < \tau_1$, approx)
- **No overshoot = Effects in same direction (k_1 and k_2 same sign)**

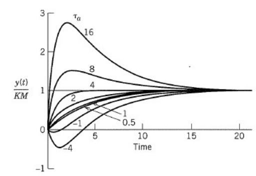
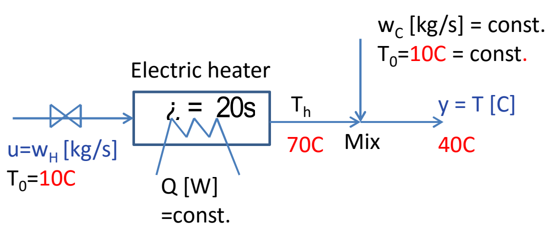


Figure 5.3 Step response of an overdamped second-order system (Eq. 5-14) for different values of τ_a (τ_1 = 4, τ_2 = 1).

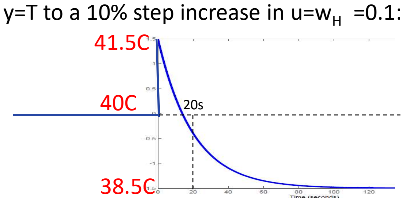
Transfer function of PID-controller

Examples of dynamic model structures

RHP-zero (inverse response)



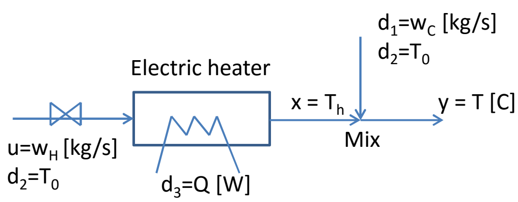
Response in y=T to a 10% step increase in u=w_H =0.1:



Two effects: 1) Direct effect of mixing: $g_1(s)=15$
2) Indirect effect of changed T_h: $g_2(s) = -30/(20s+1)$

} $g(s) = g_1 + g_2 =$ ¡ $15 \frac{¡\,20s+1}{20s+1}$

Model derivation



1. Model. Assume:
   Mass m [kg] in heater constant
   c_P constant

Energy balance heater + mixer:

$\frac{d(m c_P T_h)}{dt} = w_h c_P (T_0 ¡ T_h) + Q$

$T = \frac{w_h T_h + w_c T_c}{w_c + w_h}$

2. Linearize:

$y = ¢T; x = ¢T_h; u = ¢w_h$

$¿\frac{dx}{dt} = ¡x + ku$

$y = Cx + Du$

$k = \frac{T_0^¤ ¡ T_h^¤}{w_h^¤}$

$¿ = m=w_h^¤$

$C = \frac{w_h^¤}{w_c^¤ + w_h^¤}$

$D = \frac{T_h^¤ ¡ T^¤}{w_c + w_h^¤}$

3. Nominal steady-state data:

T_0 = 10C; T_h = 70C; T = 40C

$w_h = w_c$ = 1kg=s; m = 20kg

Gives:

$k = \frac{T_0^¤ ¡ T_h^¤}{w_h^¤} = \frac{10¡70}{1} = ¡60$

$¿ = m=w_h^¤ = 20=1 = 20$

$C = \frac{w_h^¤}{w_c^¤ + w_h^¤} = 0{:}5$

$D = \frac{T_h^¤ ¡ T^¤}{w_c + w_h^¤} = \frac{70¡40}{2} = 15$

4. Transfer function:

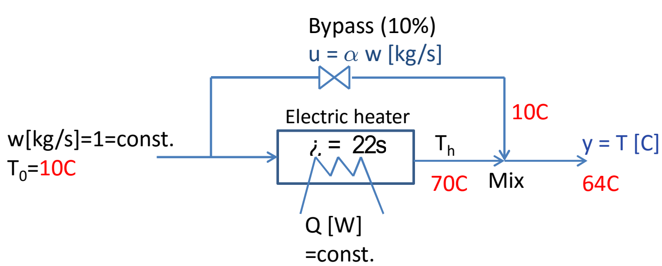
$y(s) = G(s)u(s)$

$G(s) = C\frac{k}{¿s+1} + D$

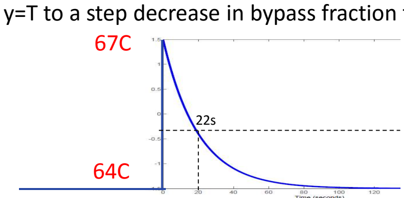
$= 0{:}5\frac{¡60}{20s+1} + 15$

$= ¡15\frac{¡20s+1}{20s+1}$

Zero at 0 (no steady-state effect)



