Input-output Controllability Analysis

Idea: Find out how well the process can be controlled - without having to design a specific controller

Note: Some processes are impossible to control

Reference: S. Skogestad, ``A procedure for SISO controllability analysis - with application to design of pH neutralization processes'', *Comp.Chem.Engng.*, **20**, 373-386, 1996.

Example: First-order with delay process

$$g(s) = k\frac{e^{-\theta s}}{1+\tau s}; \quad G_d(s) = k_d\frac{e^{-\theta_d s}}{1+\tau_d s}$$

$+$ Measurement delays: $\theta_m, \quad \theta_{md}$.

Problem: What values are desired for good controllability?

Qualitative results:

	Feedback control	Feedforward control
k	Large	Large
τ	Small	Small
θ	Small	Small
k_d	Small	Small
τ_d	Large	Large
θ_d	No effect	Large
θ_m	Small	No effect
θ_{md}	No effect	Small

WANT TO QUANTIFY!

Rules

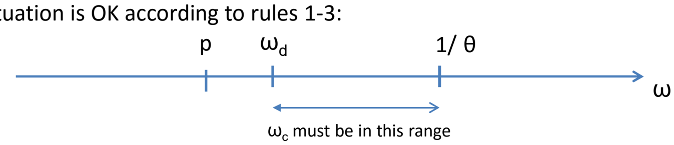
- Rules 1-3: speed of response
  - Rule 1: Fast response required to reject large disturbance
  - BUT (rule 2): Response time is limited by effective time delay
  - Rule 3: Fast response needed for stabilization
- Rule 4: Input constraints
  - Large disturbances may give input saturation

Rules for speed of response (assuming control with integral action)

- Define $\omega_c = 1/\tau_c$ = closed-loop bandwidth = where $|L|$ is approx. 1
- Define ω_d as frequency where $|g_d|=1$ (scaled model, frequency where $|y|=1$ for $|d|=1$)

- Rule 1: Fast response required to reject large disturbance
  - Need $\omega_c > \omega_d$ $(\tau_c < 1/\omega_d)$
    - Rule 1 is for typical case where $|g_d|$ is highest at low frequencies
    - The more exact rule is: We need $|Sg_d|<1$, or approximately: $|L|>|g_d|$ at frequencies where $|g_d|>1$.
- Rule 2: Response time is limited by effective time delay
  - Need $\omega_c < 1/\theta$ $(\tau_c > \theta$. SIMC-rule!)
    - Where θ is effective time delay
- Rule 3: Fast response needed for stabilization
  - Need $\omega_c > p$ $(\tau_c < 1/p)$
    - Where p is unstable pole, g(s) =k/(s-p)…
- Rule 4: Input constraints: Large disturbances may give input saturation
  - With scaled model: Need $|G| > |G_d|$ at frequencies where $|G_d|>1$

This situation is OK according to rules 1-3:



Comment: Ideal controller inverts the plant

- y = g(s)u + d
- **Ideal controller inverts the plant g(s):**
  - Think feedforward, $u = c_{ff}(s)\,(y_s - d)$
  - *Perfect* control: want $y = y_s \rightarrow c_{ff} = 1/g(s) = g^{-1}$

Limitations on perfect control: Inverse cannot always be realized:

1. Input saturation , $|u| > |u_{max}|$
2. Time delay, $g = e^{-\theta s}$.
   - $g^{-1} = e^{\theta s}$ = prediction (not possible)
   - Solution: Omit
3. Inverse response, g = -Ts+1.
   - $g^{-1} = 1/(-Ts+1)$ = unstable (not possible as u will be unbounded)
   - Solution: Omit
4. More poles than zeros, $g = 1/(\tau s+1)$,
   - $g^{-1} = \tau s + 1$ = pure differentiation (not possible as u will be unbounded).
   - Solution: Replace by: $(\tau s + 1)/(\tau_c s+1)$ where $\tau_c < \tau$ is a tuning parameter

- Example. $g(s) = 5\,(-0.5s+1)\,e^{-2s} / (3s+1)$.
  - Realizable inverse (feedforward): $0.2\,(3s+1)/(\tau_c s+1)$. E.g. choose $\tau_c = 0.5$

- **So we know what limits us from having perfect control**
  - **Same limitations apply to feedback control**
- **Controllability analysis: Want to find out what these limitations imply in terms of "acceptable control", $|y-y_s| < y_{max}$**

