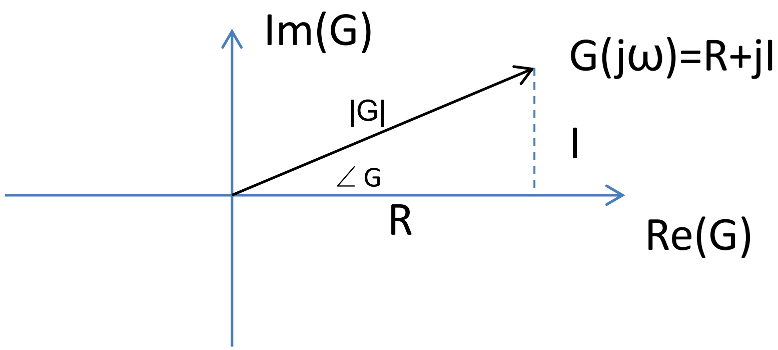
# Ch. 14
# Frequency analysis

# Mathematics. Complex numbers, $j^2=-1$




$$s = j\omega \qquad G(j\omega) = R + Ij$$

$$|G| = AR = \sqrt{R^2 + I^2}$$

$$\phi = \angle G = \arctan\frac{I}{R}$$

Polar form

Polar form:

$$G = R + jI = |G|(cos\angle G + jsin\angle G) = |G|e^{j\angle G}$$

Note: $e^{j\pi} = -1$

Polar form

Multiply complex numbers:

Multiply magnitudes and add phases

$$G = G_1 \cdot G_2 \cdot G_3$$

$$|G| = |G_1| \cdot |G_2| \cdot |G_3|$$

$$\angle G = \angle G_1 + \angle G_2 + \angle G_3$$

$$Similar - for - ratio:$$

$$G = \frac{G_1}{G_2}$$

$$|G| = |G_1| / |G_2|$$

$$\angle G = \angle G_1 - \angle G_2$$

Force linear system with sinusoidal input: $u(t) = u_0 \sin \omega t$

Output has same frequency: $y(t) = y_0 \sin(\omega t + \varphi)$

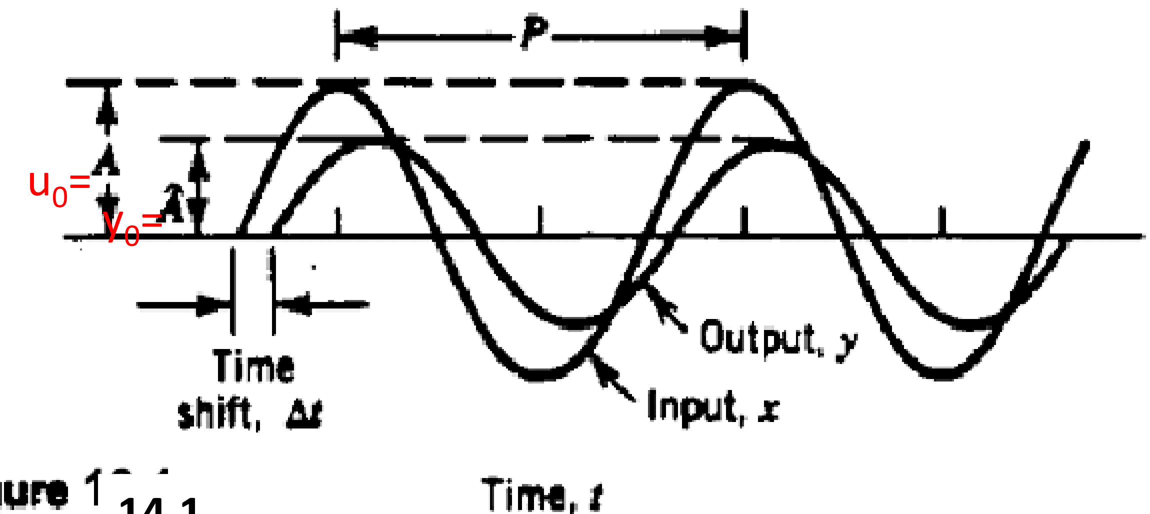



**Figure 14.1**

Attenuation and time shift between input and output sine waves ($K = 1$). The phase angle $\phi$ of the output signal is given by $\phi = -\Delta t / P \times 360$, where $\Delta t$ is the time (period) shift and $P$ is the period of oscillation.

Frequency: $\omega$ [rad/s]

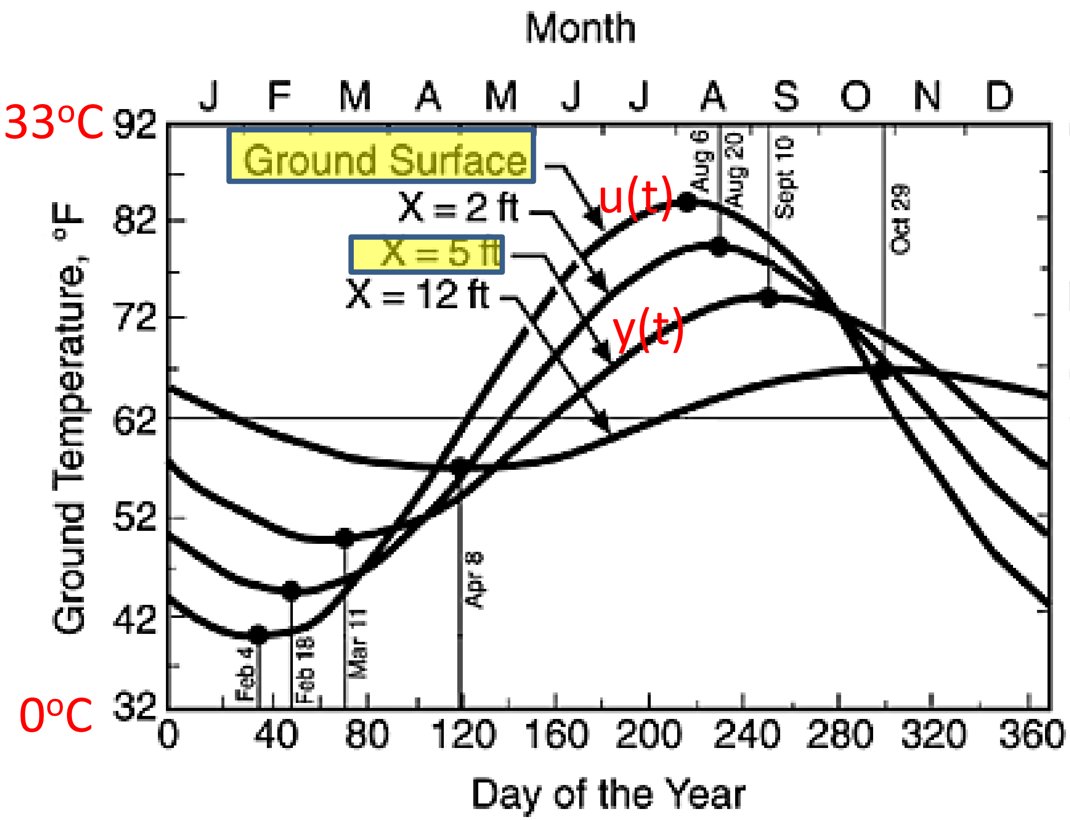
Period: $P[s] = 2\pi / \omega$

Phase shift: $\varphi$ [rad]

Time shift $\Delta t\ [s] = -\varphi / \omega$

Amplitude ratio (gain): $AR = y_0/u_0$

# Example: Ground temperature phase shift




Surface temperature:

$$u(t) = u_{avg} + u_0 \sin(\omega(t-t_0))$$

Ground temperature at X=5ft:

$$y(t) = y_{avg} + y_0 \sin(\omega(t-t_0) + \varphi)$$

Note:

- Average: $u_{avg} = y_{avg} = 62$F
  (Usually deviation variables, so average=0)
- $t_0 = 120$ d (where u crosses zero from below). Usually, $t_0 = 0$.

**Problem:**

- **Find $u_0$, $y_0$, P, $\omega$ , $\varphi$ and gain**

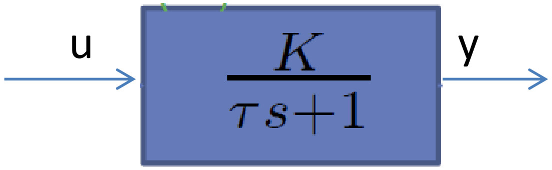
Solution for X=5ft.

- $u_0$ = 62-40=22 F, $y_0$ = 62-50=12 F, Gain = AR = 12/22 = 0.55
- P = 365d, $\omega = 2\pi/P = 2\pi/365 = 0.017$ rad/d,

Phase shift:

- Summer: $\Delta t$ = 35 days from Aug. 6 (hottest day) to Sep. 10 (hottest in ground)
- Winter: $\Delta t$ = 35 days from Feb. 4 (coldest day) to Mar. 11 (coldest in ground),
- $\varphi = -\Delta t\, \omega = -35\text{d} * 0.017 \text{ rad/d} = -0.602 \text{ rad} = -34.5^o$

u(t) = A sin(ωt)




As t →∞:
y(t) = AR *A*sin(ωt+φ)

Note: A is the same as $u_0$

General (VERY SIMPLE).
Set s=jω in G(s). Then
AR = |G(jω)|
φ = ∠ G(jω)

### 4.2.3 Sinusoidal Response

As a final example of the response of first-order processes, consider a sinusoidal input $u_{sin}(t) = A \sin \omega t$, with transform given by Eq. (4-15):

$$u(s) = A\frac{\omega}{s^2+\omega^2}$$

$$y(s) = \frac{KA\omega}{(\tau_s + 1)(s^2 + \omega^2)} \qquad \text{(4-23) (5-22)}$$

$$= \frac{KA}{\omega^2\tau^2+1}\left(\frac{\omega\tau^2}{\tau s+1} - \frac{s\omega\tau}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2}\right) \qquad \text{(4-24)}$$

Inversion gives

$$y(t) = \frac{KA}{\omega^2\tau^2 + 1}(\omega\tau e^{-t/\tau} - \omega\tau \cos \omega t + \sin \omega t) \qquad \text{(4-25)}$$

or, by using trigonometric identities,

$$y(t) = \frac{KA\omega\tau}{\omega^2\tau^2+1} e^{-t/\tau} + \frac{KA}{\sqrt{\omega^2\tau^2+1}} \sin(\omega t + \phi) \qquad \text{(4-26)}$$

where

$$\phi = -\tan^{-1}(\omega\tau) \qquad \text{(4-27)}$$

Notice that in both (4-25) and (4-26) the exponential term goes to zero as $t \to \infty$, leaving a pure sinusoidal response. This property is exploited in Chapter 13 for frequency response analysis.

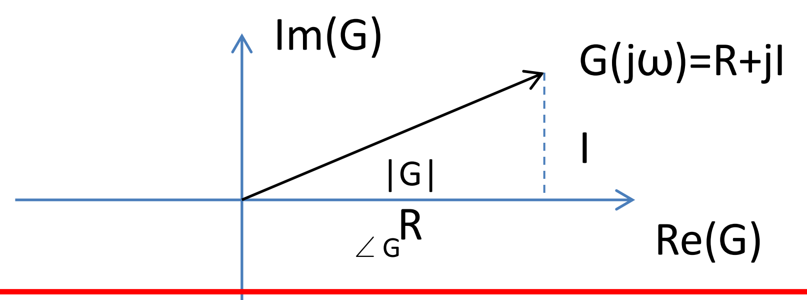
# General: Simple method to find sinusoidal response of system G(s)

1. Input signal to linear system: $u = u_0 \sin(\omega t)$
2. Steady-state ("persistent", $t \to \infty$) output signal: $y = y_0 \sin(\omega t + \varphi)$
3. What is $AR = y_0/u_0$ and $\varphi$?

Solution (extremely simple!)

1. Find system transfer function, G(s)
2. Let $s=j\omega$ (imaginary number, $j^2=-1$) and evaluate $G(j\omega) = R + j\,I$ (complex number)
3. Then ("believe it or not!")

| | |
|---|---|
| $AR = \lvert G(j\omega) \rvert$ | (magnitude of the complex number) |
| $\varphi = \angle G(j\omega)$ | (phase of the complex number) |




Proof: $y(s) = G(s)u(s)$ where $u(s) = \frac{u_0\omega}{s^2+\omega^2} = \frac{u_0\omega}{(s-j\omega)(s+j\omega)}$, etc...

(poles of $G(s)$ "die out" as $t \to \infty$)

Term $\frac{1}{s-j\omega}$ gives $G(j\omega)$ with partial fraction expansion

## Example 14.1:

1. $$G(s) = \frac{1}{\tau s + 1}$$

2. $$G(j\omega) = \frac{1}{1+\tau j\omega} \cdot \frac{1-\tau j\omega}{1-\tau j\omega} \qquad (j^2 = -1)$$

$$G(j\omega) = \underbrace{\frac{1}{1+\omega^2\tau^2}}_{R} - \underbrace{\frac{\omega\tau}{1+\omega^2\tau^2}}_{I} j$$

3. $$|G| = AR = \sqrt{R^2 + I^2} = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

$$\phi = \angle G = \arctan\frac{I}{R} = -\arctan(\omega\tau)$$

Gain and phase shift of sinusoidal response!

