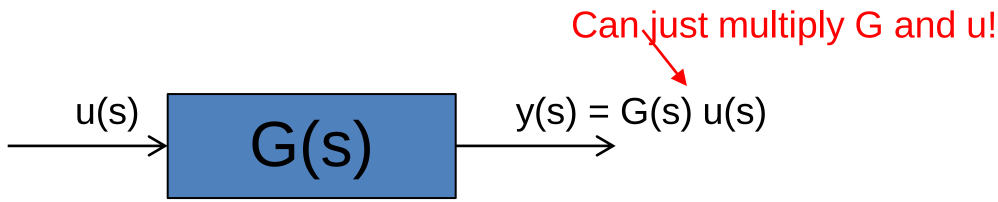
General procedure in this course

1. Nonlinear dynamic model. $dx/dt = f(x,u,d)$
2. Steady state model. $dx^*/dt=0 \rightarrow f(x^*,u^*,d^*)=0$
   – Use to find missing data
3. Introduce deviation variables and linearize
   – $d\Delta x/dt = \Delta f = A\,\Delta x(t) + B\,\Delta u(t) + B_d\,\Delta d(t)$
4. Laplace of both sides of linear model* ($t \rightarrow s$)
   – $sx(s) = A\,x(s) + B\,u(s) + B_d\,d(s)$
5. Algebra $\rightarrow$ Transfer function, $G(s)$
6. Block diagram
7. Controller design

*Note: We will only use Laplace for linear systems!

Transfer function



G(s) = transfer function of linear dynamic system
u and y: deviation variables
s: Laplace variable (replaces t as independt variable).
Note (may be confusing): s has units s^{-1} = second^{-1}

Some typical transfer functions:

1. First-order with delay process, $G(s)=k\, e^{-\theta s}/(\tau s + 1)$
   – Many examples! Heated tank, y=T, u=Q
2. Integrating process, $G(s)=k'/s$
   – Example: level (y) with u=qin
3. PID-controller , $C(s) = K_c(1 + 1/(\tau_I s) + \tau_D\, s)$

$$= K_c \frac{\tau_I \tau_D s^2 + \tau_I s + 1}{\tau_I s}$$

General* Transfer Matrix

General system with n differential equations in n state variables $x(t)$ (where x, u, y are vectors and A, B, C, D are matrices):

$$\frac{dx(t)}{dt} = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$

Laplace transform with zero intitial condition, $x(0) = 0, u(0) = 0$ (deviation variables):

$$sI\ x(s) = A\,x(s) + B\,u(s)$$
$$(sI - A)\,x(s) = B\,u(s)$$
$$x(s) = (sI - A)^{-1}\ B\,u(s)$$

Get $y(s) = G(s)u(s)$ where transfer matrix is:

$$G(s) = C\ (sI - A)^{-1}\ B + D$$

Here

$$(sI - A)^{-1} = \frac{adj(sI - A)}{det(sI - A)}$$

where $det(SI - A) =$

$$d(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

is a n'th order polynomial in n,

The n roots (generally complex) of the polynomial d(s) are the same as the eigenvalues of the state matrix A, and are known as the «poles» of the system

*Warning: Not completely general. Does not include time delay, which cannot be written as a polynomial in s.

Initial and final values for step response

- Transfer function g(s)
- Consider response y(t) to step of magnitude M in input. u(s)=M/s
- Deviation variables for y(t) and u(t)

$$\text{Steady-state gain: } \frac{y(\infty)}{M} = g(0)$$

$$\text{Initial gain: } \frac{y(0^+)}{M} = g(\infty)$$

$$\text{Initial slope: } \frac{y'(0^+)}{M} = \lim_{s\to\infty} sg(s)$$

Proof: Note that $y(s) = g(s)\frac{M}{s}$

Final value theorem: $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sy(s) = \lim_{s\to 0} sg(s)\frac{M}{s} = g(0)M$

Initial value theorem: $\lim_{t\to 0} y(t) = \lim_{s\to\infty} sy(s) = g(\infty)M$