We use scaled model

Rules 1 and 4

Original model:

$$y' = g'u' + g_d' d'$$

Scaled model

$$\underbrace{\frac{y'}{y_{max}}}_{y} = \underbrace{g'\frac{u_{max}}{y_{max}}}_{g}\underbrace{\frac{u'}{u_{max}}}_{u} + \underbrace{g_d'\frac{d_{max}}{y_{max}}}_{g_d}\underbrace{\frac{d'}{d_{max}}}_{d}$$

or

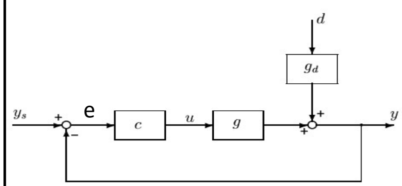
$$y = gu + g_d d$$

where $|y| < 1, |u| < |1, |d| < 1.$

Rule 1

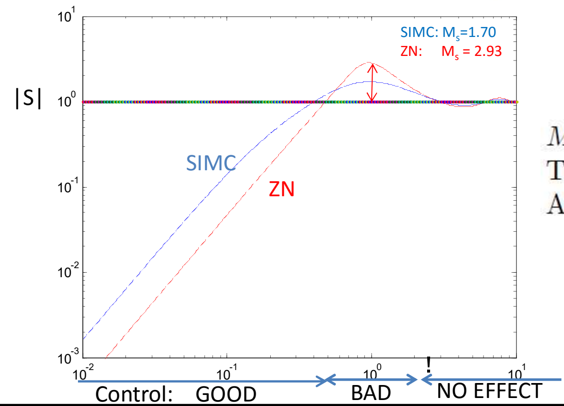
Recall:

Closed-loop frequency response (S)



No control ($c = 0$): $e_{OL} = y_s - y = y_s - g_d d$
With control: $e = y_s - y = Sy_s - Sg_d d = Se_{OL}$
$S = \frac{1}{1+L}$ - sensitivity function = effect of control
$L = gc$ - loop transfer function

Low ω where $|S| < 1$: Control reduces error
Intermediate ω where $|S| > 1$: Control increases error
High ω where $S = 1$ ($L \to 0$): Control has no effect



M_s = peak of $|S|$
Typical requirement: $M_s < 2$
At stability limit: $M_s \to \infty$

```
w = logspace(-2,1,1000);
[mag1,phase]=bode(1/(1+L1),w);
[mag2,phase]=bode(1/(1+L2),w);
figure(1), loglog(w,mag1(:),'red',w,mag2(:),'blue',w,1,'-.')
axis([0.01,10,0.001,10])
```

SCALED MODEL **MAIN REASON FOR CONTROL: DISTURBANCES!** Rule 1

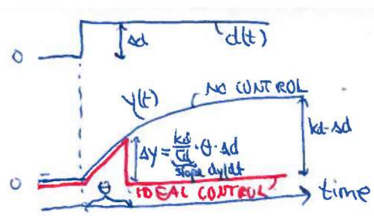
Disturbances and Loop gain L

- S = 1/(1+L) where L = gc
- No control («open-loop»): y = g u + g_d d
- With control: y = S g_d d
- Scaled variables: Want $|Sg_d|<1$ at all ω
- Approximation at low frequencies where |L| is large: S = 1/L
- So want (in scaled variables): $|L| > |g_d|$
  - Up to about frequency ω_c where |L|=1

Rule 1

Time domain

- Consider response to step disturbance, $g_d = k_d/(\tau_d s+1)$
  - Output reaches $\Delta y = (k_d\, \theta /\tau_d)\, \Delta d$ at time θ (approximately)
  - If this is larger than acceptable then we are in trouble
  - If $\Delta d=1$ and requirement is $|\Delta y|< 1$ then we must require $\underbrace{k_d /\tau_d}_{\omega_d} < 1/\theta$ (combined rule 1+2)



- Easier to generalize in frequency domain
  - Consider disturbance $d(t)=\sin\omega t$

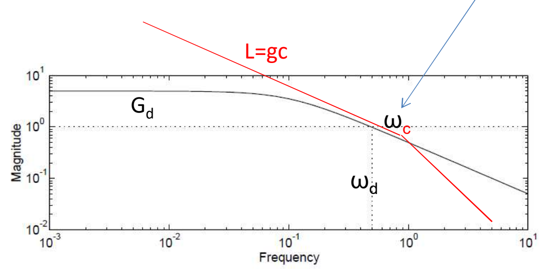
SCALED MODEL

Rule 4

MAIN REASON FOR CONTROL: DISTURBANCES!