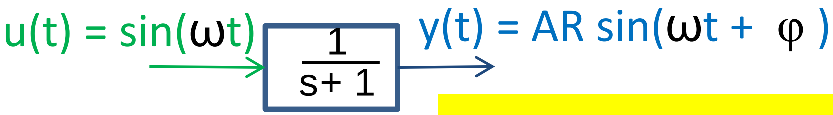
This method is not really recommended



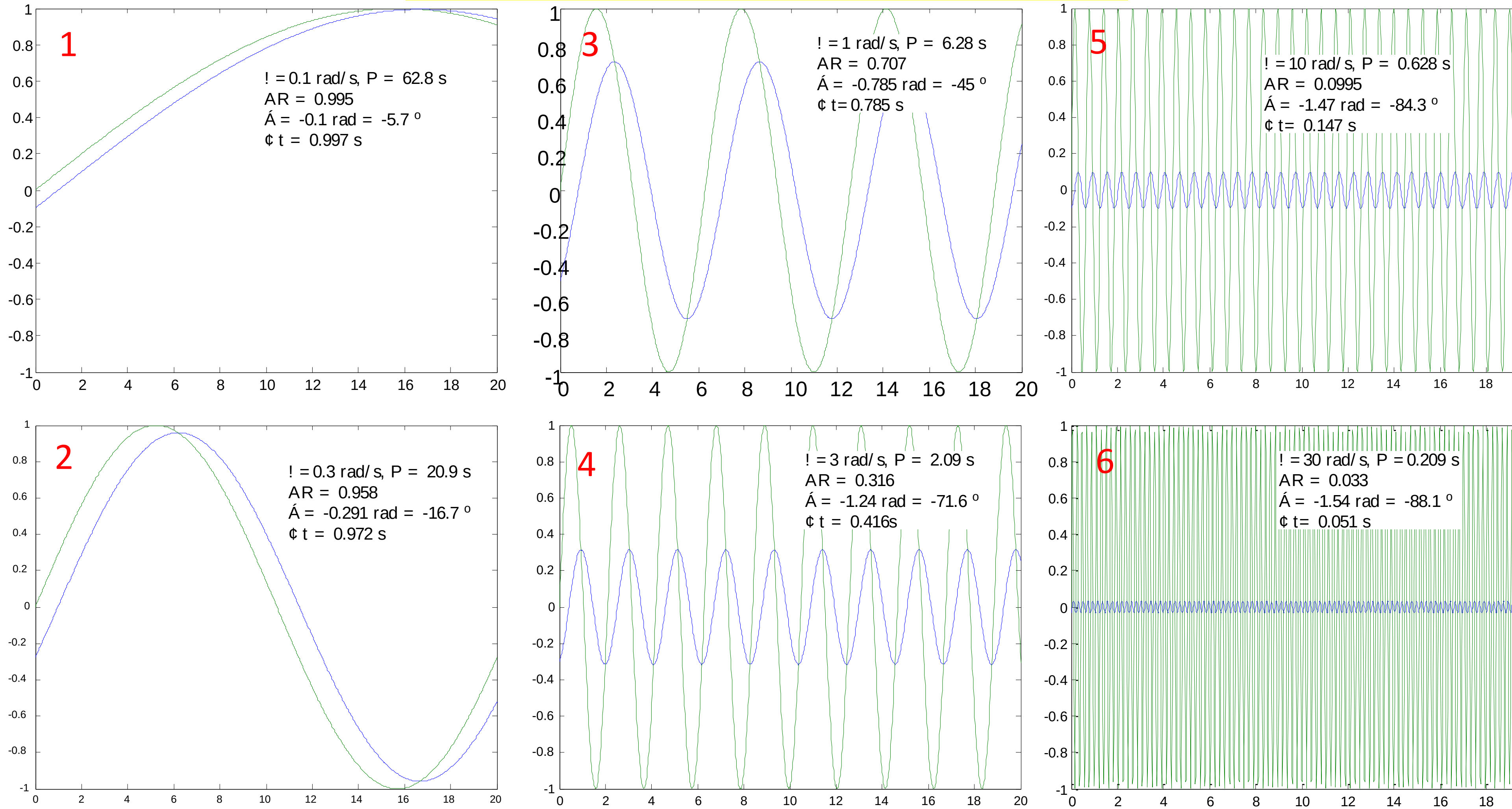
**SIMPLER**: Use polar form of complex numbers! G=G1/G2, where G1=1, G2=tau*s+1. set s=jw. Get |G|=1/|G2|=1/sqrt((w*tau)^2+1), angle(G)=0-angle(G2) = –arctg(w*tau)

# SINUSOIDAL RESPONSE OF FIRST-ORDER SYSTEM

k = 1, τ = 1 [s]




**6 Plots: Increase ω from 0.1 to 30 rad/s**




```
w=0.3; tau=1; t = linspace(0,20,1000);
u = sin(w*t);
AR = 1/sqrt((w*tau)^2+1)
phi = - atan(w*tau), phig=phi*180/pi, dt=-phi/w
y = AR*sin(w*t+phi);
plot(t,y,t,u)
```

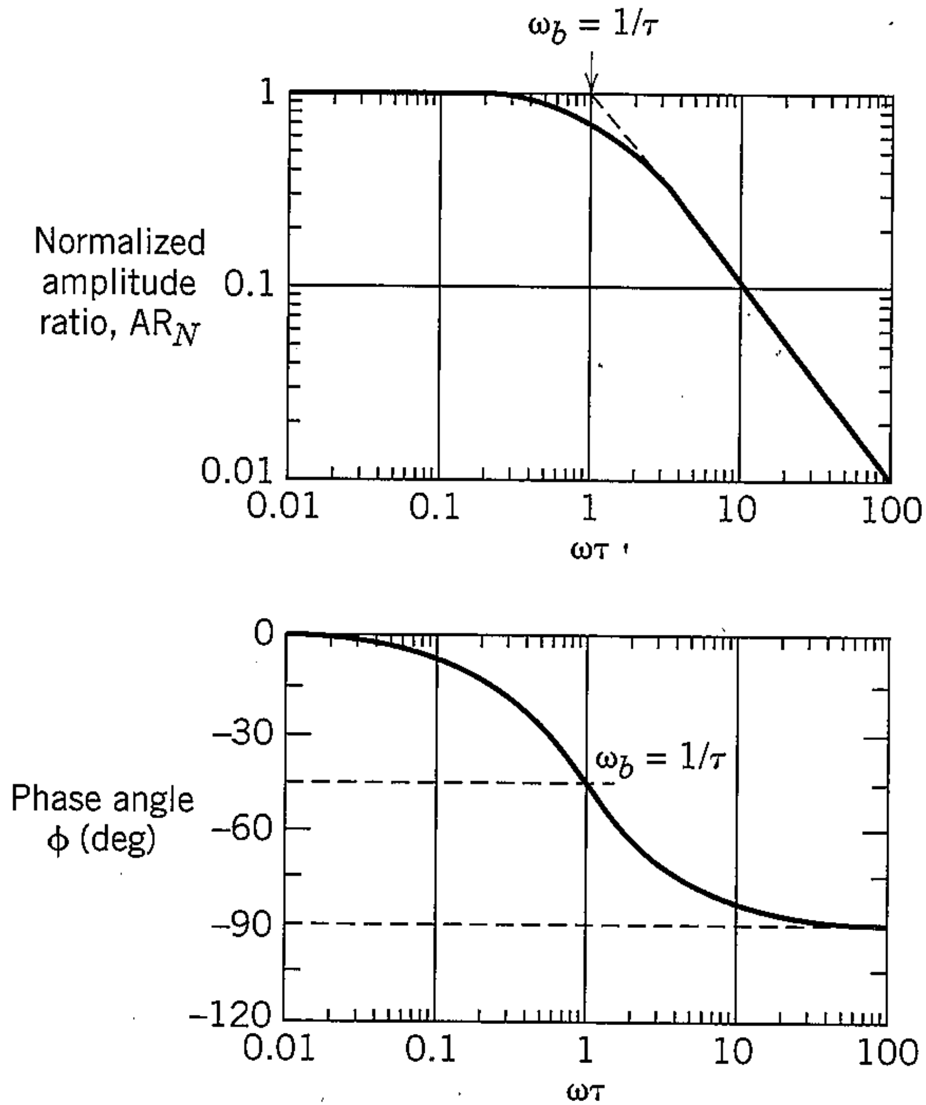



**Figure 14.2** Bode diagram for a first-order process

$$\text{AR} = |G(j\omega)| = \frac{1}{\sqrt{(\omega\tau)^2+1}}$$

$$\phi = \angle G(j\omega) = -\arctan(\omega\tau)$$

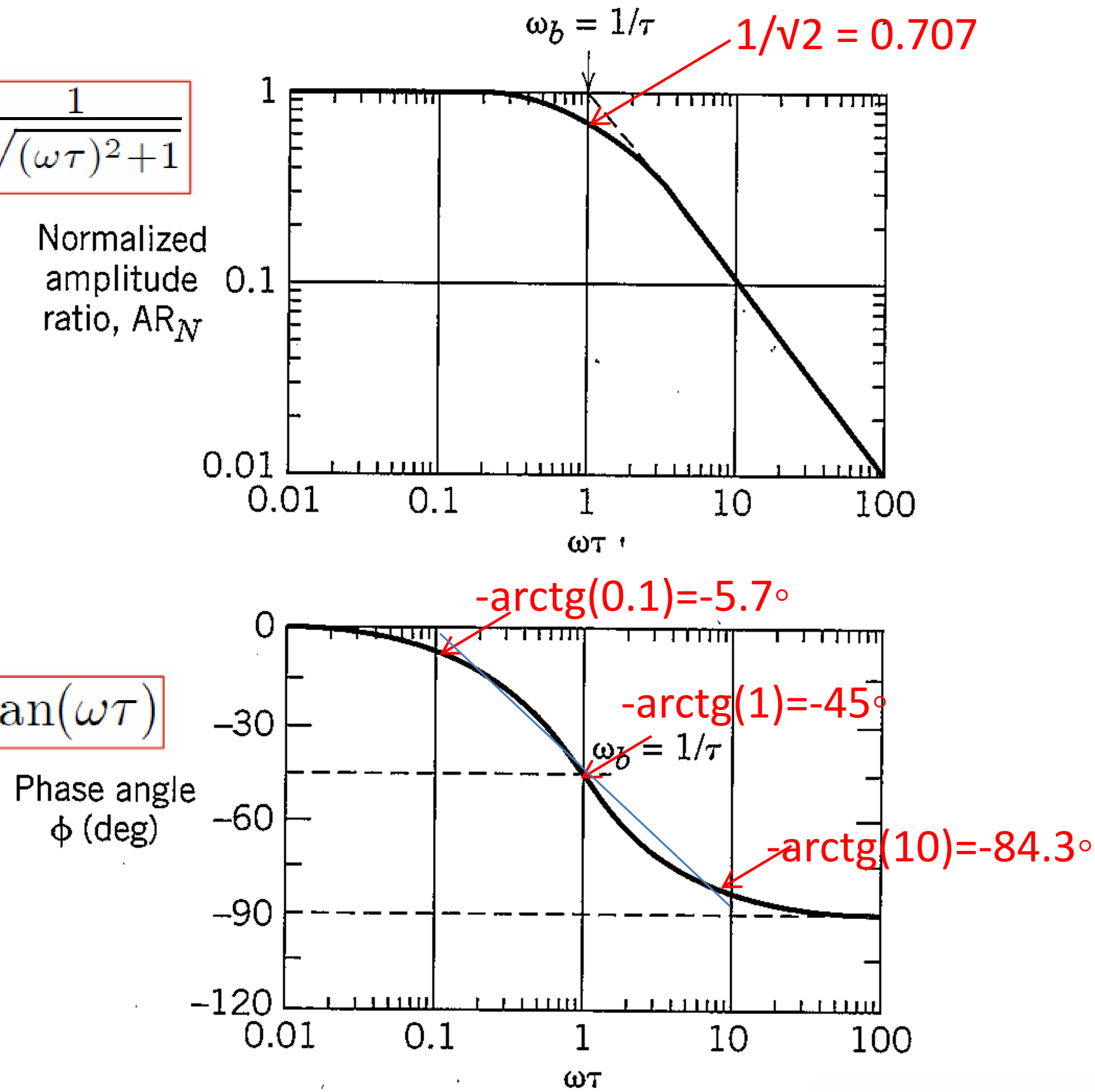



**Figure 14.2** Bode diagram for a first-order process

$$G(s) = \frac{1}{\tau s+1}$$

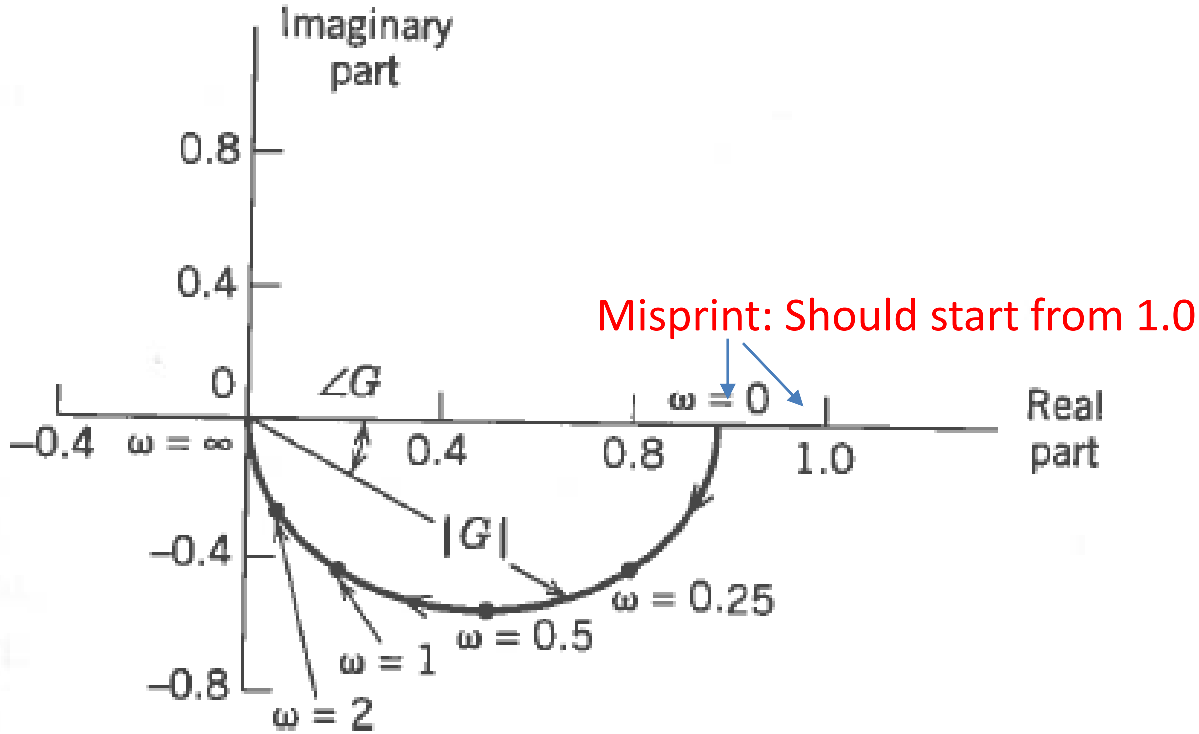



**Figure 13.12** The Nyquist diagram for $G(s) = 1/(2s + 1)$ plotting $\mathrm{Re}(G(j\omega))$ and $\mathrm{Im}(G(j\omega))$.