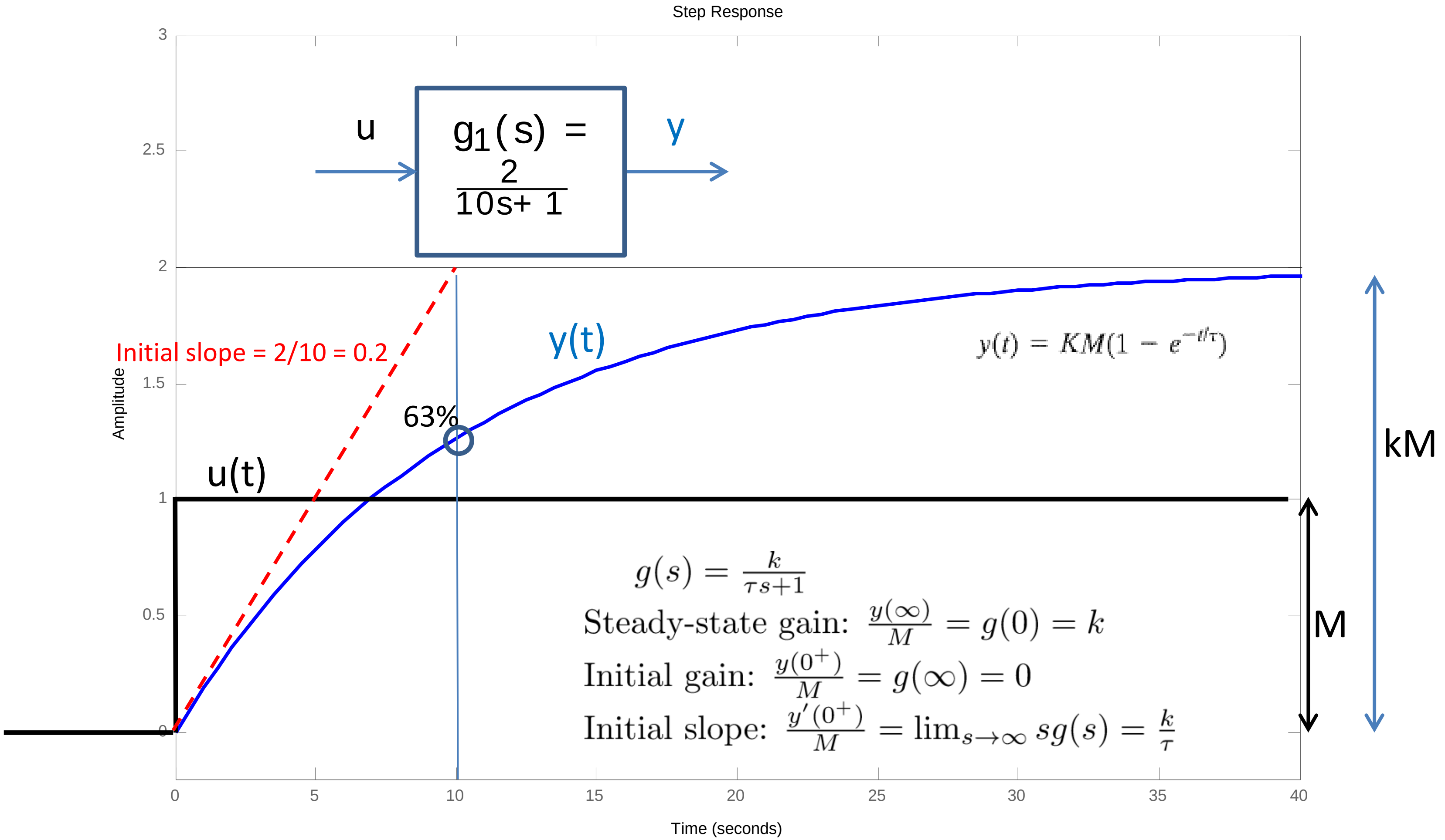
Initial value theorem: $\lim_{t\to 0} y'(t) = \lim_{s\to\infty} s(sy(s)) = \lim_{s\to\infty} sg(s)M$

Initial value theorem: $\lim_{t\to 0} y^{(n)}(t) = \lim_{s\to\infty} s^n(sy(s)) = \lim_{s\to\infty} s^n g(s)M$

First-order system responses

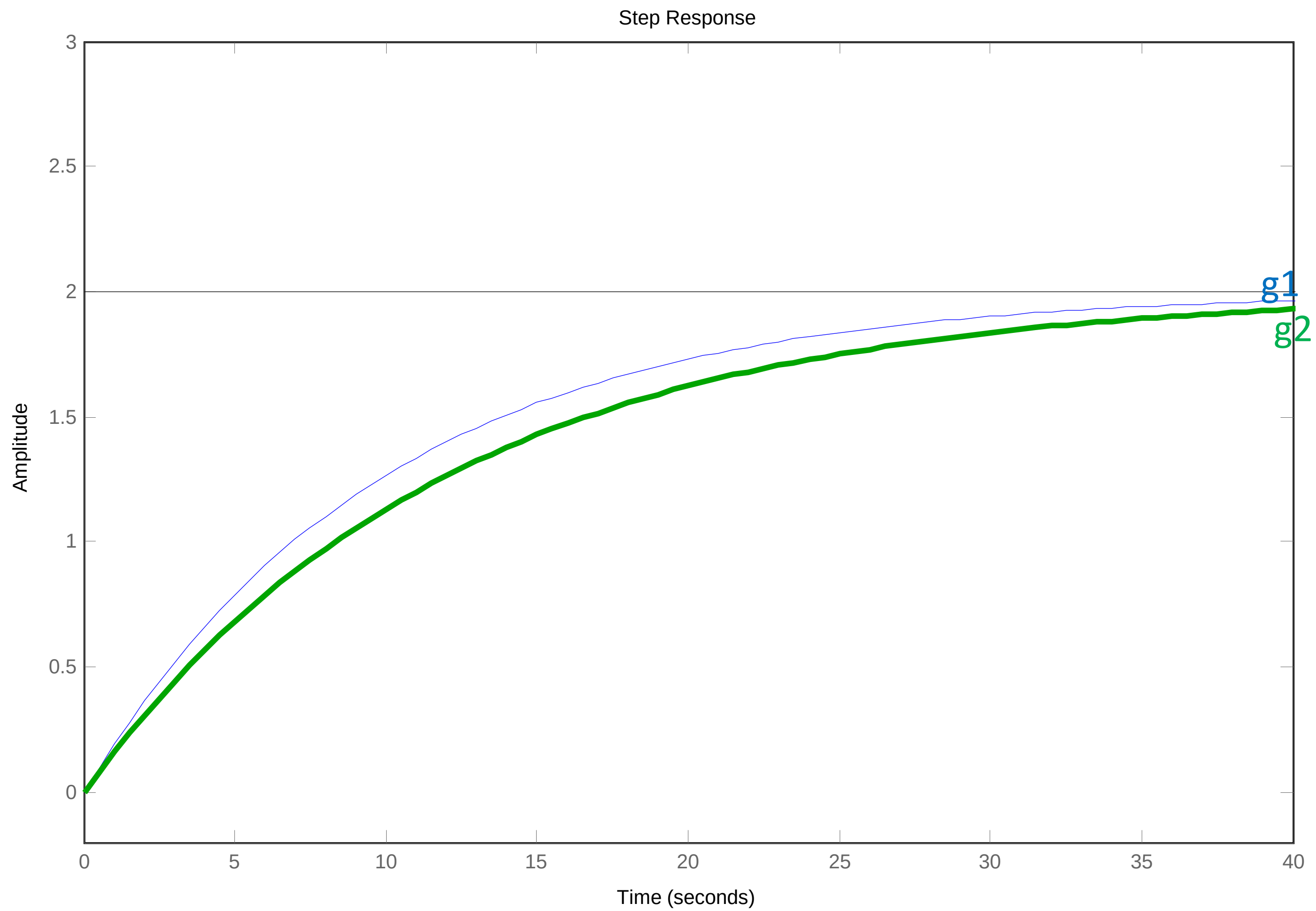
Example g1 (k=2, τ=10)

```
s=tf('s')
g1 = 2/(10*s+1)
step(g1,50)
axis([0 40 -0.2 3])
```