Response in y=T to a step decrease in bypass fraction from 0.1 to 0.05:

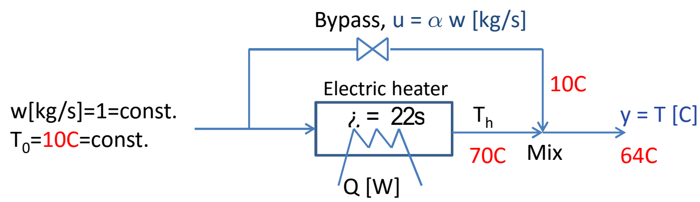


Two effects: 1) Direct effect of mixing: $g_1(s)$=-60

2) Indirect effect of changed T_h: $g_2(s)$ = 60/(22s+1)

$\}$ $g(s) = g_1 + g_2 = ¡60\frac{22s}{22s+1}$

Model derivation



1. Model. Assume:
   Mass m [kg] in heater constant
   c_P constant

Energy balance heater + mixer:

$\frac{d(m c_P T_h)}{dt} = (1-\alpha) w c_P (T_0 - T_h) + Q$

$T = (1-\alpha) T_h + \alpha T_c$

2. Linearize:

$y = \Delta T; x = \Delta T; u = \alpha$

$\tau \frac{dx}{dt} = -x + ku$

$y = Cx + Du$

$k = -\frac{T_o^* - T_h^*}{(1-\alpha^*)}$

$\tau = m/w_h^*$

$C = (1-\alpha^*)$

$D = (T_o^* - T_h^*)$

3. Nominal steady-state data:

$T_0 = 10C; T_h = 70C; T = 64C$

$w = 1kg/s; \alpha = 0.1; m = 20kg$

Gives:

$k = -\frac{T_o^* - T_h^*}{(1-\alpha^*)} = -\frac{10-70}{0.9} = 66.67$

$\tau = m/w_h^* = 20/0.9 = 22$

$C = (1-\alpha^*) = 0.9$

$D = (T_o^* - T_h^*) = -60$

4. Transfer function:

$y(s) = G(s)u(s)$

$G(s) = C\frac{k}{\tau s + 1} + D$

$= 0.9\frac{66.67}{22s+1} - 60$

$= 60(\frac{1}{22s+1} - 1) = -60\frac{22s}{22s+1}$

Summary poles and zeros

- $G(s) = n(s) / d(s) = k'(s-z_1) / (s-p_1)(s-p_2)$..
- Example: $G(s) = 4 (3s-1)/(s^2+s-2)$,

  Get: $k'=12$, $z_1=1/3$, $p_1=-2$ $p_2=1$

- Poles p (=eigenvalues of A)
  - Determine speed of response, exp(p*t)
  - Negative sign in d(s) $\Rightarrow$ p_2 in RHP: unstable, $\exp(p_2 * t) \rightarrow \infty$ (NEED control)
  - P complex: oscillating response
- Zeros z
  - Determine shape of response
  - Negative sign in n(s) $\Rightarrow$ z_1 in RHP: inverse response (BAD for control)

Skogestad Half Rule*

OBTAINING THE EFFECTIVE DELAY θ

Basis (Taylor approximation):

$$e^{-\theta s} \approx 1 - \theta s \quad \text{and} \quad e^{-\theta s} = \frac{1}{e^{\theta s}} \approx \frac{1}{1+\theta s}$$

Effective delay =

"true" delay

+ inverse reponse time constant(s)

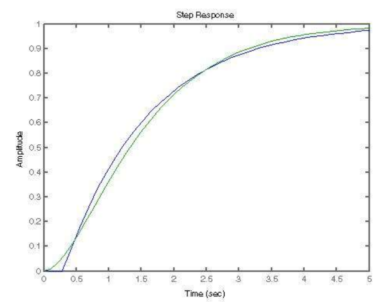
+ half of the largest neglected time constant (the "half rule") (this is to avoid being too conservative)

+ all smaller high-order time constants

The "other half" of the largest neglected time constant is added to τ_1 (or to τ_2 if use second-order model).