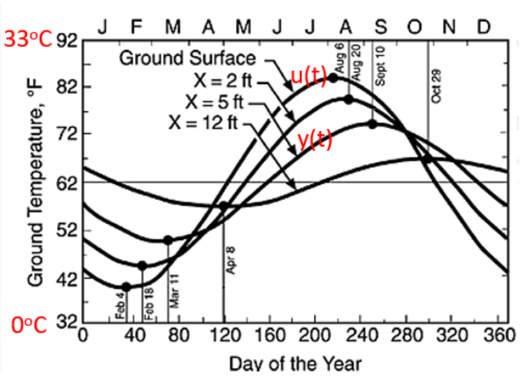
Note: Nyquist plot is not included in last edition

# Example: Ground temperature phase shift.

## What is τ if assume a first-order response from u to y? g(s) = k/(τs+1)

$$\mathrm{AR} = \frac{y_0}{u_0} = \frac{k}{\sqrt{(\omega\tau)^2+1}}$$

$$\phi = \arctan(-\omega\tau)$$




Data: $u_0$ = A= 22, $y_0$ = 12, ω=0.017 rad/d, φ = – 35°

Solution:

- We know from physics that the gain k=1. So g(s) = 1/(τs+1)

1. From amplitude data: AR = $y_0/u_0$ = 0.545.

   Get:
$$\tau = \frac{1}{\omega}\sqrt{\frac{1}{\mathrm{AR}^2}-1} = \frac{1}{0.017}\sqrt{\frac{1}{0.545^2}-1} = 90.5d$$

2. From phase shift data. φ = – 35°

   Get:
$$\tau = -\frac{1}{\omega}\tan\phi = -\frac{1}{0.017}\tan(-0.568) = 37.4d$$

Conclusion: This system is more complex than first order (no big surprise!)

It's described by partial differential equations and can be approximated by a high-order system with many poles and zeros.

For example, g(s) = ($\tau_2$s+1) / ($\tau_1$s+1) ($\tau_3$s+1) where $\tau_1 > \tau_2 > \tau_3$

# Frequenc response of time delay

$g=e^{-\theta s}$

Gain = $|g(j\omega)| = 1$

Phase shift = $\varphi$ = angle($g(j\omega)$) = $-\omega\theta$ [rad]

Alternative proof: Time domain

u(t)

y(t)

General:

$$g(s) = k \, \frac{g_1 \; g_2}{g_3 \; g_4} \, e^{-\theta s}$$

$$|g| = k \, \frac{|g_1||g_2|}{|g_3| \; |g_4|}$$

$$\angle g| = \angle g_1 + \angle g_2 - \angle g_3 - \angle g_4 - \omega\theta$$

Consider term:

$$g_a = Ts + 1$$

Set $s = j\omega$ and evaluate complex number $g_a(j\omega)$ with magnitude $|g_a|$ and phase $\angle g_a$. Get:

$$|g_a(j\omega)| = \sqrt{\omega^2 T^2 + 1};$$
$$\angle g_a = \arctan \omega T$$

Example 2

$$g(s) = \frac{k(Ts+1)}{(\tau_1 s+1)(\tau_2 s+1)} = \frac{g_1 \; g_2}{g_3 \; g_4}$$

$$g_1 = k$$
$$g_2 = Ts + 1$$
$$g_3 = \tau_1 s + 1$$
$$g_4 = \tau_2 s + 1$$

1. DERIVATIVE

$$g_1(s) = s$$

Frequency response: $g(j\omega) = j\omega = 0 + j\omega$

$$|g_1(j\omega)| = \omega$$
$$\angle g_1(j\omega) = 90^o = \pi/2 \text{ rad (purely complex at all } \omega)$$

Check:

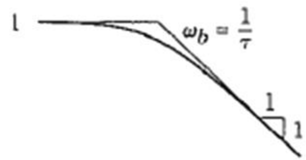
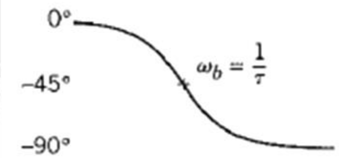
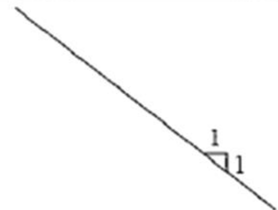
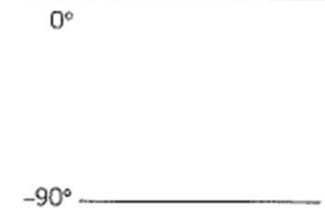
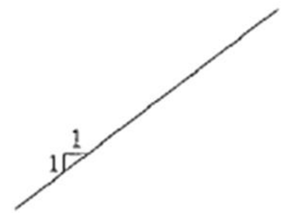
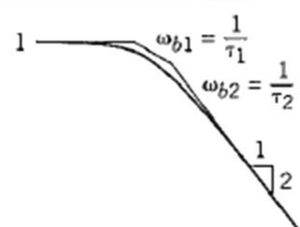
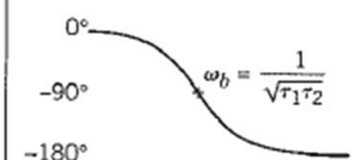
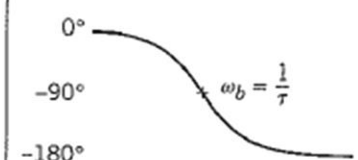
$$u(t) = u_0 sin(\omega t)$$
$$y(t) = u'(t) = u_0 \omega cos(\omega t) = \omega u_0 sin(\omega t + \pi/2)$$ OK!

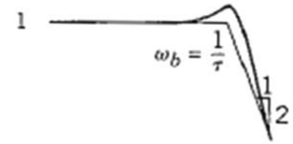
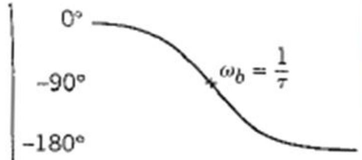
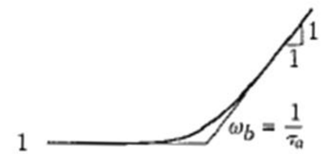
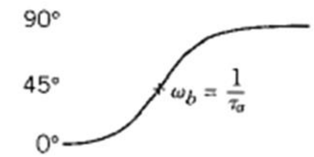
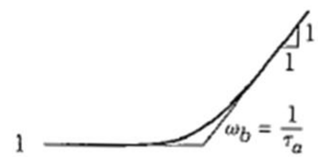
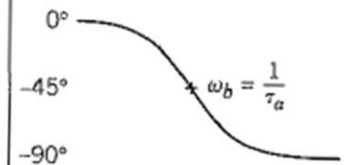
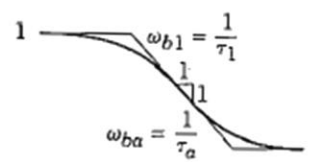
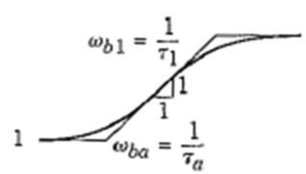
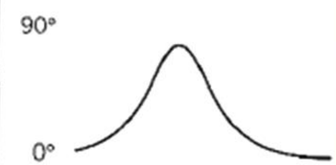
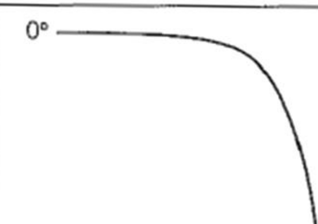
2. INTEGRATOR

$$g_2(s) = \frac{1}{s} = \frac{1}{g_1}$$

$$|g_2(j\omega)| = \frac{1}{|g_1|} = \frac{1}{\omega}$$
$$\angle g_2(j\omega) = 0^o - \angle g_1 = -90^o = -\pi/2 \text{ rad}$$

**Table 13.2** Frequency Response Characteristics of Important Process Transfer Functions

| Transfer Function | $G(s)$ | $AR = \lvert G(j\omega)\rvert$ | Plot of log $AR_N$ vs. log $\omega$ | $\phi = \angle G(j\omega)$ | Plot of $\phi$ vs. log $\omega$ |
|---|---|---|---|---|---|
| 1. First-order | $\dfrac{K}{\tau s + 1}$ | $\dfrac{K}{\sqrt{(\omega\tau)^2 + 1}}$ |   | $-\tan^{-1}(\omega\tau)$ |   |
| 2. Integrator | $\dfrac{K}{s}$ | $\dfrac{K}{\omega}$ |   | $-90°$ |   |
| 3. Derivative | $Ks$ | $K\omega$ |   | $+90°$ |   |
| 4. Overdamped second-order | $\dfrac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}$ | $\dfrac{K}{\sqrt{(\omega\tau_1)^2 + 1}\sqrt{(\omega\tau_2)^2 + 1}}$ |   | $-\tan^{-1}(\omega\tau_1) - \tan^{-1}(\omega\tau_2)$ |   |
| 5. Critically damped second-order | $\dfrac{K}{(\tau s + 1)^2}$ | $\dfrac{K}{(\omega\tau)^2 + 1}$ |   | $-2\tan^{-1}(\omega\tau)$ |   |

| | | | | | |
|---|---|---|---|---|---|
| 6. Underdamped second-order | $\dfrac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$ | $\dfrac{K}{\sqrt{(1-(\omega\tau_1)^2)^2+(2\zeta\omega\tau)^2}}$ |   | $-\tan^{-1}\left[\dfrac{2\zeta\omega\tau}{1-(\omega\tau)^2}\right]$ |   |
| 7. Left-half plane (positive) zero | $K(\tau_a s + 1)$ | $K\sqrt{(\omega\tau_a)^2+1}$ |   | $+\tan^{-1}(\omega\tau_a)$ |   |
| 8. Right-half plane (negative) zero | $-\tau_a s + 1$ | $K\sqrt{(\omega\tau_a)^2+1}$ |   | $-\tan^{-1}(\omega\tau_a)$ |   |
| 9. Lead–lag unit ($\tau_a < \tau_1$) | $K\dfrac{\tau_a s+1}{\tau_1 s+1}$ | $K\dfrac{\sqrt{(\omega\tau_a)^2+1}}{\sqrt{(\omega\tau_1)^2+1}}$ |   | $+\tan^{-1}(\omega\tau_a) - \tan^{-1}(\omega\tau_1)$ |   |
| 10. Lead–lag unit ($\tau_a > \tau_1$) | $K\dfrac{\tau_a s + 1}{\tau_1 s + 1}$ | $K\dfrac{\sqrt{(\omega\tau_a)^2+1}}{\sqrt{(\omega\tau_1)^2+1}}$ |   | $+\tan^{-1}(\omega\tau_a) - \tan^{-1}(\omega\tau_2)$ |   |
| 11. Time delay | $Ke^{-\theta s}$ | $K$ |   | $-\omega\theta$ |   |

