Reference group process control:
Need 3 students (volunteers)

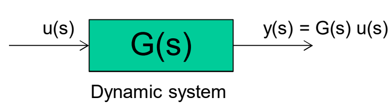
Appendix A

Laplace Transforms

1. Standard notation in dynamics and control for linear systems (shorthand notation)
   - Independent variable: Change from t (time) to s (complex variable; inverse time)
   - Just a mathematical change in variables: Like going from x to y=log(x)

2. Converts differential equations to algebraic operations

3. Advantageous for block diagram analysis.
   Transfer function, G(s):



General procedure in this course

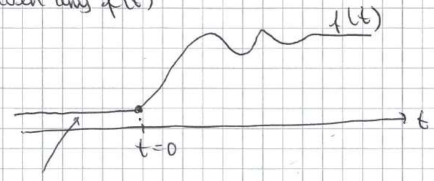
1. Nonlinear model
2. Introduce deviation variables and linearize*
3. Laplace of linear model (t → s)
4. Algebra → Transfer function, G(s)
5. Block diagram
6. Controller design

*Note: We will only use Laplace for linear systems!

DEFINITION LAPLACE TRANSFORM

Laplace transforms

Given any $f(t)$



Usually at steady state for $t<0$

Laplace transform of $f(t)$

$$F(s) = \mathcal{L}(f(t)) \overset{\text{def.}}{=} \int_0^\infty f(t)e^{-st}\,dt$$

I usually misuse notation and write $f(s)$

"$f(t)$ weighted with e^{-st} and then integrated from $t=0$ to ∞ so that s becomes new independent variable instead of time t"

s [time^{-1}]: Complex number (new indep. variable instead of t)

$$f(t) \underset{\text{Inverse, } \mathcal{L}^{-1}}{\overset{\text{Laplace}}{\rightleftarrows}} f(s)$$

Inverse, $\mathcal{L}^{-1}$:

1. Complex contour integration
2. Split into sum of simple terms where f(t) is known (partial fraction expansion)

Most important property of Laplace for us:

$$\mathcal{L}\left(\frac{df(t)}{dt}\right) = s f(s) - \underbrace{f(t=0)}_{\text{usually}=0}$$

(e.g., deviation variables!)

DIFFERENTIATION IN TIME REPLACED BY MULTIPLICATION BY s → ALGEBRA!

Appendix A

Laplace Transform

Definition*: $F(s) = L(f(t)) = \int_0^\infty f(t)e^{-st}dt$

Usually f(t) is in deviation variables so f(t=0) = 0

Examples

1. f(t)=a (step change)

a/s: Laplace of step a

$$f(s) = \int_0^\infty ae^{-st}dt = -\frac{a}{s}\left[e^{-st}\right]_0^\infty = 0 - \left[-\frac{a}{s}\right] = \frac{a}{s}$$

2. $f(t)=e^{-bt}$

$$f(s) = \int_0^\infty e^{-bt}e^{-st}dt = \int_0^\infty e^{-(b+s)t}dt = \frac{1}{b+s}\left[-e^{-(b+s)t}\right]_0^\infty = \frac{1}{s+b}$$

*Will often misuse notation and write f(s) instead of F(s)

Very important property for us:

$$L\left(\frac{df}{dt}\right) = \int_0^\infty \frac{df}{dt} e^{-st} dt = sL(f) - f(0) = s\,F(s) - f(0)$$

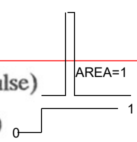
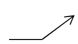
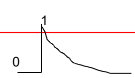
Differentiation: replaced by multiplication with s
Integration: replaced by multiplication with 1/s