1. DISTURBANCES (speed of response)

Need control up to frequency w_d where $|G_d|=1$ -> Need $w_c > w_d$ (w_c is frequency where $|L|=1$)



Note: Have $\omega_c=1/\tau_c$

SCALED MODEL

Rule 4

2. INPUT CONSTRAINTS

Process model

$$y = Gu + G_d d$$

1. Worst-case disturbance: $|d| = 1$. To achieve *perfect control* ($e = 0$) with $|u| < 1$ we must require

Rule 4: $\boxed{|G| > |G_d|}$ at frequencies where $|G_d| > 1$ (3)

2. Worst-case reference: $|r| = R_{max}$. To achieve perfect control ($y = r$) with $|u| < 1$ we must require

$$\boxed{|G| > |R_{max}|} \quad \forall \omega \leq \omega_r \qquad (4)$$

Rules for speed of response (assuming control with integral action)

- Define $\omega_c = 1/\tau_c$ = closed-loop bandwidth = where $|L|$ is approx. 1
- Define ω_d as frequency where $|g_d|=1$ (scaled model, frequency where $|y|=1$ for $|d|=1$)

- Rule 1: Fast response required to reject large disturbance
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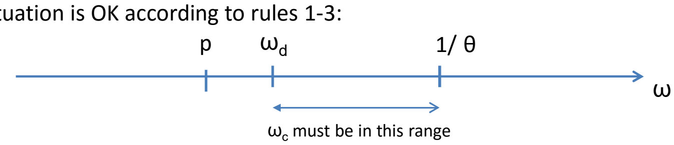
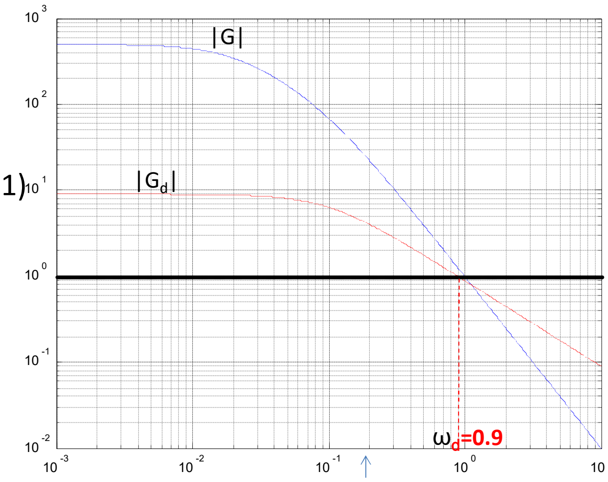


SCALED MODEL

EXAMPLE, g = 500/((50*s+1)*(10*s+1))

gd = 9/(10*s+1)