Peak goes to infinity when ξ=0

Phase increases for LHP zero

Oops! Phase drops for RHP zero

## Bide plot of time delay.

$G = \exp(-\theta s)$

$|G(j\omega)| = 1$

$\angle G(j\omega) = -\omega\theta$

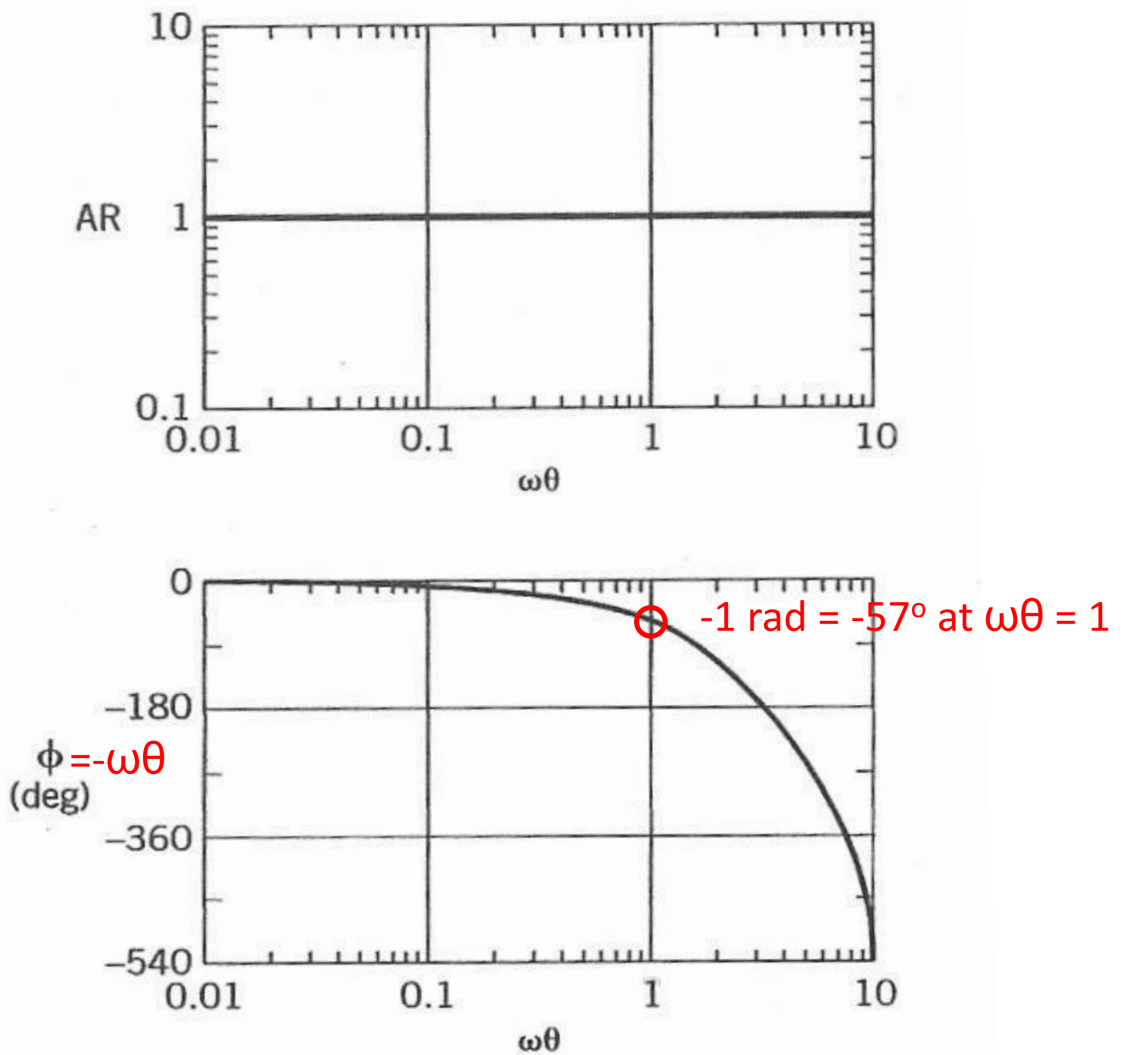


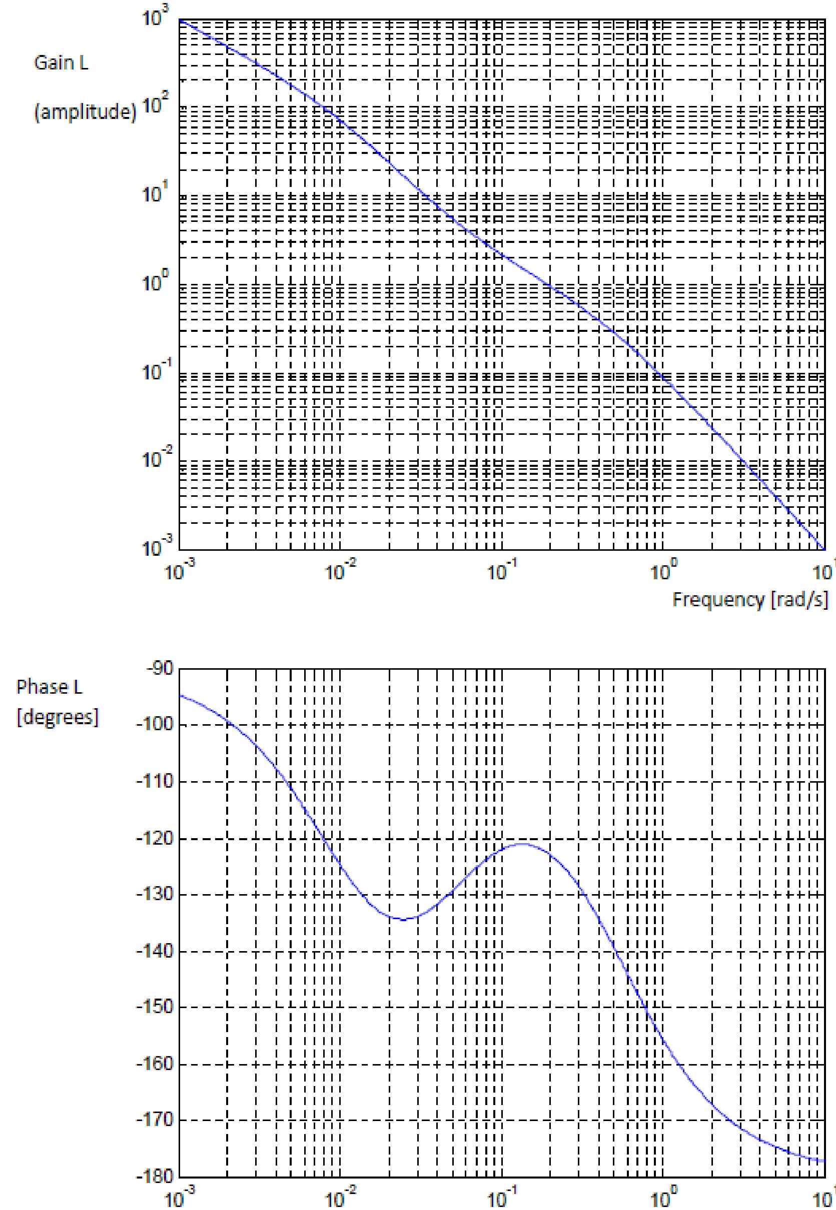

Figure 14.4 Bode diagram for a time delay, $e^{-\theta s}$.

# Example

## EXAMPLE

$$L(s) = \frac{20s+1}{s(100s+1)(2s+1)}$$

L(s)=G(s)C(S):
Loop transfer function for
SIMC PI-control with $\tau_c$=4 for
G(s) = 1/(100s+1)(2s+1)




```
s=tf('s')
L = (20*s+1)/[s*(100*s+1)*(2*s+1)]
figure(3), bode(L) % gives AR in dB
w = logspace(-3,1,1000)
[mag,phase]=bode(L,w)
figure(1), loglog(w,mag(:)), grid on, axis([0.001 10 0.001 1000])
figure(2), semilogx(w,phase(:)), grid on
```

# ASYMPTOTES

Frequency response of term (Ts+1): set s=jω.
Asymptotes:

$(j\omega T + 1) \sim 1$ for $\omega T << 1$ (slope n=0, phase=0)

$(j\omega T + 1) \sim j\omega T$ for $\omega T >> 1$ (slope n=1, phase=$90^o$)

Gain slope n: $|G| \sim \omega^n$

**Rule for asymptotic Bode-plot, L = k(Ts+1)/(τs+1)….. :**

1. Start with low-frequency asymptote (s→0)
   (a) If constant (L(0)=k):
   Gain=k (slope=0)
   Phase=$0^o$
   (b) If integrator (L=k'/s):
   Gain slope= -1 (on log-log plot). Need one fixed point, for example, gain=1 at ω=k'
   Phase: $-90^o$.
2. Break frequencies (order from large T to small T):

| | Change in gain slope | Change in phase |
|---|---|---|
| ω=1/T (zero) | +1 | $+90^o$ ($-90^o$ if T negative) |
| ω=1/**τ** (pole) | -1 | $-90^o$ ($+90^o$ if **τ** negative) |

3. Time delay, $e^{-\theta s}$. Gain: no effect, Phase contribution: $-\omega\theta$ [rad] (-1 rad = $-57^o$ at $\omega=1/\theta$)

SOLUTION

$$L(s) = \frac{20s+1}{s(100s+1)(2s+1)}$$

L(s): SIMC PI-control with $\tau_c$=4 for g(s) = 1/(100s+1)(2s+1)

Low-frequency asymptote ($s = j\omega \rightarrow 0$) is integrator: $L = \frac{1}{j\omega} = -\frac{1}{\omega}j$

Gain $= \frac{1}{\omega}$ (slope -1 on log-log),

Phase = -90$^o$

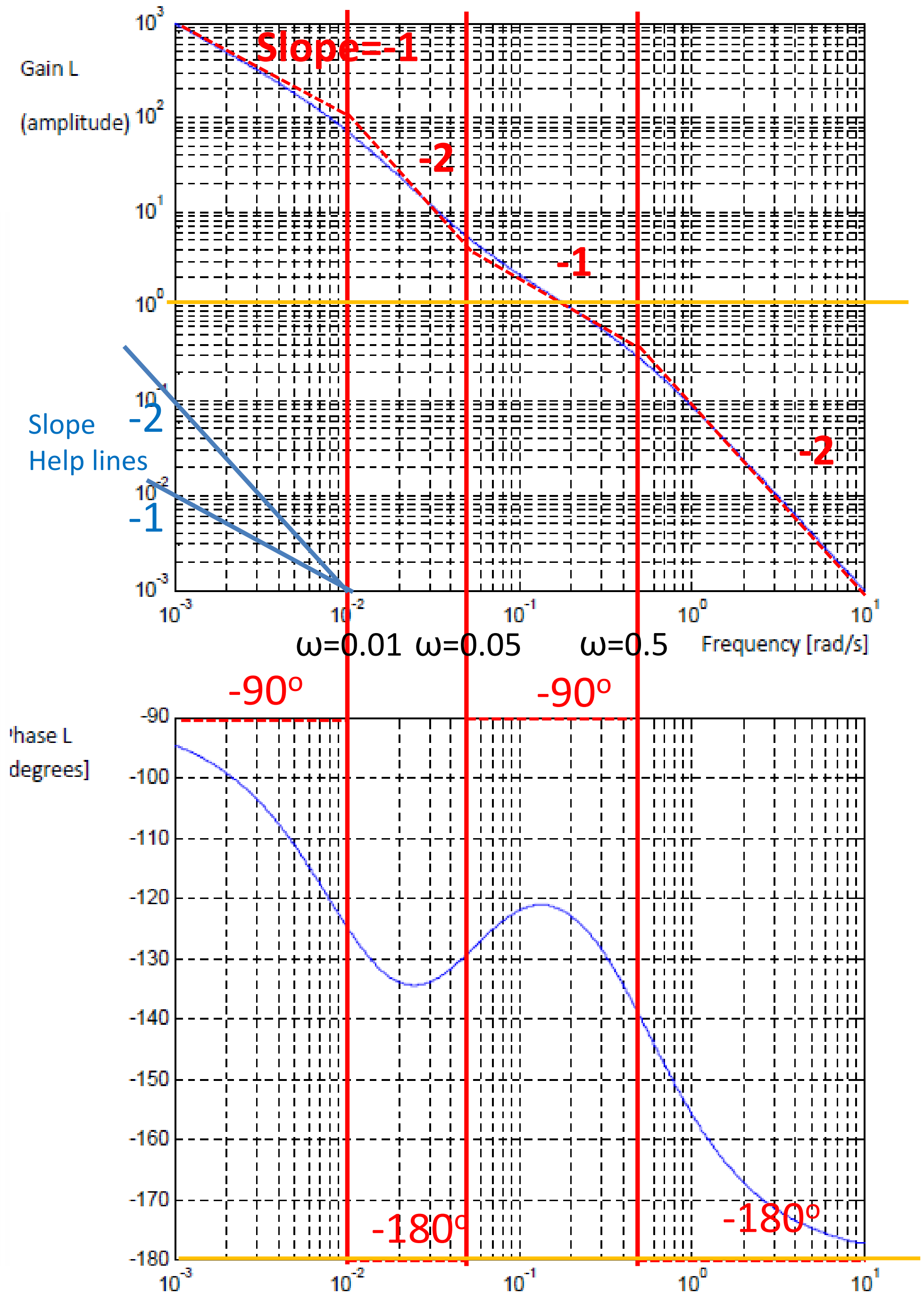
Asymptotes: Start at low frequency, $\omega \rightarrow 0$: $|L(j\omega)| = 1/\omega$. So: $|L|=10^3$ at $\omega=10^{-3}$

Break frequencies:
$\omega$ = 1/100=0.01 (pole), 1/20=0.05 (zero), 1/2=0.5 (pole)

First break frequency (at 0.01) is a pole:
Slope changes by -1 to -2 (log-log)
$\Rightarrow$ gain drops by factor 100 when $\omega$ increases by factor 10
Phase drops by -90$^o$ to -180$^o$
Asymptote $= \frac{1}{100(j\omega)^2} = -\frac{1}{100\omega^2}$