* S. Skogestad, "Simple analytic rules for model reduction and PID controller design", *J.Proc.Control*, Vol. 13, 291-309, 2003 (Also reprinted in MIC)

Example 1



The second-order process

$$g_0(s) = \frac{1}{(1s+1)(0.6s+1)}$$

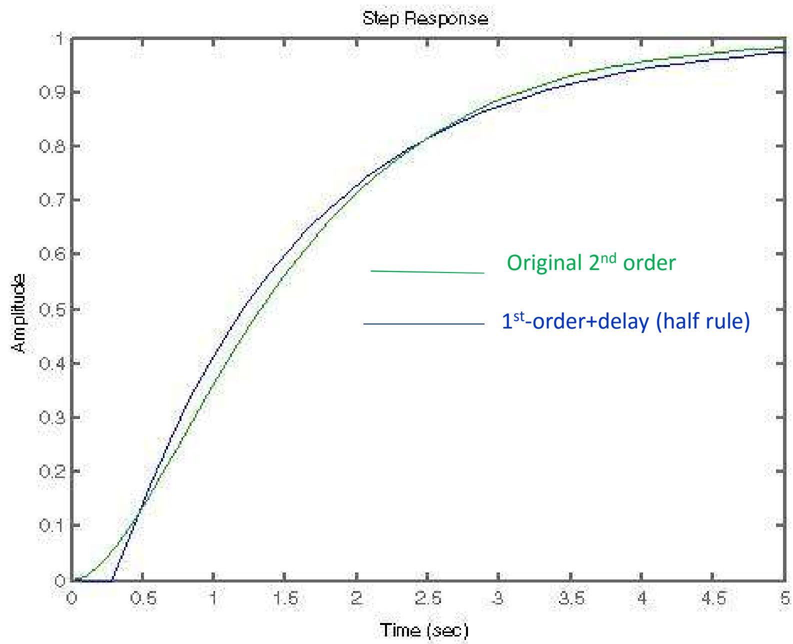
Half rule

is approximated as a first-order with delay process

$$g(s) = k\frac{e^{-\theta s}}{\tau_1 s + 1}$$

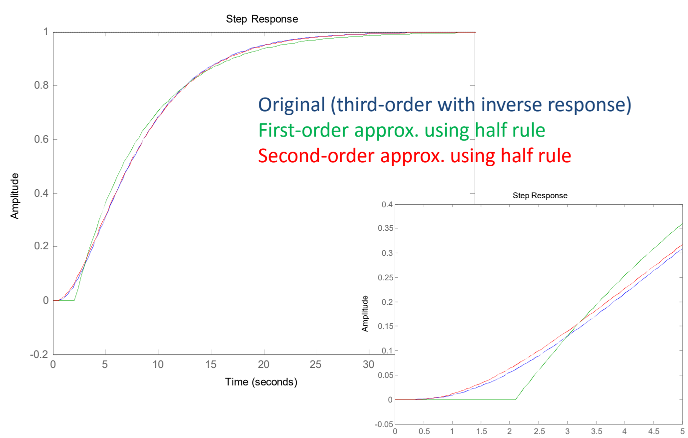
with

$$k = 1; \quad \tau_1 = 1 + 0.6/2 = 1.3; \quad \theta = 0.6/2 = 0.3;$$



Example 2

```
s=tf('s')
g=(-0.1*s+1)/[(5*s+1)*(3*s+1)*(0.5*s+1)]
g1 = exp(-2.1*s)/(6.5*s+1)
g2 = exp(-0.35*s)/[(5*s+1)*(3.25*s+1)]
step(g,g1,g2)
```



Example 3. Integrating process

$g_0(s) = \frac{k'}{s(\tau_{20}s+1)}$

Half rule gives

$g(s) = \frac{k' e^{-\theta s}}{s}$ with $\theta = \frac{\tau_{20}}{2}$

Proof:

