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Multivariable control using single loops

- Interactions
- Choice of pairings (RGA)

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Multivariable process

Distillation column

"Increasing reflux L from 1.0 to 1.1 changes y_D from 0.95 to 0.97, and x_B from 0.02 to 0.03"

"Increasing boilup V from 1.5 to 1.6 changes y_D from 0.95 to 0.94, and x_B from 0.02 to 0.01"

Steady-State Gain Matrix

$$\begin{pmatrix} \Delta Y_D \\ \Delta x_B \end{pmatrix} = G(0) \begin{pmatrix} \Delta L \\ \Delta V \end{pmatrix}$$

$$G(0) = \begin{bmatrix} g_{11} & g_{12}(0) \\ g_{21} & g_{22}(0) \end{bmatrix} = \begin{bmatrix} \frac{0.97-0.95}{1.1-1.0} & \frac{0.94-0.95}{1.6-1.5} \\ \frac{0.03-0.02}{1.1-1.0} & \frac{0.01-0.02}{1.6-1.5} \end{bmatrix} = \begin{bmatrix} 0.2 & -0.1 \\ 0.1 & -0.1 \end{bmatrix}$$

Effect of input 1 (ΔL) on output 2 (Δx_B)

Can also include dynamics :

$$G(s) = \begin{bmatrix} \frac{0.2}{1+50s} & -\frac{0.1}{1+50s} \\ \frac{0.1}{1+40s} & -\frac{0.1}{1+40s} \end{bmatrix} \begin{matrix} \rightarrow \Delta y_D \\ \rightarrow \Delta x_B \end{matrix}$$

(Time constant 50 min for y_D)

(time constant 40 min for x_B)

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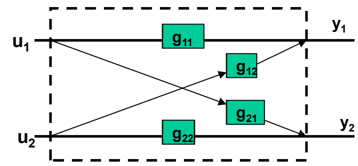
Analysis of Multivariable processes

Process Model 2x2

"Open – loop"

$$y_1(s) = g_{11}(s)u_1(s) + g_{12}(s)u_2(s)$$
$$y_2(s) = g_{21}(s)u_1(s) + g_{22}(s)u_2(s)$$

INTERACTIONS: Caused by nonzero g_{12} and/or g_{21}



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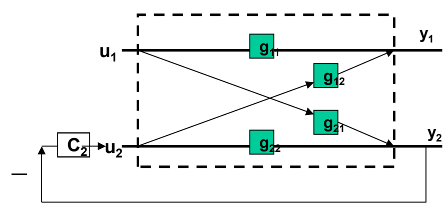
RGA: Consider effect of u_1 on y_1



1) **"Open-loop" ($C_2 = 0$):** $y_1 = g_{11}(s) \cdot u_1$

2) **"Closed-loop" (close loop 2, $C_2 \neq 0$):** $y_1 = \left(g_{11}(s) - \frac{g_{12} g_{21} \cdot C_2}{1 + g_{22} \cdot C_2} \right) u_1$

Change caused by "interactions"

Derivation.
Close loop 2: $u_2 = -c_2(y_2 - y_{2s})$
Here: $y_2 = g_{21}u_1 + g_{22}u_2$ and assume $y_{2s} = 0$:
$\Rightarrow u_2 = -c_2(g_{21}u_1 + g_{22}u_2) \Rightarrow u_2 = \frac{-c_2 g_{12}}{1+g_{22}c_2} u_1$
Effect of u_1 on y_1 with loop 2 closed is then:

$$y_1 = g_{11}u_1 + g_{12}u_2 = \underbrace{g_{11}\left(1 - \frac{g_{12}g_{21}c_2}{1 + g_{22}c_2}\right)}_{\hat{g}_{11}} u_1$$

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Limiting Case $C_2 \to \infty$ (perfect control of y_2) (steady state)

$$y_1 = \left(g_{11}(s) - \frac{g_{12}\, g_{21}}{g_{22}} \right) u_1 = g_{11}(1/\lambda_{11}) \cdot u_1$$

How much has "gain" from u_1 to y_1 changed by closing loop 2 with perfect control?

$$\text{Relative Gain} = \frac{(y_1/u_1)_{OL}}{(y_1/u_1)_{CL}} = \frac{g_{11}}{g_{11} - \frac{g_{12}\, g_{21}}{g_{22}}} = \frac{1}{1 - \frac{g_{12}\, g_{21}}{g_{11}\, g_{22}}} \overset{def}{=} \lambda_{11}^{RGA}$$