Change time constant from 10 (g1, blue) to 12 (g2, green)
…gives smaller initial slope & slower dynamics

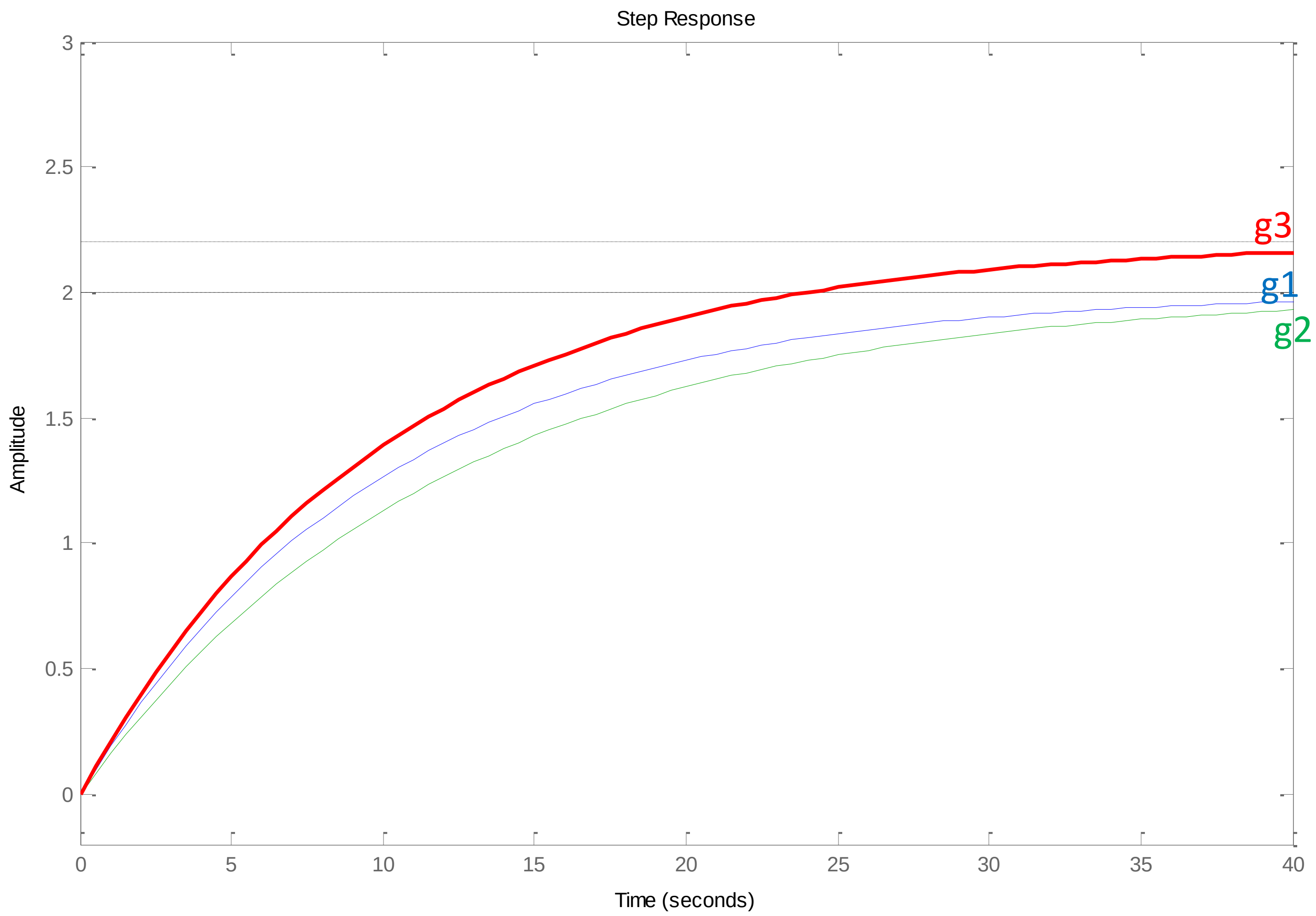
```
s=tf('s')
g1 = 2/(10*s+1), step(g1,50)
axis([0 40 -0.2 3]); hold on,
g2 = 2/(12*s+1), step(g2,50)
```



g3: Larger steady-state gain (k=2.2) (red).

Gives larger initial slope (but dynamics are <u>not</u> faster than g1, because also steady-state is larger)

```
s=tf('s')
g1 = 2/(10*s+1), step(g1,50)
axis([0 40 -0.2 3]); hold on,
g2 = 2/(12*s+1), step(g2,50)
g3 = 2.2/(10*s+1), step(g3,50)
```