Note that integrating process corresponds to an infinite time constant

Write

$g_0(s) = \frac{k'\tau_1}{\tau_1 s(\tau_{20}s+1)} = \frac{k'\tau_1}{(\tau_1 s+1)(\tau_{20}s+1)}$ where $\tau_1 \to \infty$

and then apply half rule as normal, noting that $\tau_1 + \frac{\tau_{20}}{2} \approx \tau_1$:

$g(s) \approx \frac{k'\tau_1 e^{-\frac{\tau_{20}}{2}s}}{(\tau_1+\frac{\tau_{20}}{2})s} = k'\frac{e^{-\frac{\tau_{20}}{2}s}}{s}$

Approximation of LHP-zeros

To make these rules more general (and not only applicable to the choice $\tau_c=\theta$): Replace θ (time delay) by τ_c (desired closed-loop response time). (6 places)

$$\frac{T_0 s+1}{\tau_0 s+1} \approx \begin{cases} T_0/\tau_0 & \text{for } T_0 \geq \tau_0 \geq \cancel{\theta}\,\tau_c & \text{(Rule T1)} \\ T_0/\cancel{\theta}\,\tau_c & \text{for } T_0 \geq \cancel{\theta}\,\tau_c \geq \tau_0 & \text{(Rule T1a)} \\ 1 & \text{for } \cancel{\theta}\,\tau_c \geq T_0 \geq \tau_0 & \text{(Rule T1b)} \\ T_0/\tau_0 & \text{for } \tau_0 \geq T_0 \geq 5\cancel{\theta}\,\tau_c & \text{(Rule T2)} \\ \frac{(\tilde{\tau}_0/\tau_0)}{(\tilde{\tau}_0 - T_0)s+1} & \text{for } \tilde{\tau}_0 \stackrel{\text{def}}{=} \min(\tau_0, 5\cancel{\theta}\,\tau_c) \geq T_0 & \text{(Rule T3)} \end{cases}$$

Example E3. *For the process (Example 4 in (Astrom* et al. *1998))*

$$g_0(s) = \frac{2(15s+1)}{(20s+1)(s+1)(0.1s+1)^2} \tag{13}$$

we first introduce from Rule T2 the approximation

$$\frac{15s+1}{20s+1} \approx \frac{15s}{20s} = 0.75$$

(Rule T2 applies since $T_0 = 15$ is larger than 5θ, where θ is computed below). Using the half rule, the process may then be approximated as a first-order time delay model with

$$k = 2 \cdot 0.75 = 1.5; \quad \theta = 0.1 + \frac{0.1}{2} = 0.15; \quad \tau_1 = 1 + \frac{0.1}{2} = 1.05$$

or as a second-order time delay model with

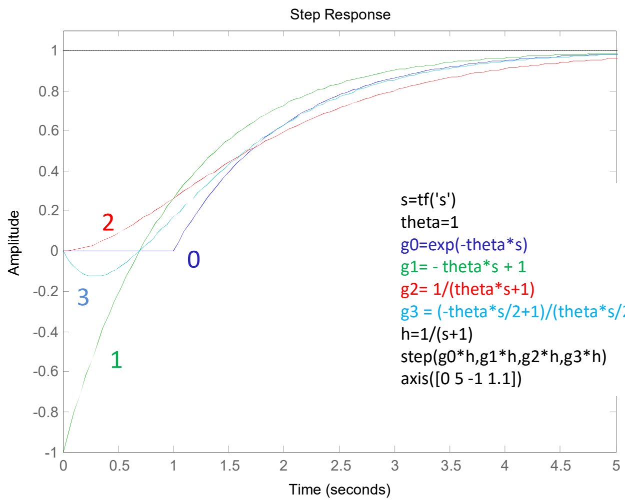
$$k = 1.5; \quad \theta = \frac{0.1}{2} = 0.05; \quad \tau_1 = 1; \quad \tau_2 = 0.1 + \frac{0.1}{2} = 0.15$$

τ_c = desired closed-loop time constant

"Going the other way"