The relative Gain Array (RGA) is the matrix formed by considering all the relative gains

$$RGA = \Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} \frac{(y_1/u_1)_{OL}}{(y_1/u_1)_{CL}} & \frac{(y_1/u_2)_{OL}}{(y_1/u_2)_{CL}} \\ \frac{(y_2/u_1)_{OL}}{(y_2/u_1)_{CL}} & \frac{(y_2/u_2)_{OL}}{(y_2/u_2)_{CL}} \end{bmatrix}$$

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Example from before

$$G = \begin{bmatrix} 0.2 & -0.1 \\ 0.1 & -0.1 \end{bmatrix}, \quad \lambda_{11} = \frac{1}{1 - \underbrace{\frac{0.1 \;\; 0.1}{0.2 \;\; 0.1}}_{0.5}} = 2$$

$$RGA = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Only acceptable pairings:

$u_1 \leftrightarrow y_1$

$u_2 \leftrightarrow y_2$

Not recommended:

$u_1 \leftrightarrow y_2$

$u_2 \leftrightarrow y_1$

With integral action: Negative RGA $\Rightarrow$ individual loop unstable OR overall system unstable when individual loops saturates

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Property of RGA:

- **Columns and rows always sum to 1**
- **RGA independent of scaling (units) for u and y.**

RGA for general case:
$[RGA]_{ij} = (g_{ij})_{OL}/(g_{ij})_{CL} = [G]_{ij}[G^{-1}]_{ji}$
$=$ element-by-element multiplication of G and ${G^{-1}}^T$.
Matlab: `RGA = G.*pinv(G).'`

Example

```
G = [5 10 1; 20 -10 0; 18 0 2]
G =

    5   10    1
   20  -10    0
   18    0    2
```

```
>> rga=G.*pinv(G).'
rga =
    0.3125    1.2500   -0.5625
    1.2500   -0.2500         0
   -0.5625         0    1.5625
```

Conclusion: of the 6 possible pairings only one has positive RGA's

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Use of RGA:

(1) Interactions

- **RGA-element (λ)> 1: Smaller gain by closing other loops ("fighting loops" gives slower control)**
- **RGA-element (λ) <1: Larger gain by closing other loops (can be dangerous)**
- **RGA-element (λ) negative: Gain reversal by closing other loops (Oops!)**

Rule 1. Avoid pairing on negative steady-state relative gain – otherwise you get instability if one of the loops become inactive (e.g. because of saturation)

Rule 2. Choose pairings corresponding to RGA-elements close to 1

Traditional: Consider Steady-state

Better (improved Rule 2): Consider frequency corresponding to closed-loop time constant

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Example

$$G = \begin{pmatrix} 16.8 & 30.5 & 4.30 \\ -16.7 & 31.0 & -1.41 \\ 1.27 & 54.1 & 5.40 \end{pmatrix}, \; RGA(G) = \begin{pmatrix} 1.50 & 0.99 & -1.48 \\ -0.41 & 0.97 & 0.45 \\ -0.08 & -0.95 & 2.03 \end{pmatrix}$$

Only diagonal pairings give positive steady-state RGA's!

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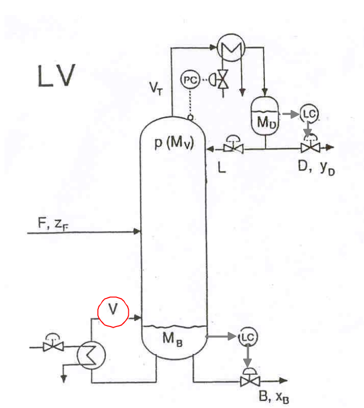
Distillation

$$y = \begin{pmatrix} y_D \\ x_B \end{pmatrix}, \quad u = \begin{pmatrix} L \\ V \end{pmatrix}$$

$$G(0) = \begin{pmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{pmatrix}, \quad RGA(0) = \begin{pmatrix} 35 & -34 \\ -34 & 35 \end{pmatrix}$$