Proof: $\int_0^\infty \frac{df}{dt} e^{-st} dt = \int_0^\infty e^{-st} df$

Set v=e^{-st} and du=df and use integration by parts

Appendix A

Table A.1 Laplace Transforms for Various Time-Domain Functions[a]

f(*t*)			*F*(*s*)
1. $\delta(t)$	(unit impulse)		1
2. $S(t)$	(unit step)		$\frac{1}{s}$
3. t	(ramp)		$\frac{1}{s^2}$
4. t^{n-1}			$\frac{(n-1)!}{s^n}$
5. e^{-bt}			$\frac{1}{s+b}$
6. $\frac{1}{\tau} e^{-t/\tau}$			$\frac{1}{\tau s + 1}$
7. $\frac{t^{n-1} e^{-bt}}{(n-1)!}$	$(n > 0)$		$\frac{1}{(s+b)^n}$
8. $\frac{1}{\tau^n (n-1)!} t^{n-1} e^{-t/\tau}$			$\frac{1}{(\tau s + 1)^n}$
9. $\frac{1}{b_1 - b_2}(e^{-b_2 t} - e^{-b_1 t})$			$\frac{1}{(s+b_1)(s+b_2)}$

Appendix A

Table A.1 Laplace Transforms for Various Time-Domain Functions[a]

	$f(t)$	$F(s)$
10.	$\frac{1}{\tau_1-\tau_2}(e^{-t/\tau_1}-e^{-t/\tau_2})$	$\frac{1}{(\tau_1 s+1)(\tau_2 s+1)}$
11.	$\frac{b_3-b_1}{b_2-b_1}e^{-b_1 t}+\frac{b_3-b_2}{b_1-b_2}e^{-b_2 t}$	$\frac{s+b_3}{(s+b_1)(s+b_2)}$
12.	$\frac{1}{\tau_1}\frac{\tau_1-\tau_3}{\tau_1-\tau_2}e^{-t/\tau_1}+\frac{1}{\tau_2}\frac{\tau_2-\tau_3}{\tau_2-\tau_1}e^{-t/\tau_2}$	$\frac{\tau_3 s+1}{(\tau_1 s+1)(\tau_2 s+1)}$
13.	$1-e^{-t/\tau}$	$\frac{1}{s(\tau s+1)}$
14.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
15.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
16.	$\sin(\omega t+\phi)$	$\frac{\omega\cos\phi+s\sin\phi}{s^2+\omega^2}$
17.	$e^{-bt}\sin \omega t$ (b, ω real)	$\frac{\omega}{(s+b)^2+\omega^2}$
18.	$e^{-bt}\cos \omega t$ (b, ω real)	$\frac{s+b}{(s+b)^2+\omega^2}$
19.	$\frac{1}{\tau\sqrt{1-\zeta^2}}e^{-\zeta t/\tau}\sin(\sqrt{1-\zeta^2}\,t/\tau)$ $(0 \le \lvert\zeta\rvert < 1)$	$\frac{1}{\tau^2 s^2+2\zeta\tau s+1}$

Table A.1 Laplace Transforms for Various Time-Domain Functions[a] (continued)

	$f(t)$	$F(s)$
20.	$1+\frac{1}{\tau_2-\tau_1}(\tau_1 e^{-t/\tau_1}-\tau_2 e^{-t/\tau_2})$ $(\tau_1 \neq \tau_2)$	$\frac{1}{s(\tau_1 s+1)(\tau_2 s+1)}$
21.	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta t/\tau}\sin[\sqrt{1-\zeta^2}\,t/\tau+\psi]$, $\psi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$, $(0 \le \lvert\zeta\rvert < 1)$	$\frac{1}{s(\tau^2 s^2+2\zeta\tau s+1)}$
22.	$1-e^{-\zeta t/\tau}[\cos(\sqrt{1-\zeta^2}\,t/\tau)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\sqrt{1-\zeta^2}\,t/\tau)]$ $(0 \le \lvert\zeta\rvert < 1)$	$\frac{1}{s(\tau^2 s^2+2\zeta\tau s+1)}$
23.	$1+\frac{\tau_3-\tau_1}{\tau_1-\tau_2}e^{-t/\tau_1}+\frac{\tau_3-\tau_2}{\tau_2-\tau_1}e^{-t/\tau_2}$ $(\tau_1 \neq \tau_2)$	$\frac{\tau_3 s+1}{s(\tau_1 s+1)(\tau_2 s+1)}$
24.	$\frac{df}{dt}$	$sF(s)-f(0)$
25.	$\frac{d^n f}{dt^n}$	$s^n F(s)-s^{n-1}f(0)-s^{n-2}f^{(1)}(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$
26.	$f(t-t_0)S(t-t_0)$	$e^{-t_0 s}F(s)$

Alternative forms of step response for 2nd order system (items 20–22)

[a]Note that $f(t)$ and $F(s)$ are defined for $t \ge 0$ only.