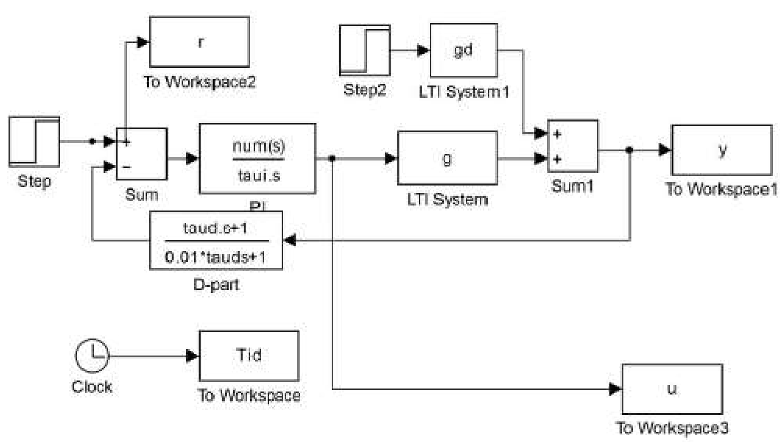
Effective delay θ **limits** ω_c

PI-control: $\omega_c < 1/\theta = 1/5 = 0.2$

PID-control: $\omega_c < 1/\theta = 1/0 = \infty$

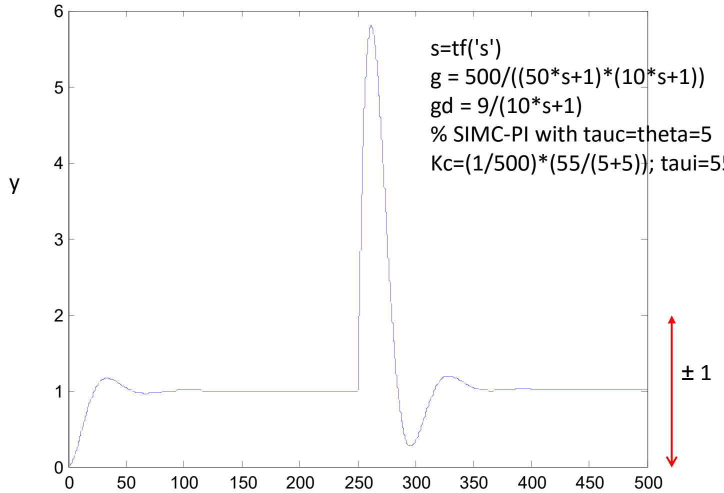
```
s=tf('s')
g = 500/((50*s+1)*(10*s+1))
gd = 9/(10*s+1)
w = logspace(-3,1,1000);
[mag,phase]=bode(g,w);
[magd,phased]=bode(gd,w);
loglog(w,mag(:),'blue',w,magd(:),'red',w,1,'black'), grid on
```

CHECK CONTROLLABILITY ANALYSIS WITH SIMULATIONS



SCALED MODEL

PI control not acceptable*

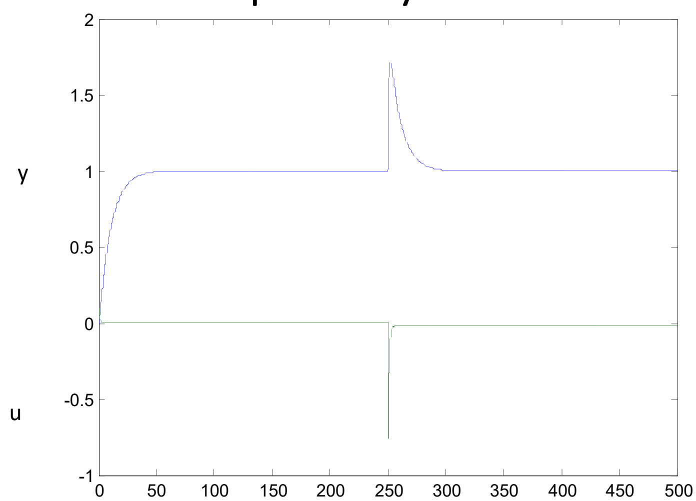


s=tf('s')
g = 500/((50*s+1)*(10*s+1))
gd = 9/(10*s+1)
% SIMC-PI with tauc=theta=5
Kc=(1/500)*(55/(5+5)); taui=55; taud=0;

*As expected since need $\omega_c > \omega_d = 0.9$, but can only achieve $\omega_c < 1/\theta = 1/5 = 0.2$

SCALED MODEL

PID control acceptable: y and u are within ±1



g = 500/((50*s+1)*(10*s+1))
gd = 9/(10*s+1)
%SIMC-PID (cascade form) with tauc=1/wd=1:
Kc=(1/500)*(50/(1+0)); taui=50; taud=10;

If process is not controllable: Need to change the design

- For example, dampen disturbance by adding buffer tank:

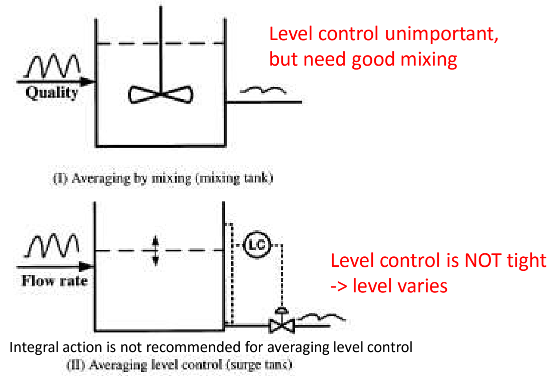


Figure 1. Two types of buffer tanks.