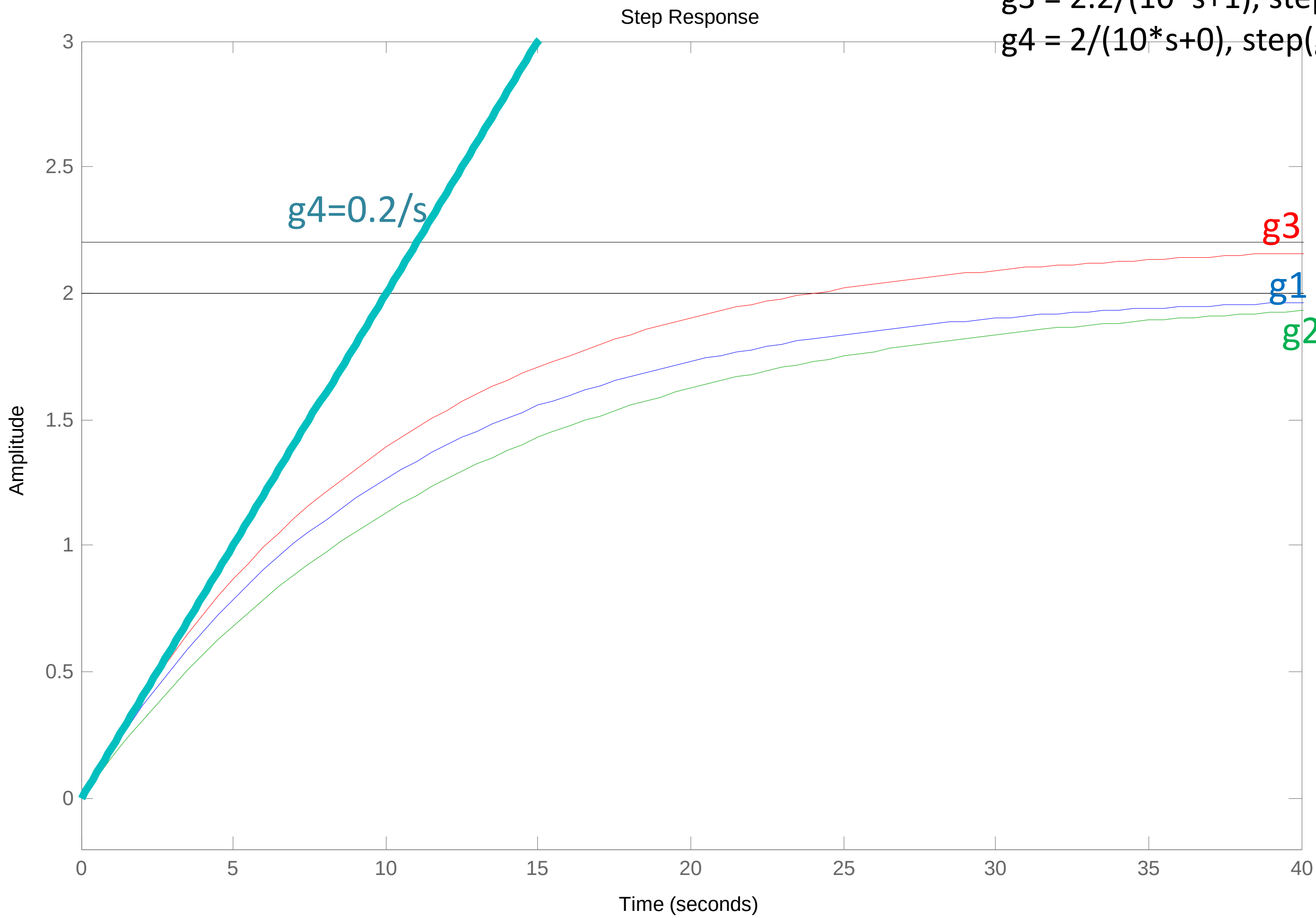
Integrating system, g(s)=k'/s

- Special case of first-order system with $\tau=\infty$ and k=∞ but slope $k'=k/\tau$ is finite
- Large τ: $g(s)=k/(\tau s+1) \approx k/(\tau s) = k'/s$
- Step response (u=M): $y(t)/M = k't$ (ramp)

g4: Integrating system =0.2/s

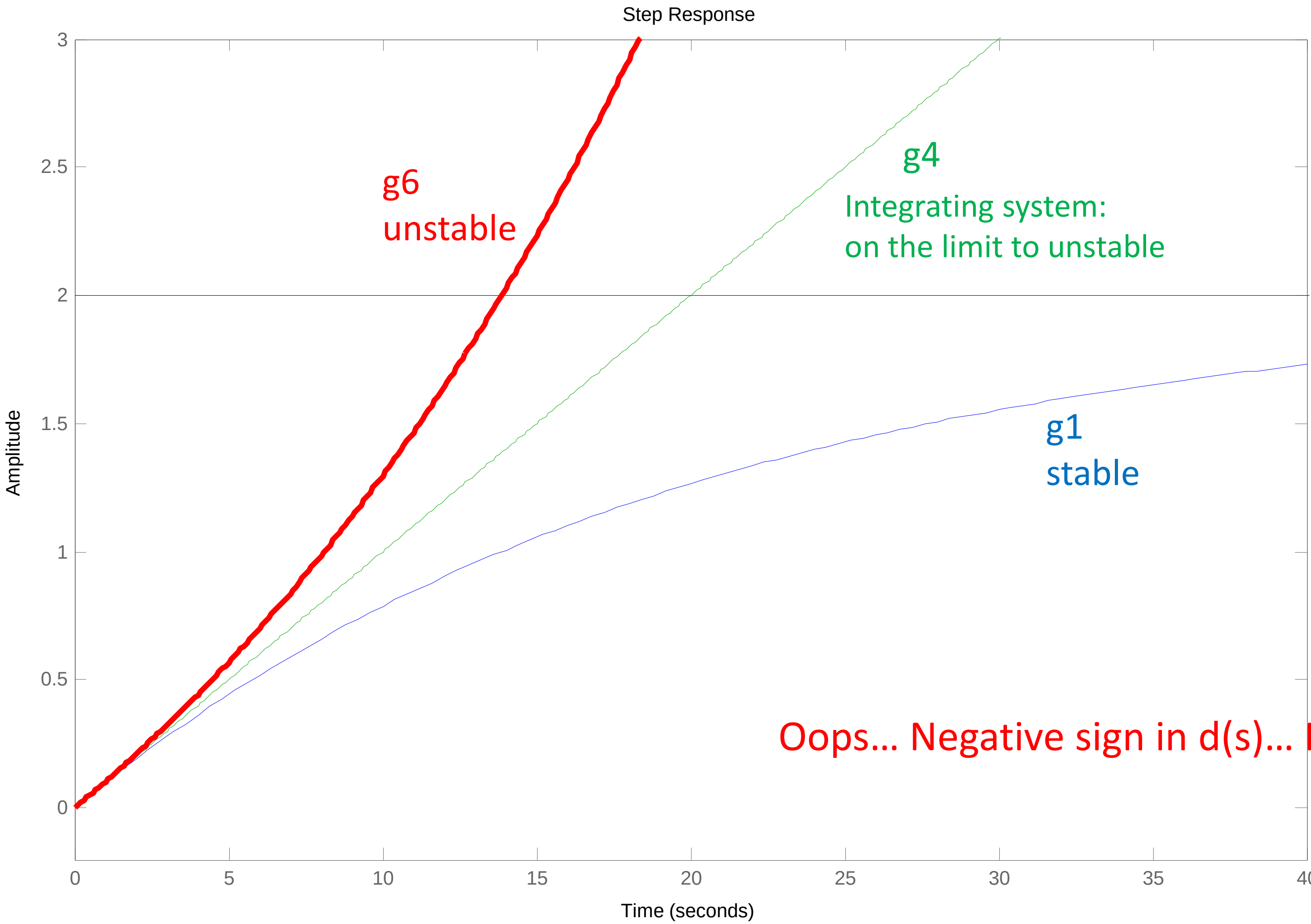
g1 & g4: Same initial response (slope = 0.2=k/τ)

```
s=tf('s')
g1 = 2/(10*s+1), step(g1,50)
axis([0 40 -0.2 3]); hold on,
g2 = 2/(12*s+1), step(g2,50)
g3 = 2.2/(10*s+1), step(g3,50)
g4 = 2/(10*s+0), step(g4,50)
```