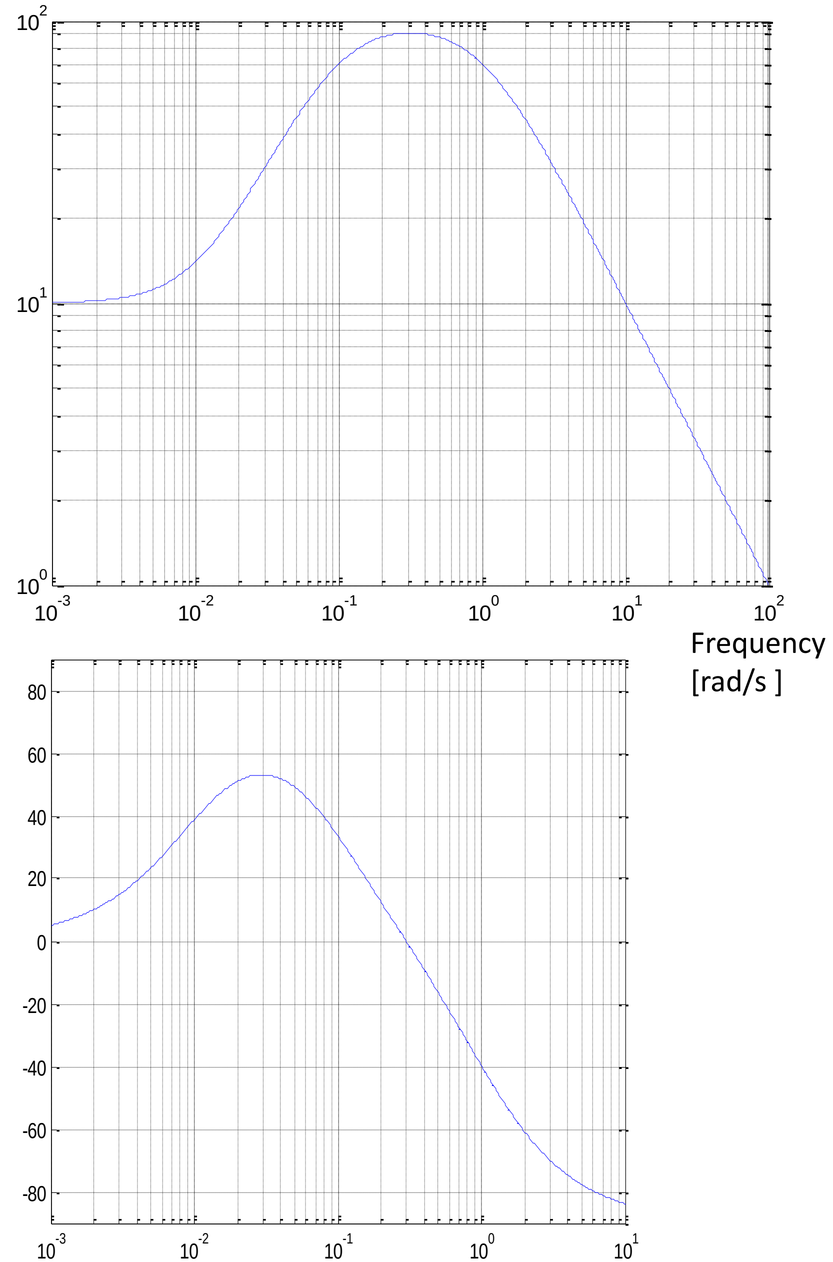
Next break frequency (at 0.05) is a zero:
Slope changes by +1 to -1 (log-log)
Phase increases by +90$^o$ to -90$^o$
Asymptote $= \frac{20}{100j\omega} = -\frac{1}{5\omega}j$

Final break frequency (at 0.5) is a pole:
Slope changes by -1 to -2 (log-log)
Phase drops by -90$^o$ to -180$^o$
Asymptote $= \frac{1}{10(j\omega)^2} = -\frac{1}{10\omega^2}$



# Example with phase lead

$$g(s) = 10\frac{100s+1}{(10s+1)(s+1)}$$



Low-frequency asymptote:

$$G_{c0} = \frac{2}{10s} = \frac{0.2}{s}.$$

"Fixed point":

$$\omega = 0.001 \Rightarrow |G_{c0}| = \frac{0.2}{0.001} = 200$$

PI-controller:

$$G_c(s) = 2\frac{10s+1}{10s}$$

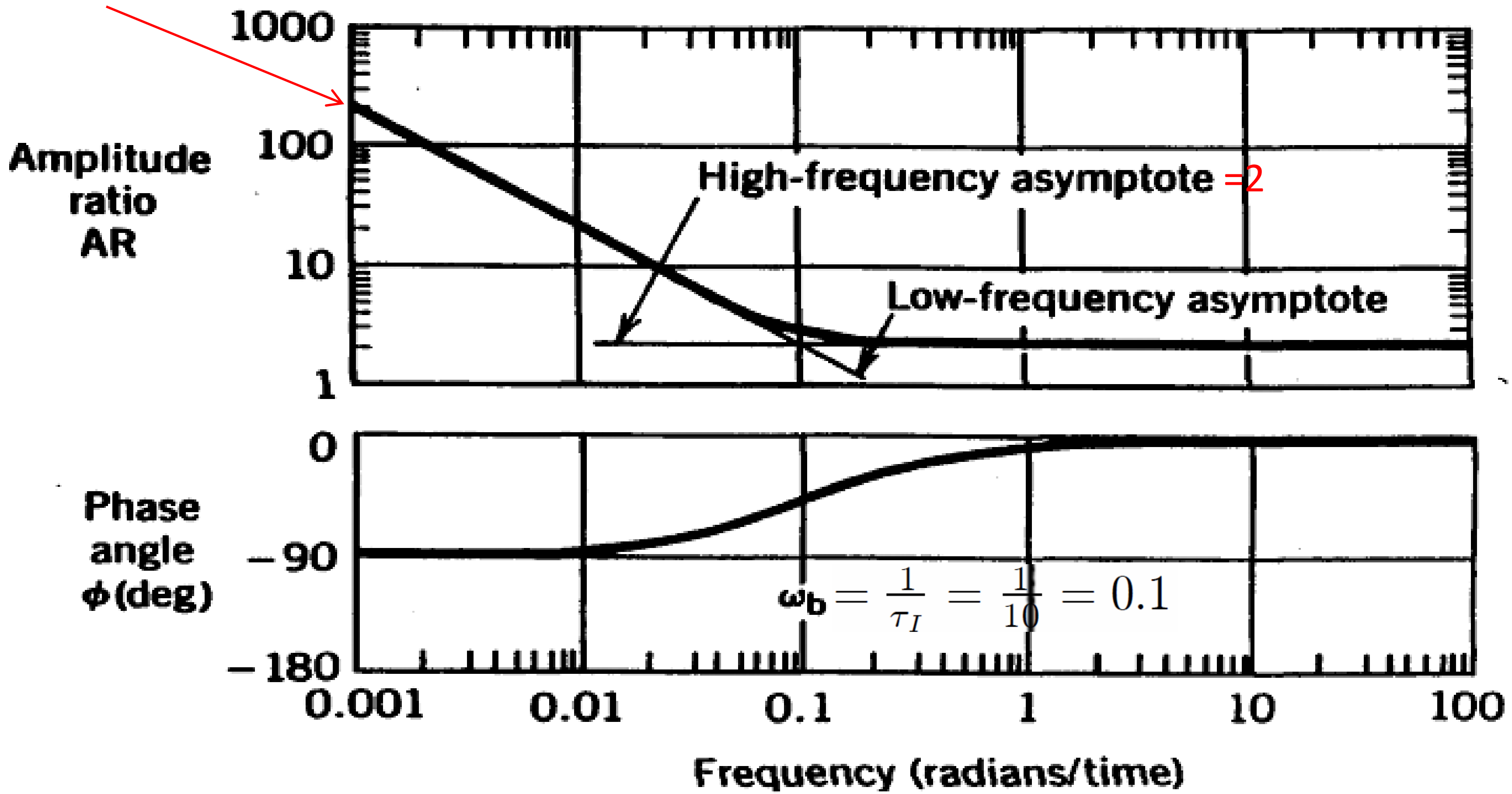



**Figure 13.9.** Bode plot for PI controller, $G_c(s) = 2\left(1 + \frac{1}{10s}\right)$

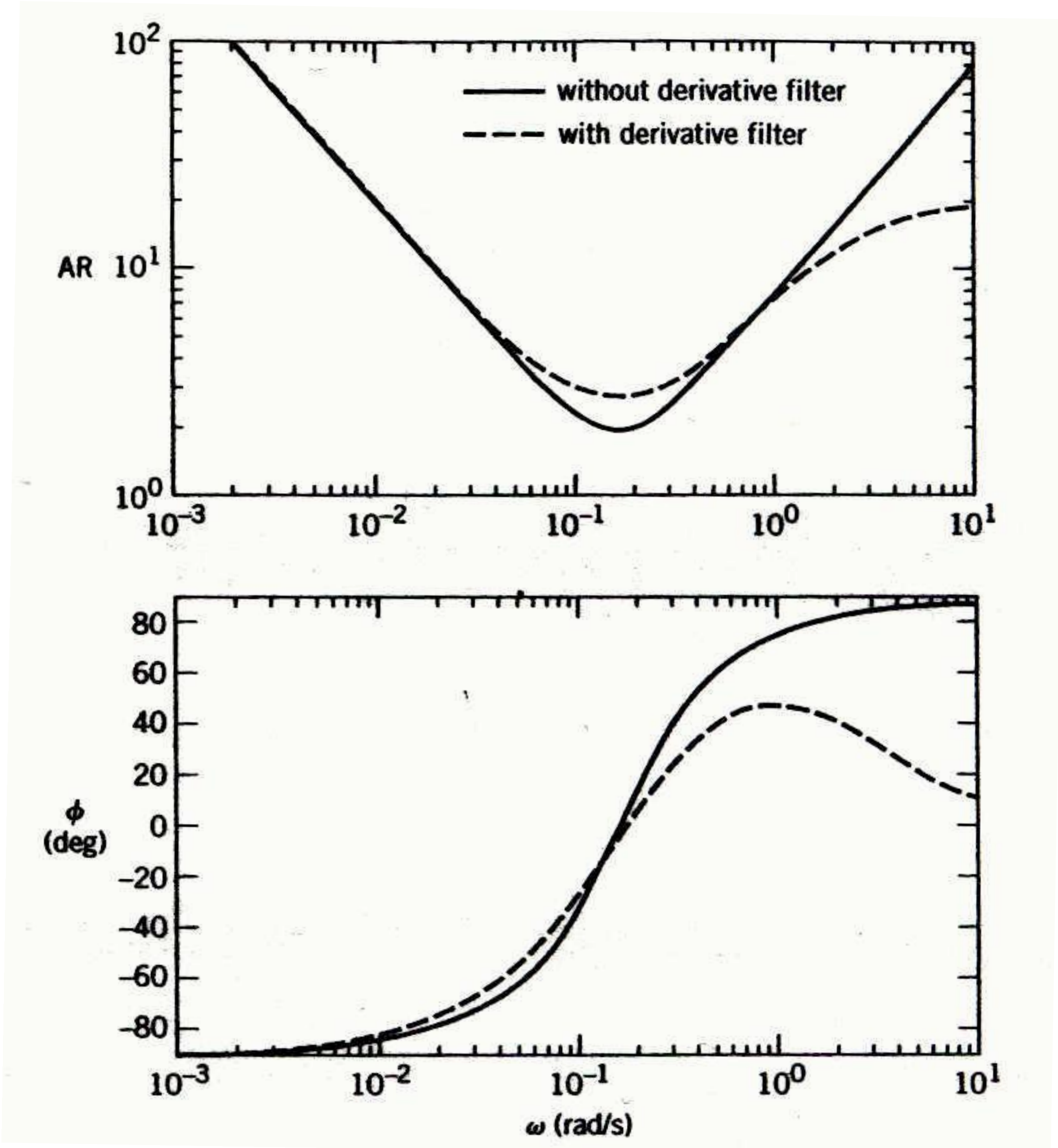



Figure 14.6 Bode plots of ideal parallel PID controller and series PID controller with derivative filter ($\alpha = 0.1$).

Ideal parallel:

$$G_c(s) = 2\left(1 + \frac{1}{10s} + 4s\right)$$

Series with Derivative Filter:

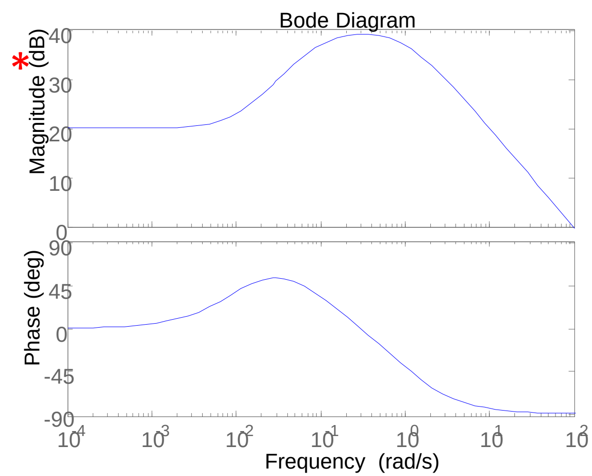
$$G_c(s) = 2\left(\frac{10s+1}{10s}\right)\left(\frac{4s+1}{0.4s+1}\right)$$

# Electrical engineers (and Matlab) use decibel for gain

- $|G|\ [dB] = 20 \log_{10}|G|$

| $\|G\|$ | $\|G\|$ [dB] |
|---|---|
| 0.1 | -20 dB |
| 1 | 0 dB |
| 2 | 6 dB |
| 10 | 20 dB |
| 100 | 40 dB |
| 1000 | 60 dB |

```
s=tf('s')
g = 10*(100*s+1)/[(10*s+1)*(s+1)]
bode(g) % gives AR in dB*
```




*To change magnitude from dB to abs: Right click + properties + units (absolute, log scale)

Other way: $|G| = 10^{|G|(dB)/20}$

GM=2 is same as GM = 6dB

# CLOSED-LOOP STABILITY

- $L = gg_m$ = loop transfer function with negative feedback
- **Bode's stability condition**: $|L(\omega_{180})|<1|$
  - Limitations
    - Open-loop stable (L(s) stable)
    - Phase of L crosses -180° only once

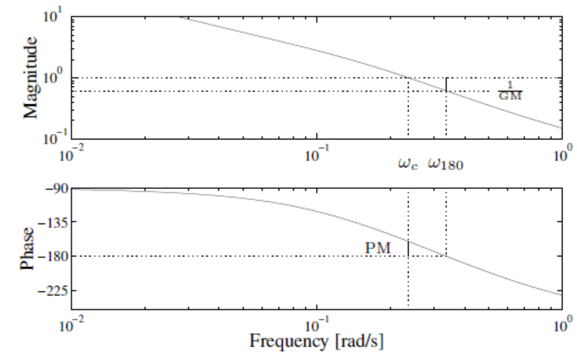
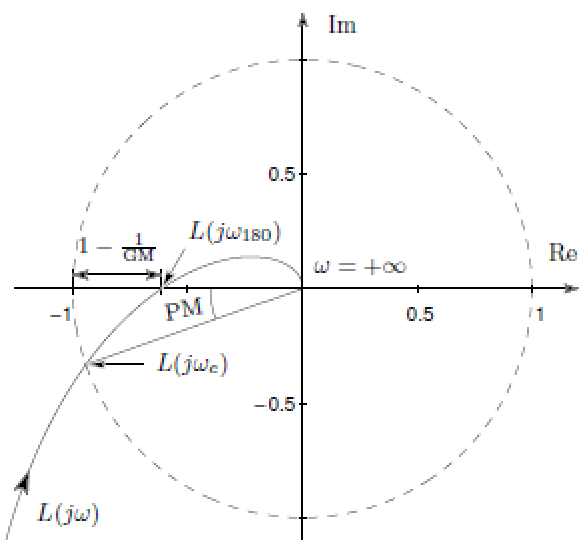



**Figure 2.12**: Typical Bode plot of $L(j\omega)$ with PM and GM indicated

- The same but more general: **Nyquist stability condition**:

Locus of $L(j\omega)$ should encircle the (-1)-point P times in the anti-clockwise direction (where P = no. of unstable poles in L).