Appendix A

Example:

$$\frac{d^3y}{dt^3}+6\frac{d^2y}{dt^2}+11\frac{dy}{dt}+6y=4$$
$$y(0)=y'(0)=y''(0)=0$$

y is deviation variable, system initially at rest (s.s.)

To find transient response for $y(t)$

1. Take Laplace Transform (L.T.)
2. Factor, use partial fraction decomposition
3. Take inverse L.T. Use table

Step 1 Take L.T. (note zero initial conditions)

$$s^3Y(s)+6s^2Y(s)+11sY(s)+6Y(s)=\frac{4}{s}$$

Appendix A

Rearranging,

$$Y(s)=\frac{4}{(s^3+6s^2+11s+6)s}$$

Step 2a. Factor denominator of Y(s)

$$s(s^3+6s^2+11s+6)=s(s+1)(s+2)(s+3)$$

Step 2b. Use partial fraction decomposition

$$\frac{4}{s(s+1)(s+2)(s+3)}=\frac{\alpha_1}{s}+\frac{\alpha_2}{s+1}+\frac{\alpha_3}{s+2}+\frac{\alpha_4}{s+3}$$

Multiply by s, set s = 0

$$\left.\frac{4}{(s+1)(s+2)(s+3)}\right|_{s=0}=\alpha_1+s\left.\left[\frac{\alpha_2}{s+1}+\frac{\alpha_3}{s+2}+\frac{\alpha_4}{s+3}\right]\right|_{s=0}$$

$$\frac{4}{1\cdot 2\cdot 3}=\alpha_1=\frac{2}{3}$$

Appendix A

For α_2, multiply by (s+1), set s=-1 (same procedure for α_3, α_4)

$$\alpha_2 = -2,\ \alpha_3 = 2,\ \alpha_4 = -\frac{2}{3}$$

Step 3. Take inverse of L.T. $\left(Y(s) = \frac{2}{3s} - \frac{2}{s+1} + \frac{2}{s+2} - \frac{2/3}{s+3}\right)$

Use table:

$$y(t) = \frac{2}{3} - 2e^{-t} + 2e^{-2t} - \frac{2}{3}e^{-3t}$$

$t \to \infty$: $y(t) \to \frac{2}{3}$ $\qquad$ $t = 0$: $y(0) = 0$. (check original ODE)

You can use this method on any order of ODE, limited only by factoring of denominator polynomial (characteristic equation)

Must use modified procedure for repeated roots, imaginary roots

Appendix A

Other properties of Laplace transform:

A. Final value theorem

$$y(t = \infty) = \lim_{s \to 0} sY(s)$$

"steady-state value"

Example: $Y(s) = \frac{1}{\tau s + 1}\frac{a}{s}$

$$y(\infty) = \lim_{s \to 0} \frac{a}{\tau s + 1} = a$$

B. Time-shift theorem

y(t)=0 $\quad t \le \theta$

$$L(y(t-\theta)) = e^{-\theta s}\, Y(s)$$

$e^{-\theta s}$: transfer function for time delay θ

Appendix A

C. Initial value theorem

$$\lim_{t \to 0} y(t) = \lim_{s \to \infty} sY(s)$$

Example

$$For \quad Y(s) = \frac{4s+2}{s(s+1)}$$

$y(0) = 4$ by initial value theorem (multiply Y(s) by s and set s=∞)

$y(\infty) = 2$ by final value theorem (multiply Y(s) by s and set s=0)

D. Initial slope property

$$\lim_{t \to 0} y'(t) = \lim_{s \to \infty} s^2 Y(s)$$