g6: Unstable system (e.g., exothermic reactor):

Note: Sign change in denominator d(s)



$$g1 = \frac{2}{10\,s + 1}$$

$$g4 = \frac{2}{10\,s}$$

$$g6 = \frac{2}{10\,s - 1}$$

Oops... Negative sign in d(s)... Pole p=0.1 U

More on transfer functions & responses

1. A bit about poles and zeros
2. **Second-order systems (lecture 14)**
   – Can have oscillations (complex poles)
3. **Closed-loop transfer function (with control) (lecture 15)**
4. **More on poles and zeros (lecture 16)**
   – Including inverse response (RHP-zeros)
5. **Approximating transfer functions (lecture 17)**
   – Time delay
   – Half rule
6. **Derivation of SIMC PID rules** (lecture 18)

Understanding transfer functions

g(s) = n(s)/d(s).

Example.

$$g(s) = \frac{12s + 6}{30s^2 + 33s + 3}$$

Standard forms:

1. Time constant form

$$g(s) = k\frac{(T_1 s + 1)\cdots}{(\tau_1 s + 1)(\tau_2 s + 1)\cdots}$$

$$g(s) = 2\frac{2s + 1}{(10s + 1)(s + 1)}$$

2. Pole-zero form

$$g(s) = c\frac{(s - z_1)\cdots}{(s - p_1)(s - p_2)\cdots}$$

$$g(s) = \frac{12}{30}\frac{s + 0.5}{(s + 0.1)(s + 1)}$$

$$z_1 = -1/T_1 = -1/2 = -0.5$$
$$p_1 = -1/\tau_1 = -1/10 = -0.1;$$
$$p_2 = -1/\tau_2 = -1/1 = -1$$

Poles and zeros

Transfer function, g(s) = n(s)/d(s)

Poles (eigenvalues): Found from d(s)=det(sI-A)=0.

- Determine speed of response
- Poles in right half plane, e.g., p=0.1 (negative sign in d(s)): Unstable

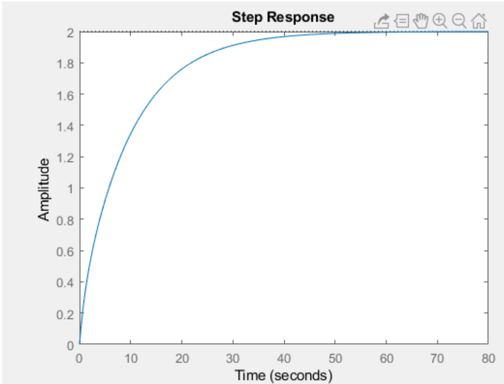
Zeros: Found from n(s)=0

- Determine shape of response
- Zeros in right half plane, e.g., z=0.5 (negative sign in n(s)): Inverse response

$$g(s) = c\frac{(s-z_1)\cdots}{(s-p_1)(s-p_2)\cdots}$$

$$g(s) = \frac{12}{30}\frac{s+0.5}{(s+0.1)(s+1)}$$

$$z_1 = -1/T_1 = -1/2 = -0.5$$
$$p_1 = -1/\tau_1 = -1/10 = -0.1;$$
$$p_2 = -1/\tau_2 = -1/1 = -1$$