Approximations of time delay

Example: Step response of first-order system plus delay




```
s=tf('s')
theta=1
g0=exp(-theta*s)                      % time delay. theta
g1= - theta*s + 1                     % RHP-zero. –Ts+1, T=theta
g2= 1/(theta*s+1)                     % first order, tau=theta
g3 = (-theta*s/2+1)/(theta*s/2+1)     % Pade-approx
h=1/(s+1)
step(g0*h,g1*h,g2*h,g3*h)
axis([0 5 -1 1.1])
```

Example of oscillating system: PI-control of integrating process (level) with small K_c (and/or small τ_I)

$g(s) = \frac{1}{s}$

$c(s) = K_c(1 + \frac{1}{\tau_I s})$

Closed-loop response

Closed-loop response to disturbance d at input and setpoint change

$$y = \frac{g}{1+gc} d + \frac{gc}{1+gc} y_s$$

PI-control of integrator:

$$g(s) = \frac{1}{s}; \quad c(s) = K_c \frac{\tau_I s + 1}{\tau_I s}$$

Get

$$y = \frac{\tau_I s}{\tau_I s^2 + K_c \tau_I s + K_c} d + \frac{K_c(\tau_I s + 1)}{\tau_I s^2 + K_c \tau_I s + K_c} y_s$$

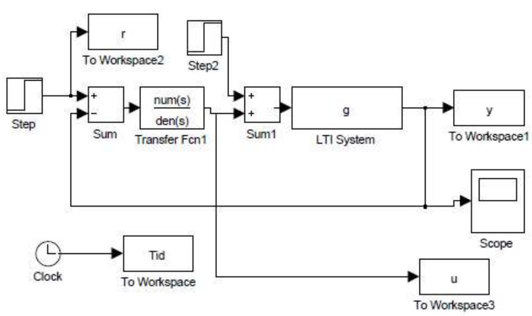
With $\tau_I = 1, K_c = 0.25$:

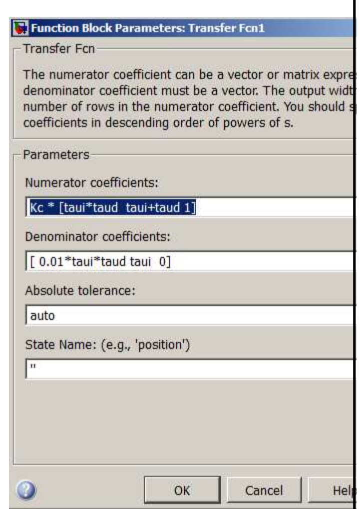
$$y = \frac{s}{s^2 + 0.25s + 0.25} d + \frac{0.25(s+1)}{s^2 + 0.25s + 0.25} y_s = \underbrace{\frac{4s}{4s^2 + s + 1}}_{h(s)} d + \underbrace{\frac{(s+1)}{4s^2 + s + 1}}_{T(s)} y_s$$

Notes:

- Steady-state gain $h(0)$ for disturbance transfer function $h(s)$ is zero (because controller has integral action)
- Steady-state gain $T(0)$ for setpoint transfer function $T(s)$ is 1 (because controller has integral action)
- Denominator is on form $\tau^2 s^2 + 2\tau\zeta s + 1$ with $\tau = 2$ and $\zeta = 0.25 < 1$, so there will be oscillations with period $P \approx 2\pi\tau$
- Initial response ($t \to 0$) to disturbance is the same as with no control ($h(s) = \frac{g}{1+gc} \to g(s)$ when $s \to \infty$ since $g(s)c(s) \to 0$ (which is the case for all real systems))