Stable plant (P=0): Closed-loop stable if L has no encirclements of -1 (=Bode's stability condition)

# Proof of Bode stability condition

- Starting point: Stability is a system property for linear systems, so if the system is stable for one signal it's stable for all signals.
- Consider a particular signal: Sinusoid with frequency $\omega_{180}$.
- With negative feedback, the total phase shift around the loop is $-360^o$, so this sinusoid comes «back in phase»
- If the gain around the loop is less than 1, the sinusoid will die out.
- Conclusion: The closed-loop system is stable if and only if $|L(jw)|<1$ at frequency $\omega_{180}$

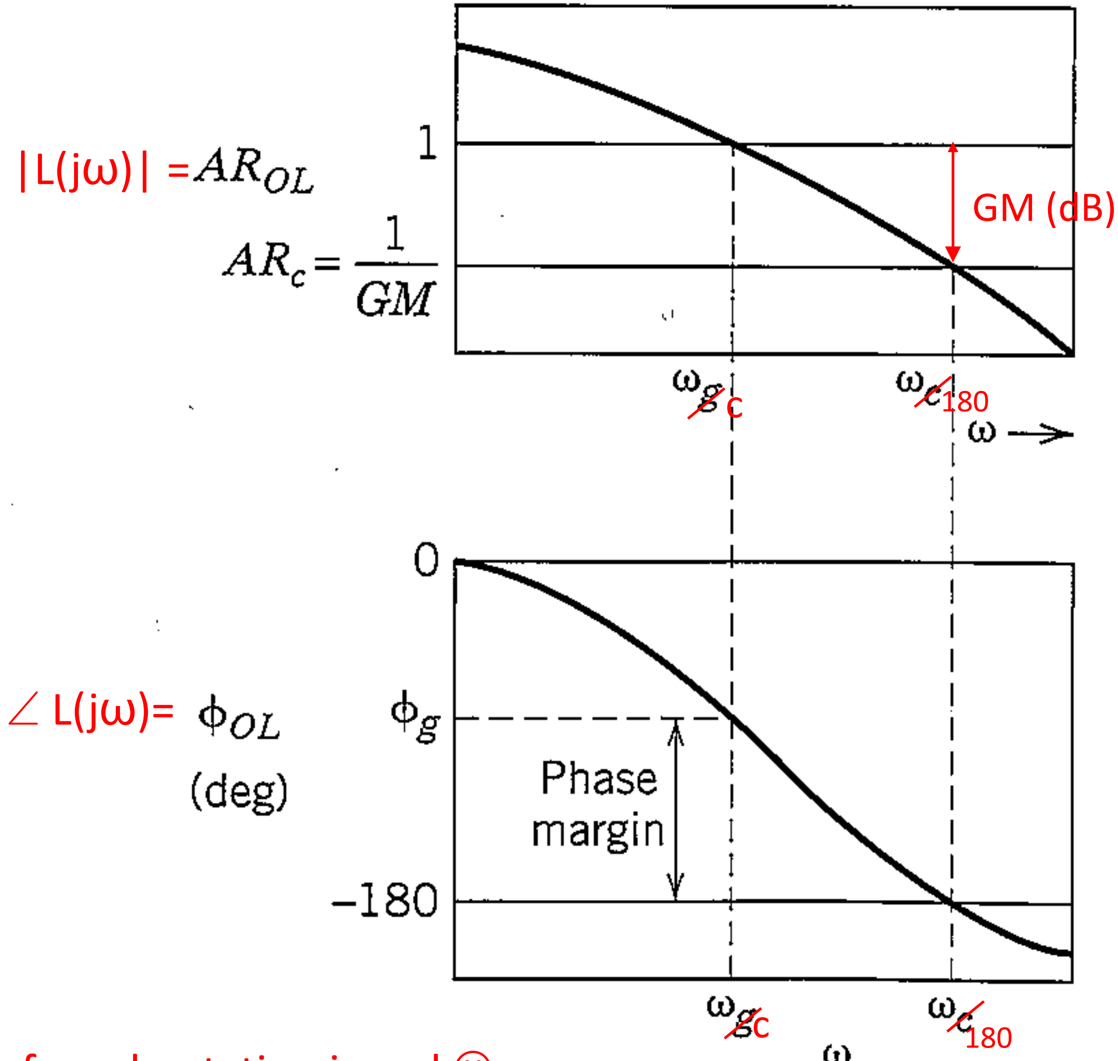
- Example 1. P-control of delay process, $g(s)=ke^{-\theta s}$. For what $K_c$ is system stable?
- Example 2. I-control of delay process. For what $K_I$ is system stable? compare with SIMC. Is SIMC robust?

Solution. Stable if and only if

1. P-control: $kK_c < 1$
2. I-control: $kK_I < \frac{\pi}{2}\frac{1}{\theta}$

Note: SIMC with $\tau_c=\theta$ gives I-control with $kK_I = \frac{1}{2\theta}$,

1. So Gain Margin (GM) $=\pi = 3.14$ (worst is 1 = 0dB),
2. Unstable if we increase delay from $\theta$ to $\pi\theta$, so Time Delay Margin (DM)= $(\pi-1)\theta$ (worst is 0)




Sigurd's preferred notation in red ☺

**Figure 14.12** Gain and phase margins on a Bode plot

$GM = 1/L(j\omega_{180})$

$\omega_{180}$ = frequency where phase shift around the loop is -180° = - π rad.

$\angle L(j\omega_{180}) = -180^\circ = -\pi$ rad

$PM = \angle L(j\omega_c) + 180^\circ$

$= \angle L(j\omega_c) + \pi$ [rad]

$\omega_c$ = frequency where loop gain is 1.

$|L(j\omega_c)| = 1$

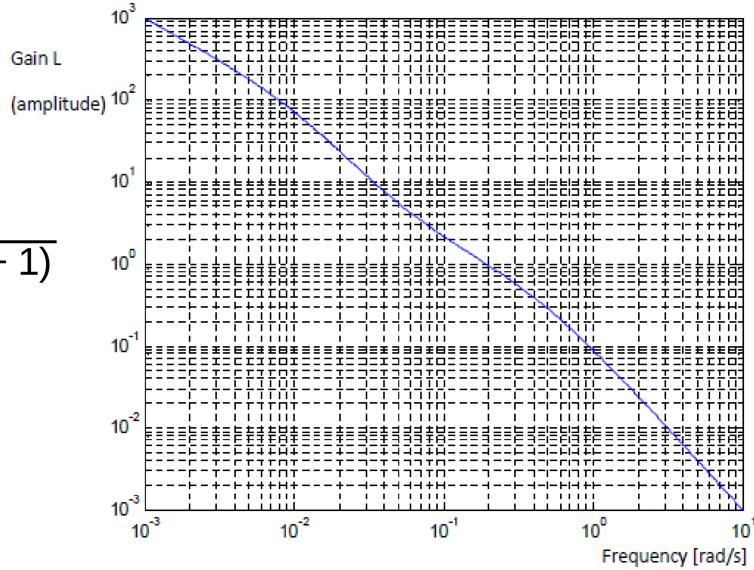
Time delay margin (DM), $\Delta\theta$= PM[rad]/$\omega_c$

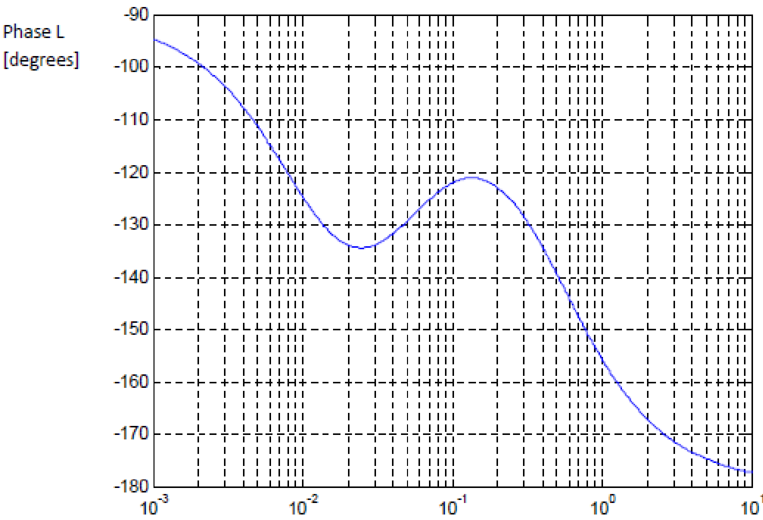
TASK 1: Bode-plot of L(s) = (20s+1)/[s(100s+1)(2s+1)] . Write on the asymptotes
TASK 2: What is GM and PM?
TASK 2: How does the plot change if we add a delay of 2 time units ( $e^{-2*s}$)
TASK 3: What is now GM and PM? How much extra time delay can we allow?

# EXAMPLE 3

$$L(s) = \frac{20s+1}{s(100s+1)(2s+1)}$$

L(s): SIMC PI-control with $\tau_c$=4 for
g(s) = 1/(100s+1)(2s+1)



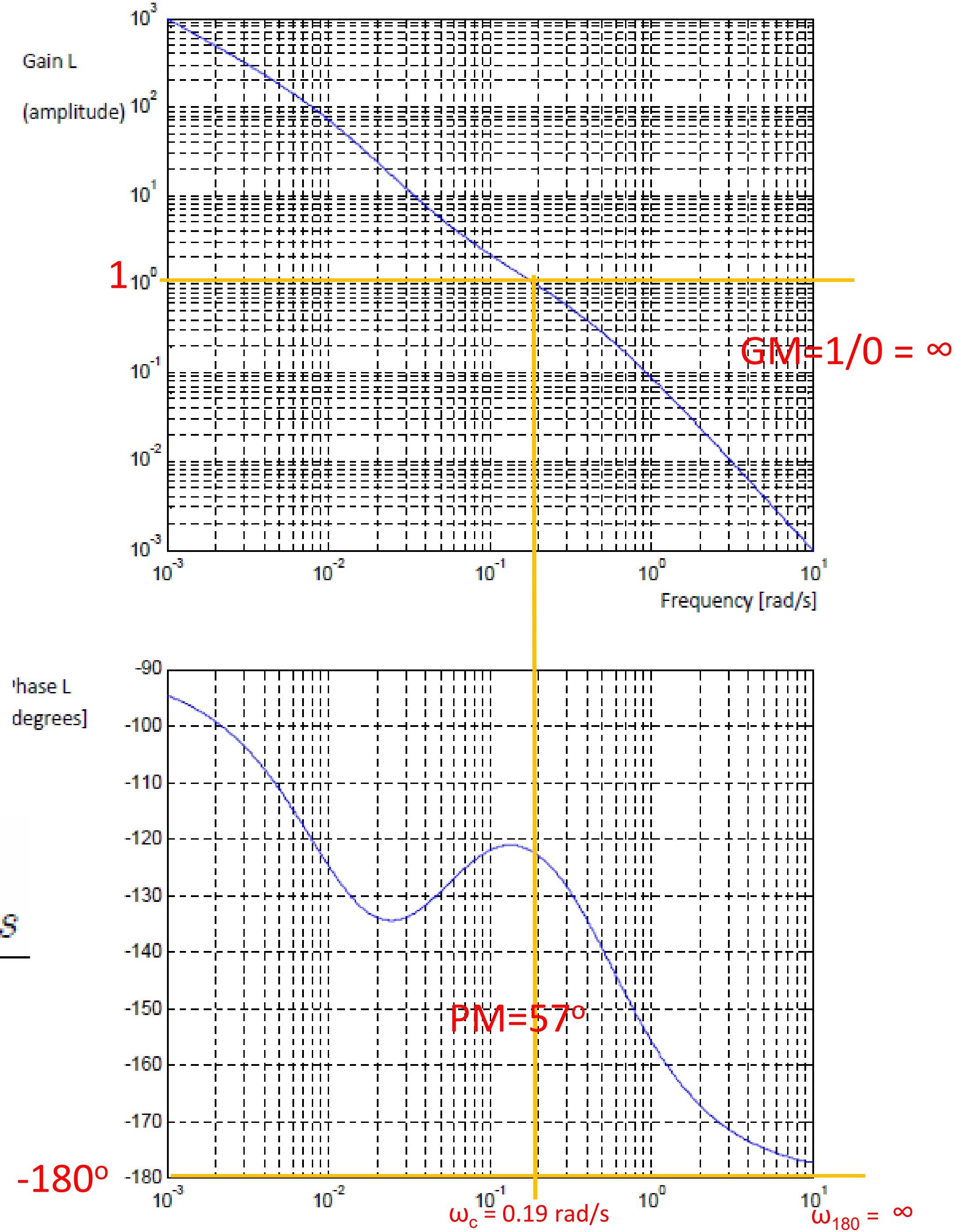




```
s=tf('s')
L = (20*s+1)/[s*(100*s+1)*(2*s+1)]
figure(3), bode(L) % gives AR in dB
w = logspace(-3,1,1000)
[mag,phase]=bode(L,w)
figure(1), loglog(w,mag(:)), grid on, axis([0.001 10 0.001 1000])
figure(2), semilogx(w,phase(:)), grid on
```