2. 2nd order system. Special case: Two first-order in series

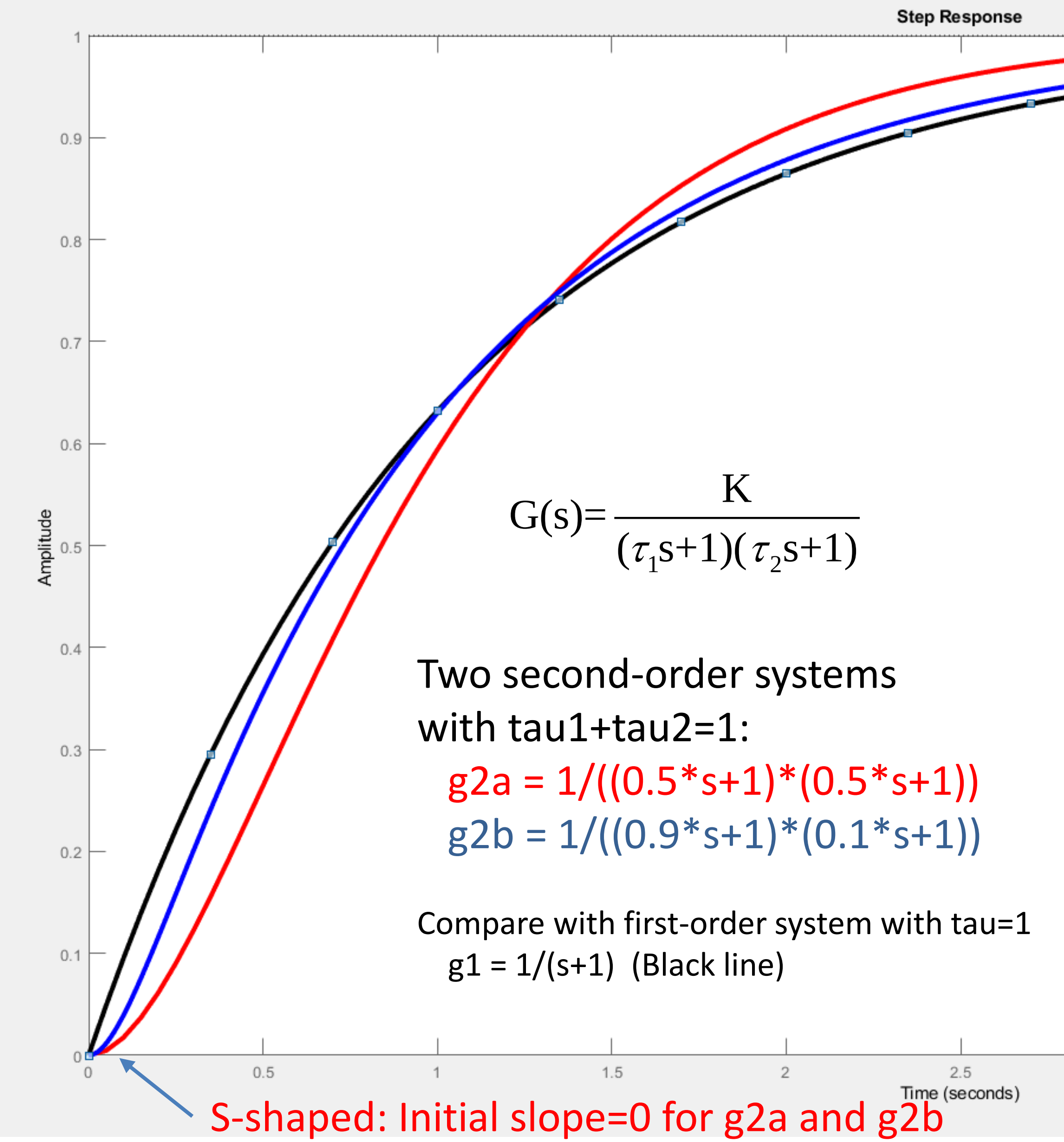
$$G(s)=\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}$$

Example: Temperature in two tanks in series, $\tau_1 = V_1/q$, $\tau_2 = V_2/q$

Step response (M = change in input):

$$y(t) = KM\left(1 - \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_1 - \tau_2}\right) \qquad (5\text{-}47)$$

Step response for two first-order in series: S-shaped response



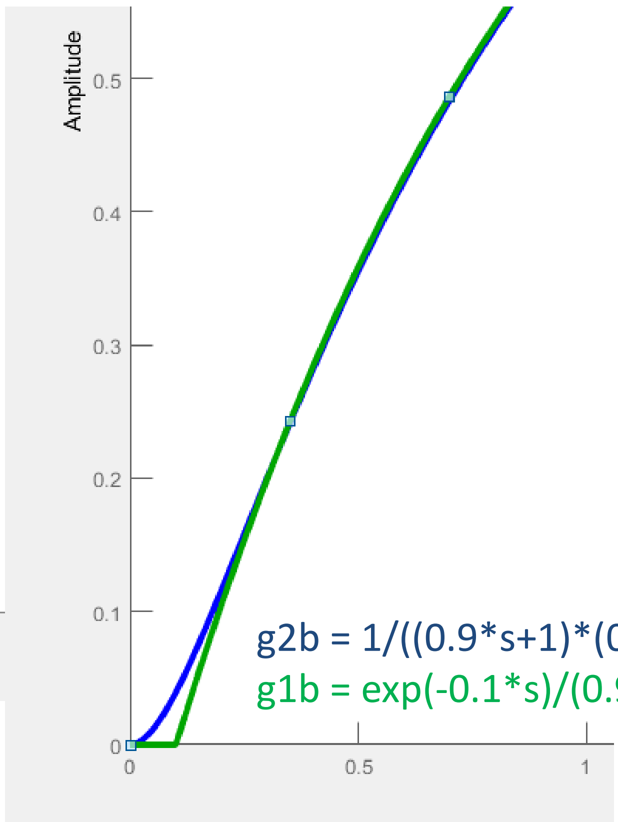
$$G(s)=\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}$$

Two second-order systems
with tau1+tau2=1:
g2a = 1/((0.5*s+1)*(0.5*s+1))
g2b = 1/((0.9*s+1)*(0.1*s+1))

Compare with first-order system with tau=1
g1 = 1/(s+1) (Black line)