SCALED MODEL

Problem 1

$$G(s) = \frac{2}{s+1} \quad G_d(s) = \frac{3}{5s+1}$$

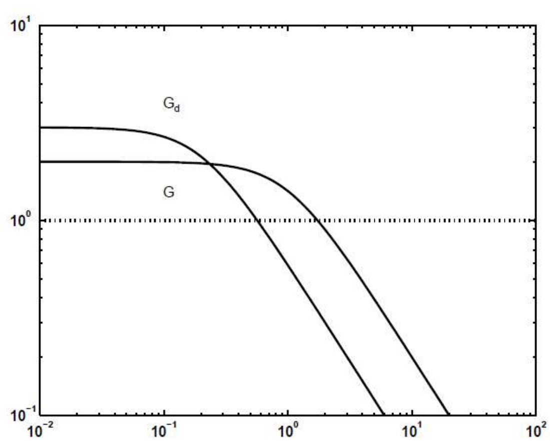
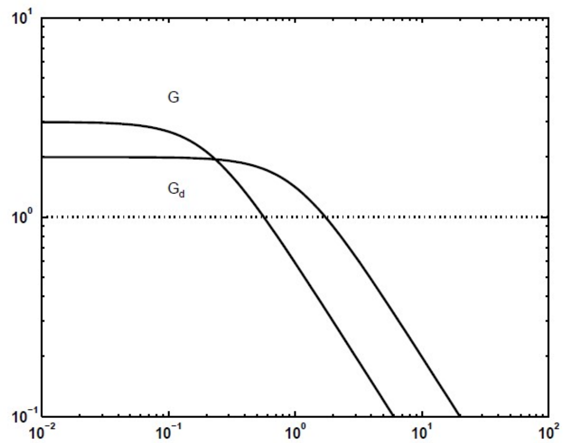


Figure 3: Magnitude of G and G_d.

SCALED MODEL

Problem 2

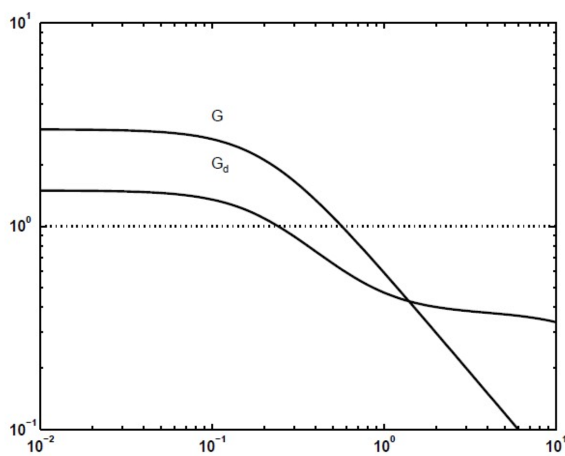
$$G(s) = \frac{3}{5s+1} \quad G_d(s) = \frac{2}{s+1}$$



SCALED MODEL

Problem 3

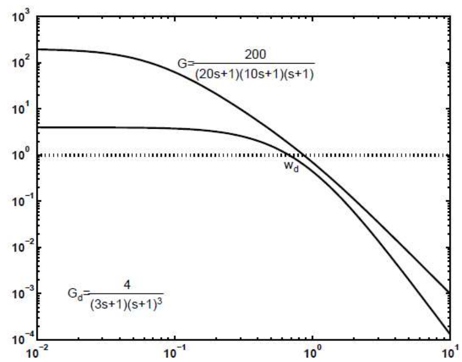
$$G(s) = \frac{3}{5s+1} \quad G_d(s) = 7.5\frac{s - 0.8}{(s+0.2)(s+20)}$$



SCALED MODEL

Problem 4

$$G(s) = \frac{200}{(20s+1)(10s+1)(s+1)} \quad G_d(s) = \frac{4}{(3s+1)((s+1)^3}$$



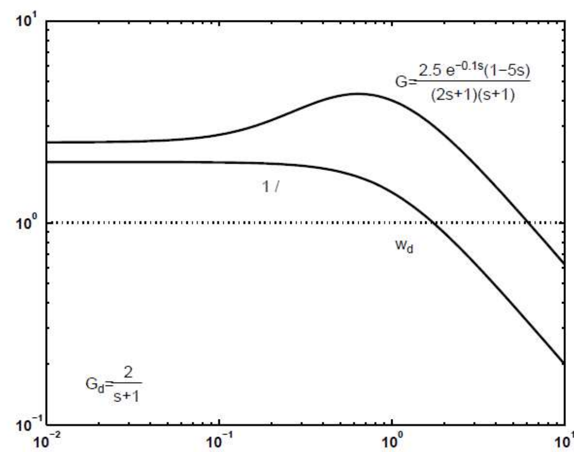
(a) Magnitude of G and Gd

g = 200/((20*s+1)*(10*s+1)*(s+1))
gd = 4/((3*s+1)*(s+1)^3)
Kc=(1/200)*20/1,taui=20,taud=10.5