SOLUTION

$$L(s) = \frac{20s+1}{s(100s+1)(2s+1)}$$

L(s): SIMC PI-control with $\tau_c$=4 for g(s) = 1/(100s+1)(2s+1)



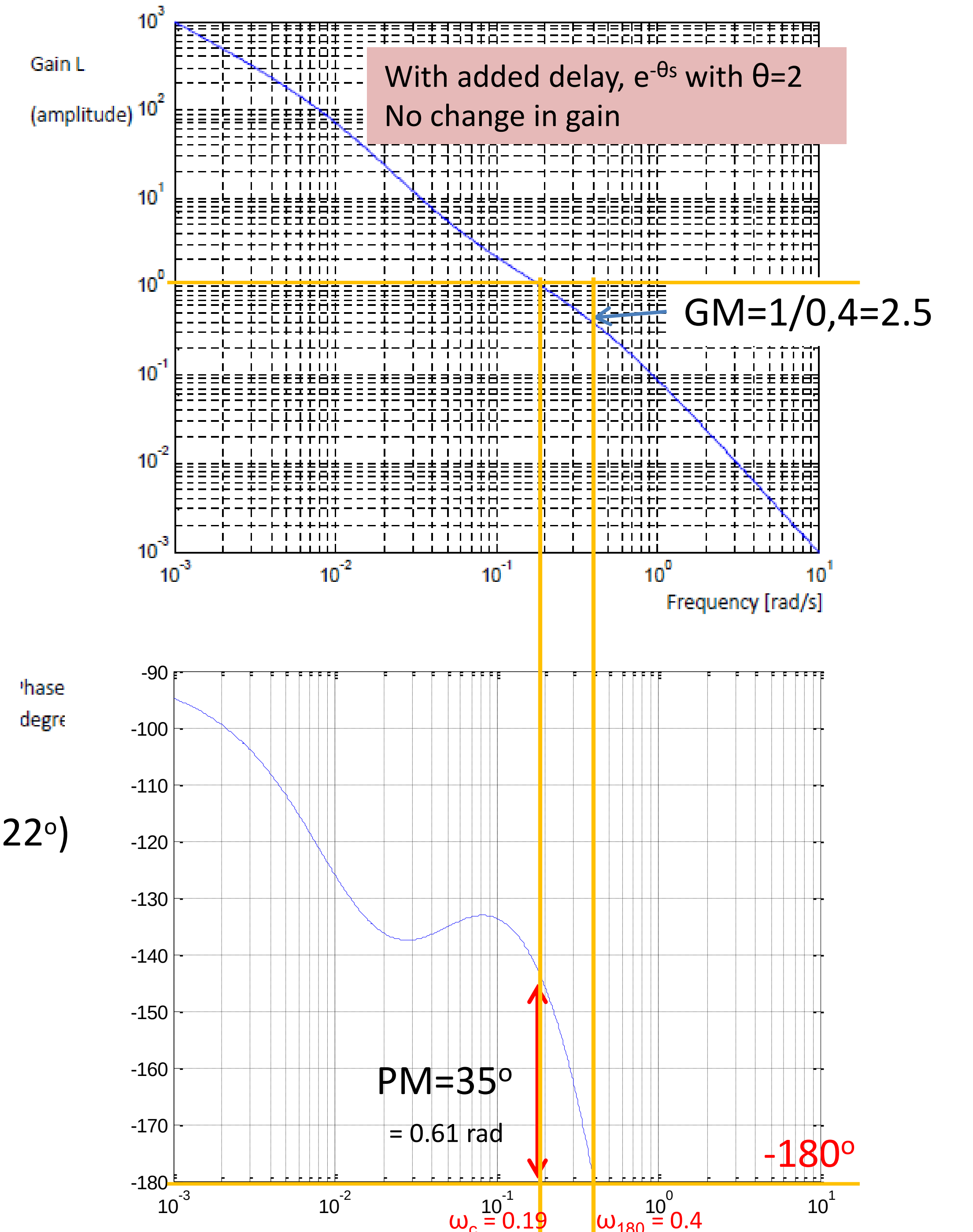

Time delay margin

$$\Delta\theta = \frac{PM[rad]}{\omega_c[rad/s]} = \frac{1}{0.19} = 5.2s$$

# EXAMPLE3': ADD 2 UNITS OF DELAY

$$L = \frac{20s+1}{s(100s+1)(2s+1)} e^{-2s}$$

Now phase crosses -180° so
GM is no longer infinity




Phase addition from delay = $-\omega\theta$

At $\omega_c$: $- -\omega_c\theta = 0.19*2 = -0.38$ rad (-22°)

So new PM = 57° (old) – 22° = 35°

New time delay margin

$$\Delta\theta = \frac{PM[rad]}{\omega_c[rad/s]} = \frac{0.61}{0.19} = 3.2s$$

# Example 4. PI-control of integrating process with delay. Compare ZN and SIMC*

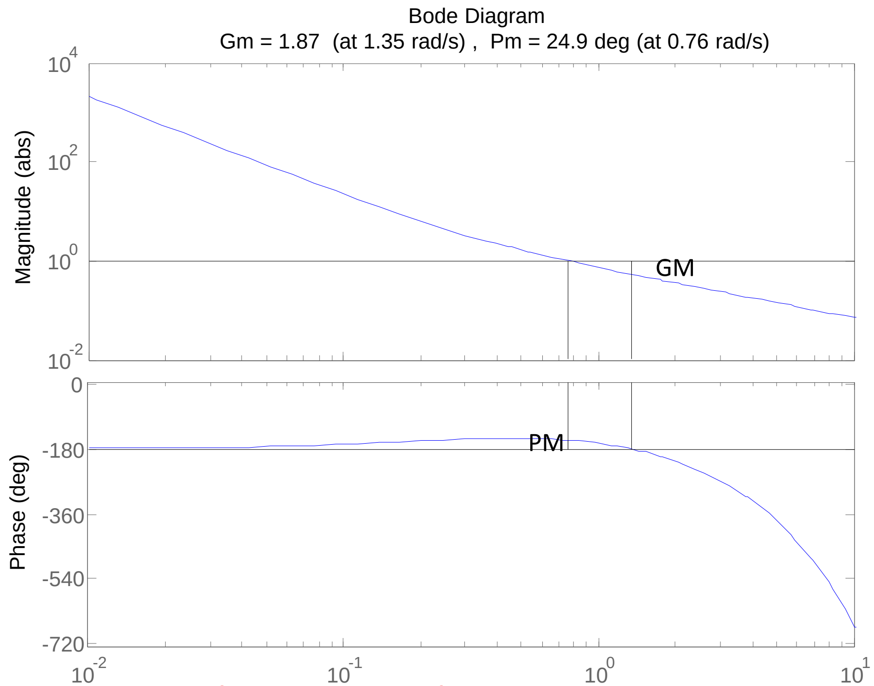
- $g(s) = k' e^{-\theta s}/s$
- ZN: Use P-control and increase $K_c$ until instability. Find:
  - $P_u = 4\theta$ and $K_u = (\pi/2)/(k'\theta)$
- PI-controller, $c(s) = K_c (1+1/(\tau_I s))$

| | $K_c$ | $\tau_I$ |
|---|---|---|
| Ziegler-Nichols | $0.45K_u = 0.707/(k'\theta)$ | $P_u/1.2 = 3.33\theta$ |
| SIMC ($\tau_c=\theta$) | $0.5/(k'\theta)$ | $8\theta$ |

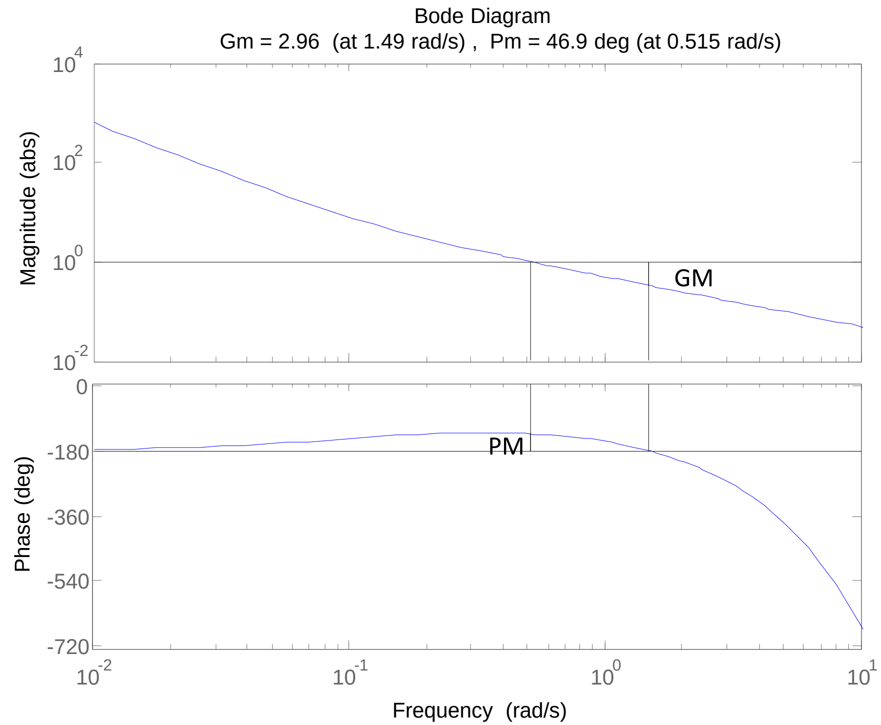
*Task: Compare Bode-plot (L=gc), robustness and simulations (use $k'=1$, $\theta=1$).

$$g(s) = \frac{e^{-s}}{s}, \quad c(s) = K_c \frac{\tau_I s + 1}{\tau_I s}$$

## Ziegler-Nichols PI




## SIMC-PI




## SIMC is a lot more robust:

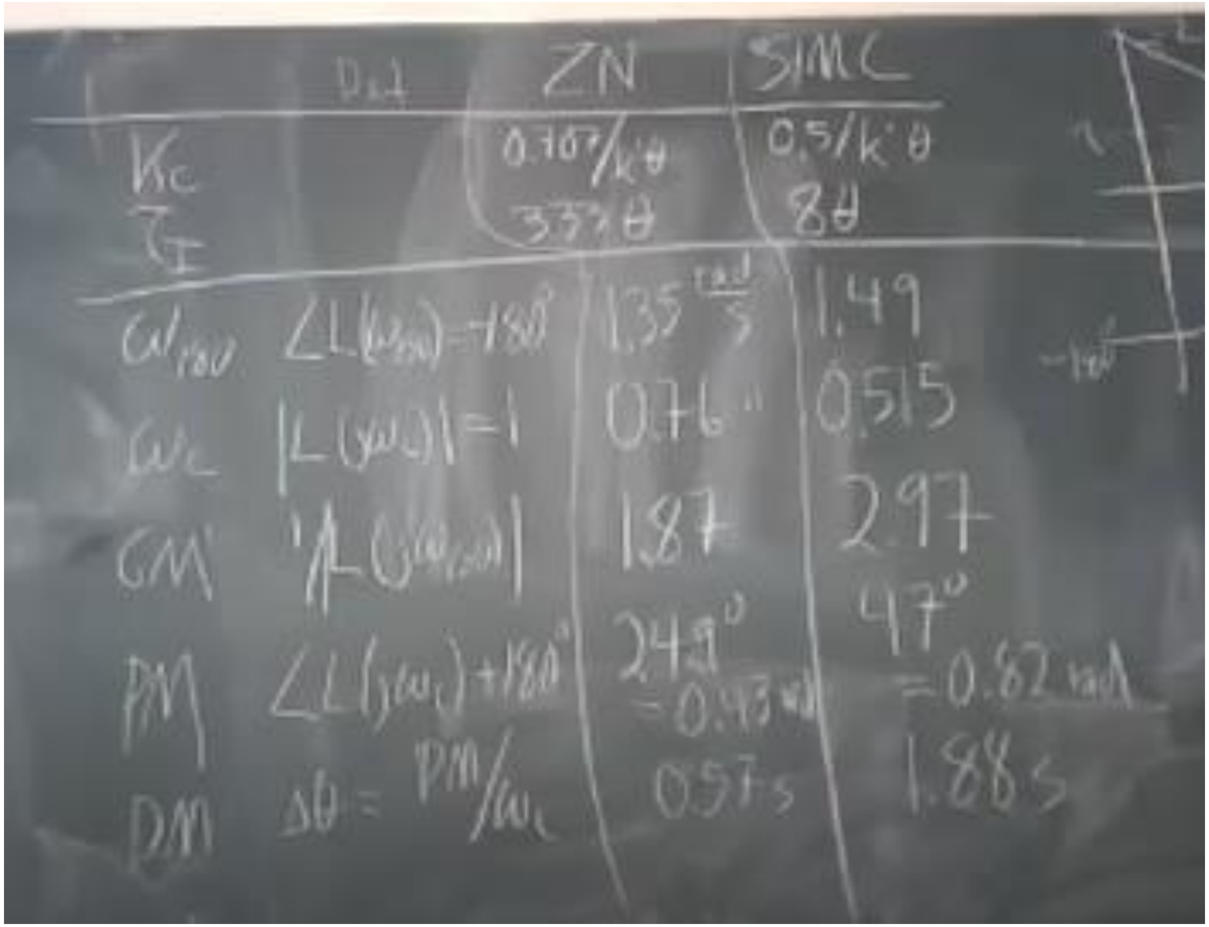
| | GM | PM | Delay margin, $\Delta\theta$ |
|---|---|---|---|
| Ziegler-Nichols | 1.87 | 24.9$^o$ | 0.57 s |
| SIMC ($\tau_c=\theta$) | 2.97 | 46.9$^o$ | 1.88 s |

```
s=tf('s')
g = exp(-s)/s
Kc=0.707, taui=3.33
c = Kc*(1+1/(taui*s))
L1 = g*c
figure(1), margin(L1) % Bode-plot with margins
% To change magnitude from dB to abs: Right click + properties + units
Kc=0.5, taui=8
c = Kc*(1+1/(taui*s))
L2 = g*c
figure(2), margin(L2)
```