S-shaped: Initial slope=0 for g2a and g2b

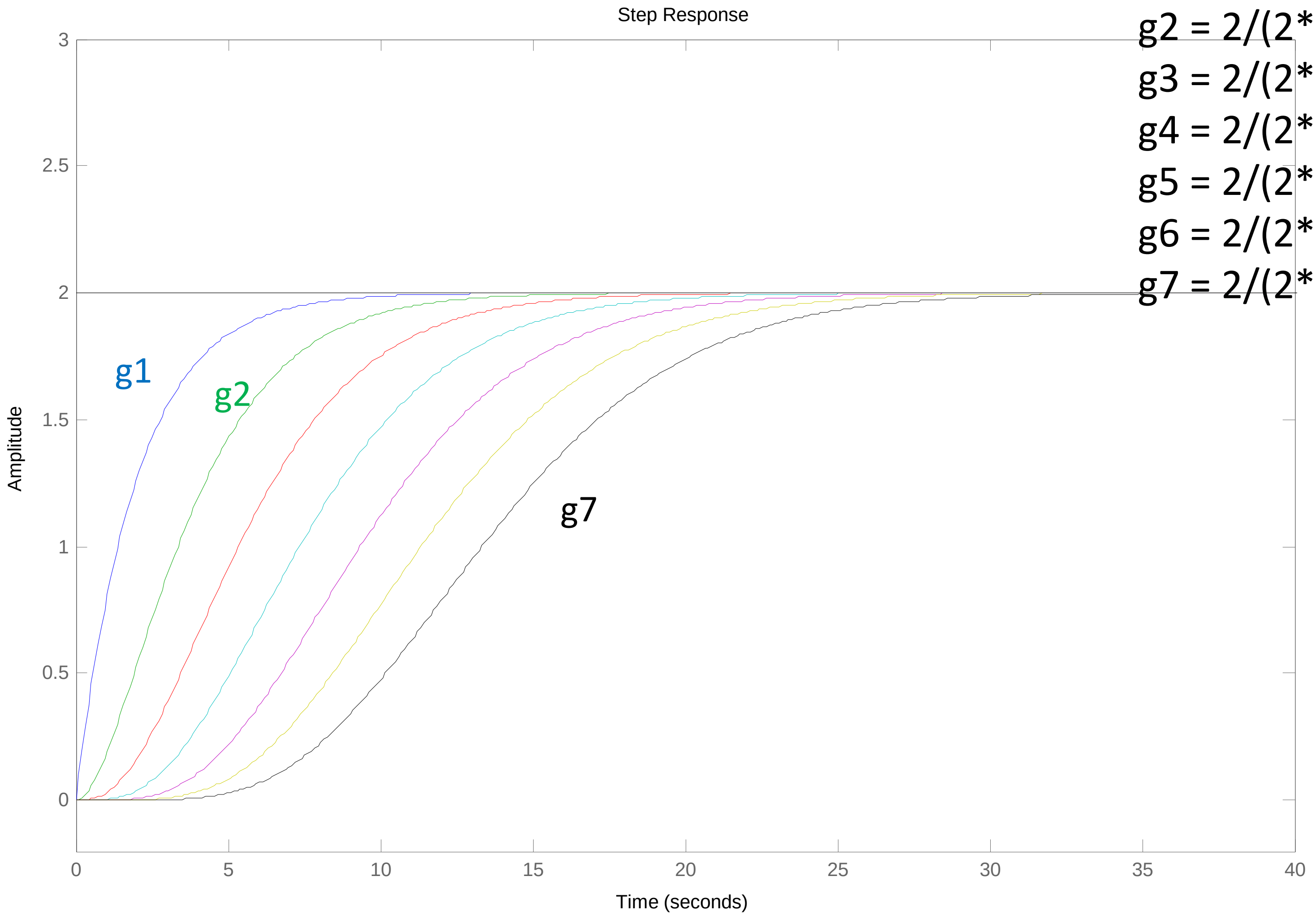
Note: Second-order system with tau1 much larger than tau2 can be approximated as first-order plus delay with delay = tau2:



g2b = 1/((0.9*s+1)*(0.1*s+1))
g1b = exp(-0.1*s)/(0.9*s+1)

n identical first-order systems in series

g1 = 2/(2*s+1), step(g1,50)
g2 = 2/(2*s+1)^2, step(g2,50)
g3 = 2/(2*s+1)^3, step(g3,50)
g4 = 2/(2*s+1)^4, step(g4,50)
g5 = 2/(2*s+1)^5, step(g5,50)
g6 = 2/(2*s+1)^6, step(g6,50)
g7 = 2/(2*s+1)^7, step(g7,50)



Note: More poles (relative to zeros) gives flatter initial step response. Proof:

$$\text{Initial value theorem: } \lim_{t\to 0} y^{(n)}(t) = \lim_{s\to\infty} s^n(sy(s)) = \lim_{s\to\infty} s^n g(s) M$$

For system with poles excess= m = n_p-n_z, we get that g(s) ~ $1/s^m$ when s goes to infinity.

Then the m'th derivative, $y^{(m)}$(t), is finite (non-zero) for step-response. The other m-1 derivatives of y(t) are zero!

Example G1(s) = 2/(2s+1). m=n_p=1. So first derivative y'(t) (initial slope) is non-zero

Example G7(s) = 2/(2s+1)^7. m=n_p=7. So six first derivatives of y(t) are zero -> Very flat initial response. Almkost like time delay.