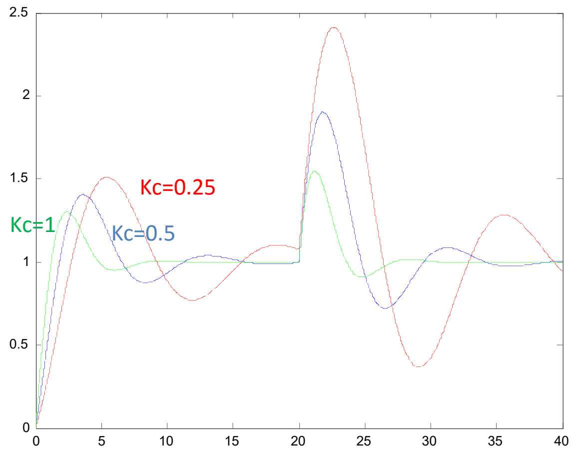
Simulink, tunepid4





PI-control of integrator (level control). G = 1/s, taui=1. **VARY Kc**

NO DELAY, theta = 0




```
%tunepid4
s=tf('s')
theta=0
g=(1/s)*exp(-theta*s) % integrating
taud=0
taui=1
Kc=0.5 % oscillations (Kc*k'*taui = 0.5 < 4)
sim tunepid4; plot(Tid,y); hold on %

Kc=0.25 % more oscillations (Kc*k'*taui = 0.25 < 4)
sim tunepid4; plot(Tid,y,'red');

Kc=1   % less oscillations (Kc*k'*taui = 1)
sim tunepid4; plot(Tid,y,'green');
hold off
```

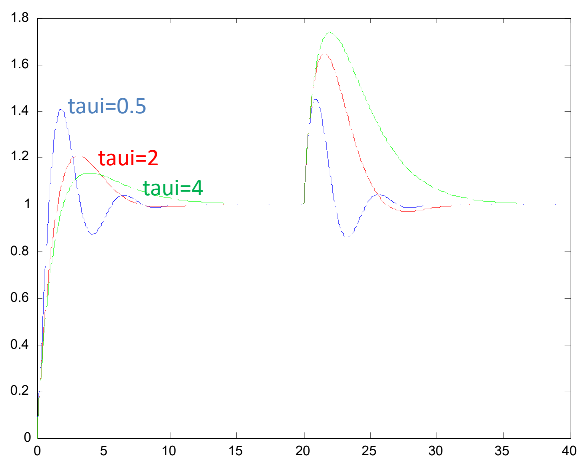
Conclusion: Get "slow" oscillations if controller gain is too small

Input disturbance at t=20

Setpoint change at t=0

PI-control of integrator (level control). G = 1/s, Kc=1. **VARY taui**

NO DELAY, theta = 0



Conclusion.
Kc=1, taui=4 looks OK
(no oscillations)

Conclusion: Get "slow" oscillations if integral time is too small

Input disturbance at t=20

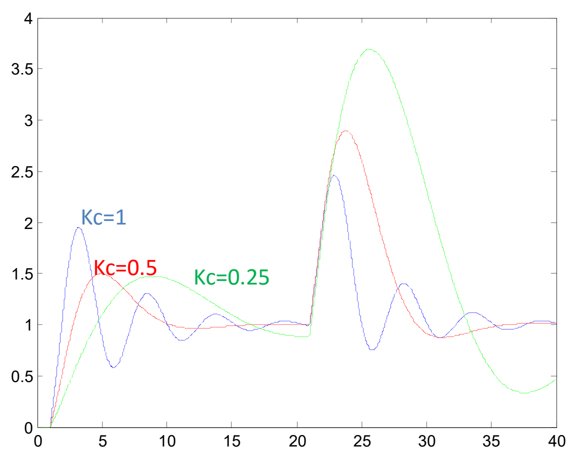
Setpoint change at t=0

Can also get oscillation if we have time delay and use large K_c

$$g(s) = \frac{1}{s} e^{-\theta s}$$
$$c(s) = K_c\left(1 + \frac{1}{\tau_I s}\right)$$

PI-control of integrator (level control). G = 1/s, taui=4

ADD DELAY, theta = 1




```
%tunepid4
s=tf('s')
theta=1
g=(1/s)*exp(-theta*s) % integrating with delay (lev
taud=0
taui=4
Kc=1 % Too high Kc.
% -> "fast" oscillations because of delay!!
sim tunepid4; plot(Tid,y); hold on %

Kc=0.5 % OK
sim tunepid4; plot(Tid,y,'red');

Kc=0.25 % Too low Kc.
% -> "slow" oscillations from integrator
sim tunepid4; plot(Tid,y,'green');
hold off
```

Comment. SIMC-rule would give, Kc=0.5, taui=8

Setpoint change at t=0

Input disturbance at t=20

CONCLUSION
Kc too small (Kc=0.25): "Slow" oscillations (integrator not stabilize
Kc too large (Kc=1): "Fast" oscillations (because of time delay)