$\Delta\theta$= PM[rad]/$\omega_c$

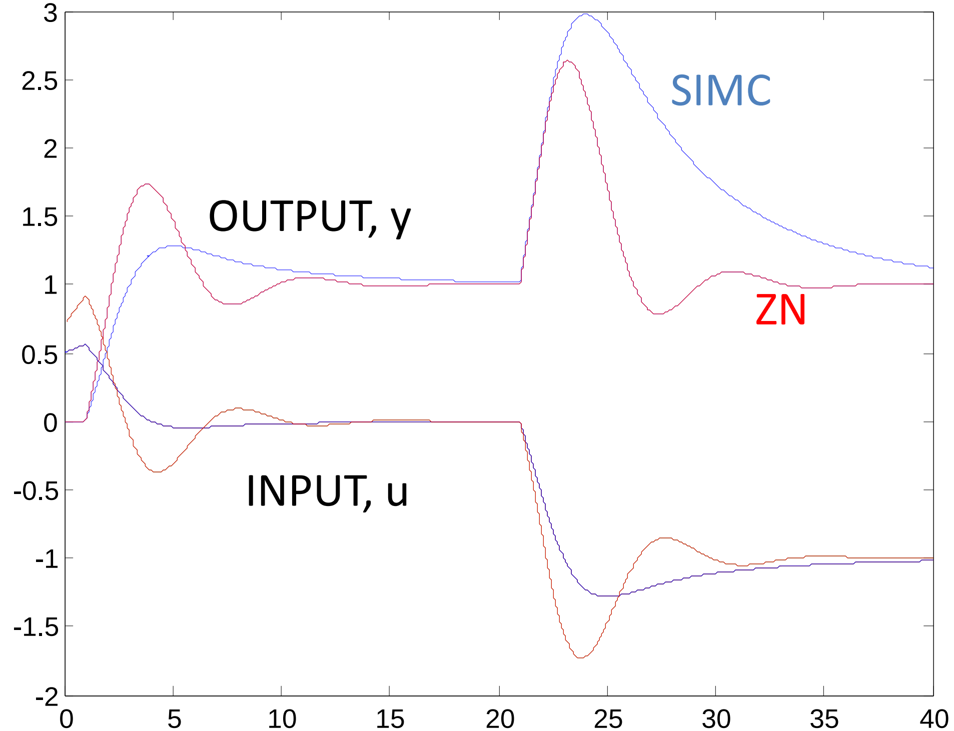
ZN: $\Delta\theta$ = 24.9*(3.14/180)/0.76 = 0.572s

SIMC: $\Delta\theta$ = 46.9*(3.14/180)/0.515 = 1.882s




Delay = 1

# Closed-loop response: PI-control of $g(s) = \frac{e^{-s}}{s}$



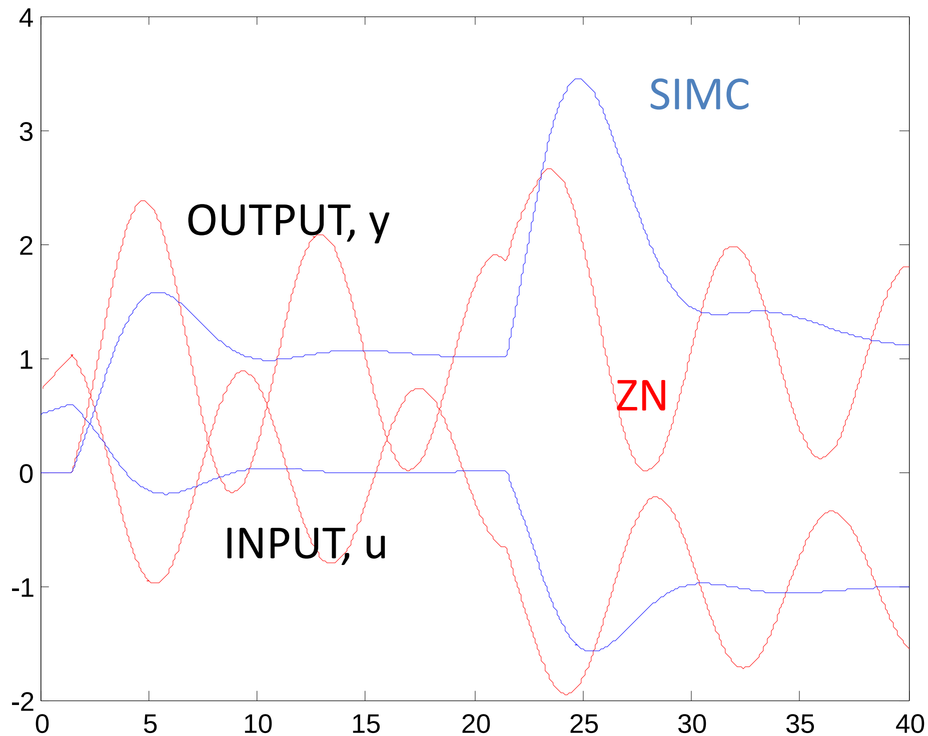

t=0: setpoint change, t=20: input disturbance

```
Simulink file: tunepid4
s=tf('s')
g = exp(-s)/s
Kc=0.707, taui=3.33, taud=0 % ZN
sim('tunepid4')
plot(Tid,y,'red',Tid,u,'red')
Kc=0.5, taui=8, taud=0 % SIMC
sim('tunepid4')
hold
plot(Tid,y,'blue',Tid,u,'blue')
hold off
```

Conclusion: Ziegler-Nichols (ZN) responds faster to the input disturbance, but is much less robust.

- ZN goes unstable if we increase delay from 1s to 1.57s.
- SIMC goes unstable if we increase delay from 1s to 2.88s.

# INCREASE DELAY: $g(s) = \frac{e^{-1.5s}}{s}$




ZN is almost unstable when the delay is increased from 1s to 1.5s.
SIMC does not change very much

# Back to Example 2.
# I-control of delay process

- Find GM, PM and DM for SIMC-controller (analytical)

- Example 2. I-control of delay process. & what is $\omega_c$, $\omega_{180}$, GM, PM and DM & give for SIMC (analytical)

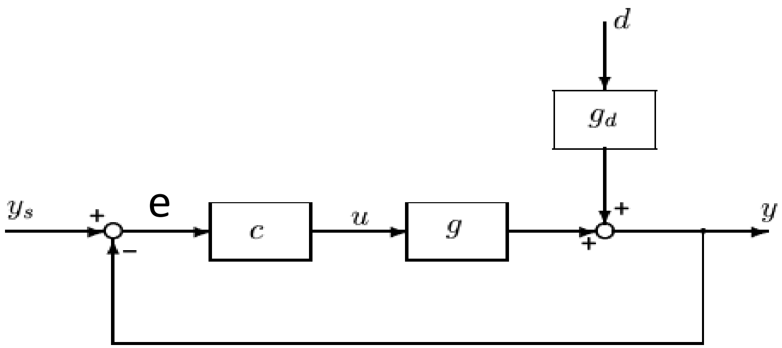
Solution
wc=k' KI, w180=(pi/2)(1/theta).
GM = w180/k'KI = (pi/2)/(k'KI theta),
PM=(pi/2)- k'KI*theta,
DM = PM/wc = (pi/2)/k'KI - theta

SIMC:
SIMC with $\tau_c=\theta$ gives $kK_I = \frac{1}{2\theta}$, so
GM $=\pi$ = 3.14.
PM = (pi-1)/2 = 1.07 rad = $61.5^o$
DM = (pi-1)*theta = 2.14 theta

# Bode stability condition. Why may D-action help in some cases?

- **Some unstable processes, for example a double integrating process, may need D-action for stabilization.** The reason is to add positive phase and therefore stabilize the system. Why does this help?
  - Recall the Bode stability condition. It says that the loop gain should be less than 1 at the frequency where the phase shift around the loop -180 degrees.
  - Another statement is that phase shift should be less than -180$^o$ at the frequency where the loop gain is 1.
  - So for stability and robustness we want as little phase shift as possible (to improve the phase margin). The things that add negative phase shift are time delay (the worst), poles and RHP-zeros.
  - LHP-zeros (D-action, (Td*s +1)) have the opposite (positive) effect on the phase, and this is why they may be added for stabilization in difficult cases, for example, an unstable process. Of course, zeros will also affect the loop gain, but at frequencies up to the break frequency, 1/Td, the positive effect on the phase is most important.
  - **So why don't we always add D-action?** One reason is that it increases the controller gain and therefore the input usage. However, the main reason is probably that it does not help very much in most cases and it makes the controller design more complicated (and easier to do mistakes).

# Closed-loop frequency response




No control ($c = 0$): $e_{OL} = y_s - y = y_s - g_d d$

With control: $e = y_s - y = S y_s - S g_d d = S e_{OL}$

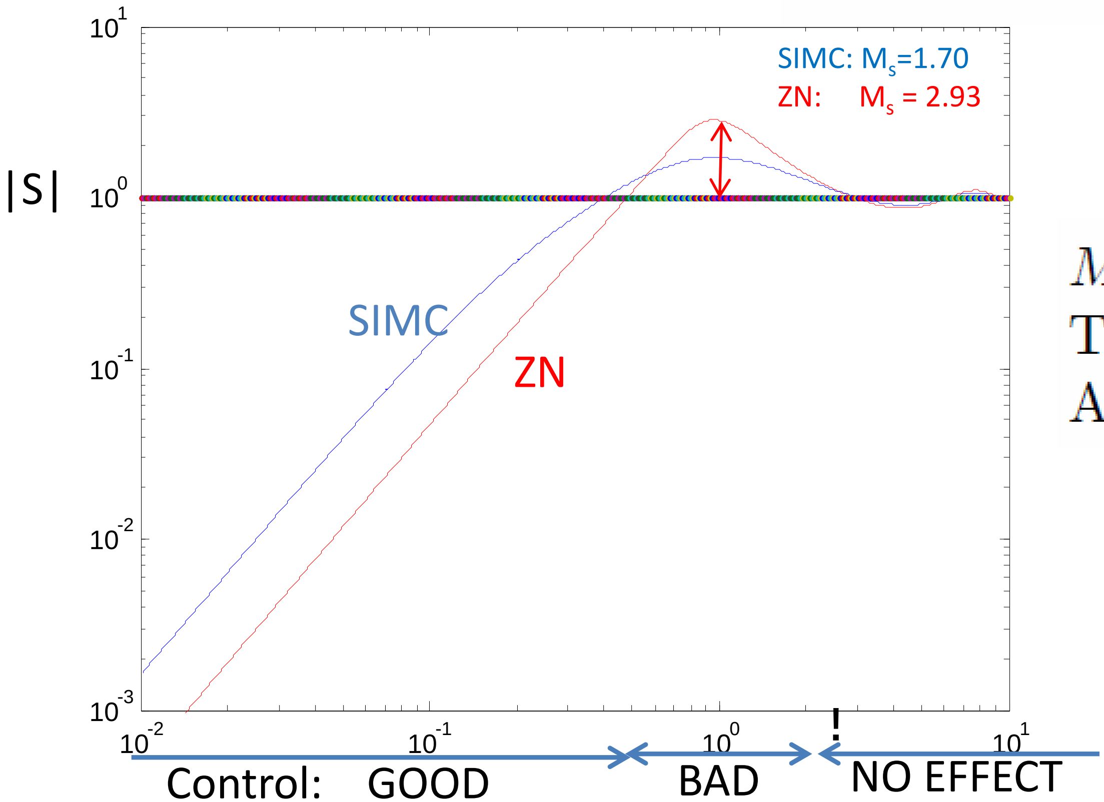
$S = \frac{1}{1+L}$ - sensitivity function = effect of control

$L = gc$ - loop transfer function

Low $\omega$ where $|S| < 1$: Control reduces error

Intermediate $\omega$ where $|S| > 1$: Control increases error

High $\omega$ where $S = 1$ $(L \to 0)$: Control has no effect




$M_s$ = peak of $|S|$

Typical requirement: $M_s < 2$

At stability limit: $M_s \to \infty$

```
w = logspace(-2,1,1000);
[mag1,phase]=bode(1/(1+L1),w);
[mag2,phase]=bode(1/(1+L2),w);
figure(1), loglog(w,mag1(:),'red',w,mag2(:),'blue',w,1,'-.')
axis([0.01,10,0.001,10])
```

# Example Ziegler Nichols

Task: Find ZN-settings for integrating+ delay process analytically

- First need to find Pu and Ku