SCALED MODEL

Problem 5

$$G(s) = \frac{2.5e^{-0.1s}(1 - 5s)}{(3s+1)((s+1)^3} \quad G_d(s) = \frac{2}{s+1}$$



SCALED MODEL

Problem 6

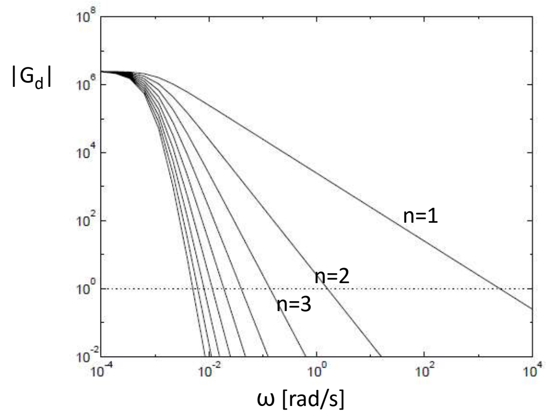
pH-Neutralization.
$y = c_{H+} - c_{OH-}$ (want=0 +- 10^{-6} mol/l,pH=7+-1)
$u = q_{base}$ (c_{OH-}=10mol/l, pH=15)
$d = q_{acid}$ (c_{H+}=10mol/l, pH=-1)

Using tanks in series,
Acid and base in tank 1.

Scaled model: k_d = 2.5e6
Each tank: τ = 1000s
Control: θ = 10s (meas. delay for pH)

Problem: How many tanks?

With n tanks: $G_d(s) = k_d/(1+\tau s)^n$.
τ: residence time in each tank.



Reference for more applications of controllability analysis: Chapter 5 in book by Skogestad and Postlethwaite (2005)

Control system

- 3 tanks: Neutralization (base addition) only in tank 1 gives large effective delay (>> 10s) because of tank dynamics in g(s)
- Suggested solution is to add (a little) base also in the other tanks:

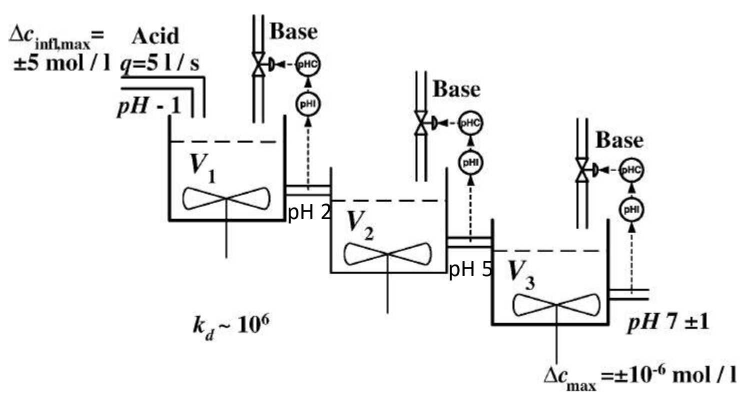


Fig. 3. Neutralization in three stages.

Conclusion

- Use controllability analysis
  - To avoid spending time on impossible control problem
  - To help design the process (e.g., size buffer tanks)
- Also useful for tuning.
  - τ_c = SIMC tuning parameter = $1/\omega_c$
  - Must for acceptable controllability have:

$$\omega_d \leq \omega_c \leq \frac{1}{\theta} \Leftrightarrow \boxed{\theta \leq \tau_c \leq \frac{1}{\omega_d}}$$

  - Agrees with SIMC-rules
    - Tight control: $\tau_c = \theta$
    - "Smooth" control: $\tau_c = 1/\omega_d$

Exam

- Wednesday 06 December 2017 from 09:00 to 13:00
- The test (questions) is in English but you may answer in Norwegian or English.
- Permitted examination support material:
  - One (1) A4 double-sided piece of paper with your handwritten notes (it does not need to be approved or stamped prior to the exam).
  - No other written material.
  - Standard calculator.
- Note: Remember to state clearly all assumptions you make.

Q&A session

Alternative: Monday 04 Dec. 12-14