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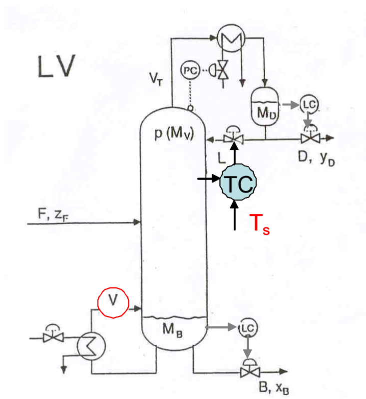
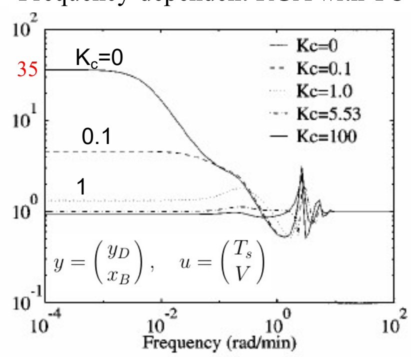


Can break interactions with cascade:
Frequency-dependent RGA with TC



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Sometimes useful: Iterative RGA

- For large processes, lots of pairing alternatives
- RGA evaluated iteratively is helpful for quick screening

$$\text{RGA}(G) = \Lambda(G) = G \times (G^{-1})^T$$

$$\Lambda^2(G) = \Lambda(\Lambda(G))$$

$$\Lambda^\infty = \lim_{k\to\infty} \Lambda^k(G)$$

- Converges to “Permuted Identity” matrix (correct pairings) for generalized diagonally dominant processes.
- Can converge to incorrect pairings, when no alternatives are dominant.
- Usually converges in 5-6 iterations

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Example of Iterative RGA

$$G = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 0.33 & 0.67 \\ 0.67 & 0.33 \end{bmatrix} \quad \Lambda^2 = \begin{bmatrix} -0.33 & 1.33 \\ 1.33 & -0.33 \end{bmatrix}$$

$$\Lambda^3 = \begin{bmatrix} -0.07 & 1.07 \\ 1.07 & -0.07 \end{bmatrix} \quad \Lambda^4 = \begin{bmatrix} 0.00 & \boxed{1.00} \\ \boxed{1.00} & 0.00 \end{bmatrix}$$

Correct pairing

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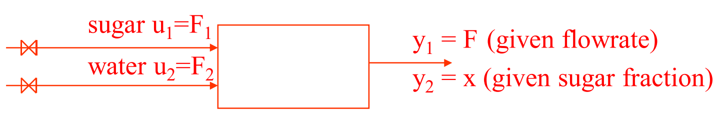
Exercise. Blending process



- Mass balances (no dynamics)
  - Total: $F_1 + F_2 = F$
  - Sugar: $F_1 = x\,F$

(a) Linearize balances and introduce: $u_1=dF_1$, $u_2=dF_2$, $y_1=F_1$, $y_2=x$,
(b) Obtain gain matrix G ($y = G\,u$)
(c) Nominal values are x=0.2 [kg/kg] and F=2 [kg/s]. Find G
(d) Compute RGA and suggest pairings
(e) Does the pairing choice agree with “common sense”?

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Solution.
(a) The balances "mass in = mass out" for total mass and sugar mass are

$$F_1 + F_2 = F; \;\; F_1 = xF$$

Note that the mixing process itself has no dynamics. Linearization yields

$$dF_1 + dF_2 = dF : \;\; dF_1 = x^* dF + F^* dx$$

With $u_1 = dF_1$, $u_2 = dF_2$, $y_1 = dF$ and $y_2 = dx$ we then get the model

$$y_1 = u_1 + u_2$$
$$y_2 = \frac{1-x^*}{F^*}u_1 - \frac{x^*}{F^*}u_2$$

where $x^* = 0.2$ is the nominal steady-state sugar fraction and $F^* = 2$ kg/s is the nominal amount.
(b,c) The transfer matrix then becomes

$$G(s) = \begin{pmatrix} 1 & 1 \\ \frac{1-x^*}{F^*} & -\frac{x^*}{F^*} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0.4 & -0.1 \end{pmatrix}$$

(d) The corresponding RGA matrix is (at all frequencies)

$$\Lambda = \begin{pmatrix} x^* & 1-x^* \\ 1-x^* & x^* \end{pmatrix} = \begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix}$$

For decentralized control, it then follows from pairing rule 1 ("prefer pairing on RGA elements close to 1") that we should pair on the off-diagonal elements; that is, use u_1 to control y_2 and use u_2 to control y_1.
(e) This corresponds to using the largest stream (water, u_2) to control the amount ($y_1 = F$), which is reasonable from a physical point of view. Also note that the RGA-elements are always between 0 and 1 for this process, and the RGA-elements are all 0.5, corrresponding to "switching" the pairings, when $x^* = 0.5$, which is when the two feed streams are equal.

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