General 2nd order system

$$G(s)=\frac{K}{\tau^2 s^2+2\zeta\tau s+1}=\frac{K}{\tau^2(s-\lambda_1)(s-\lambda_2)}$$

Roots (poles, eigenvalues): $\lambda_{1,2}=\frac{-\zeta\pm\sqrt{\zeta^2-1}}{\tau}$

$\zeta > 1$	Overdamped (two real poles)
$\zeta = 1$	Critically damped (two real identical poles)
$\lvert\zeta\rvert < 1$	Underdamped (complex poles; oscillations)
$\zeta < 0$	Unstable

Special case: Two real poles, $\zeta \geq 1$: Two first-order in series

$$G(s)=\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}=\frac{K}{\tau_1\tau_2 s^2+(\tau_1+\tau_2)s+1}$$

$$\zeta=\frac{\tau_1+\tau_2}{2\sqrt{\tau_1\tau_2}}\geq 1$$

$$\tau=\sqrt{\tau_1\tau_2}$$

$Two-real-poles:$

$\lambda_1=-1/\tau_1, \lambda_2=-1/\tau_2$

$$y(t)=KM\left(1-\frac{\tau_1 e^{-t/\tau_1}-\tau_2 e^{-t/\tau_2}}{\tau_1-\tau_2}\right) \quad (5\text{-}47)$$

Step response complex poles, $|\zeta|<1$

$$G(s)=\frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} = \frac{K}{\tau^2 (s-\lambda_1)(s-\lambda_2)}$$

$$\lambda_{1,2} = \frac{-\zeta \pm \sqrt{\zeta^2 - 1}}{\tau}$$

$$= \sigma \pm i\omega$$

y(s) = G(s) u(s) with u(s)=1/s (step).

Inverse Laplace (get terms $e^{\lambda t}$) and use Euler's formula for complex parts:

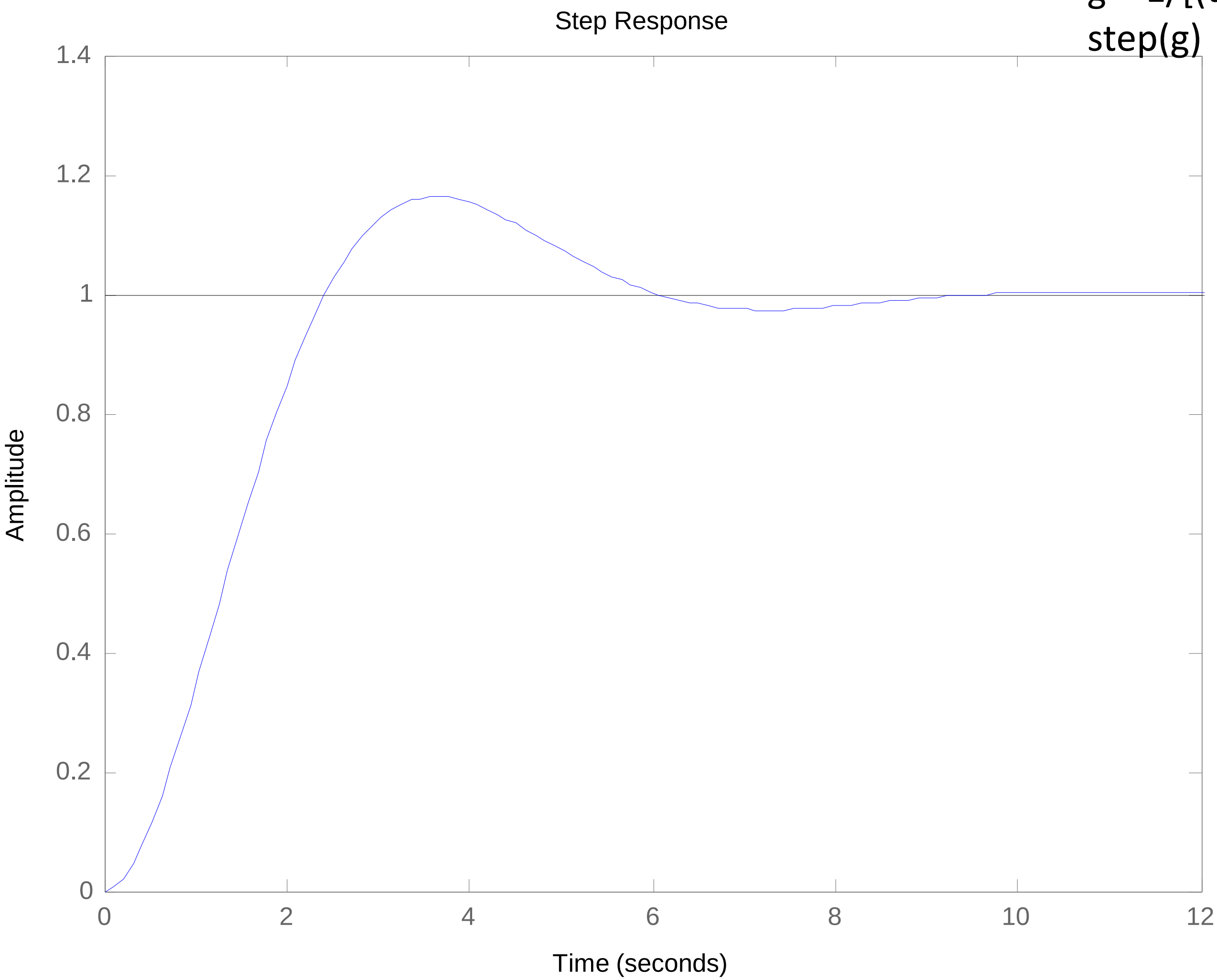
$$e^{i\omega t} = \cos \omega t + i \sin \omega t$$

Get for y(t):

$$21.\ 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin\left[\sqrt{1-\zeta^2}\, t/\tau + \psi\right]$$

$$\psi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}, \quad (0 \le |\zeta| < 1)$$

```
s=tf('s')
zeta=0.5, tau=1
g = 1/[(tau*s)^2 + 2*tau*zeta*s + 1]
step(g)
```




```
g =
        1
   -----------
   s^2 + s + 1
```

```
>> pole(g)

ans =

  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
```

$$G(s)=\frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

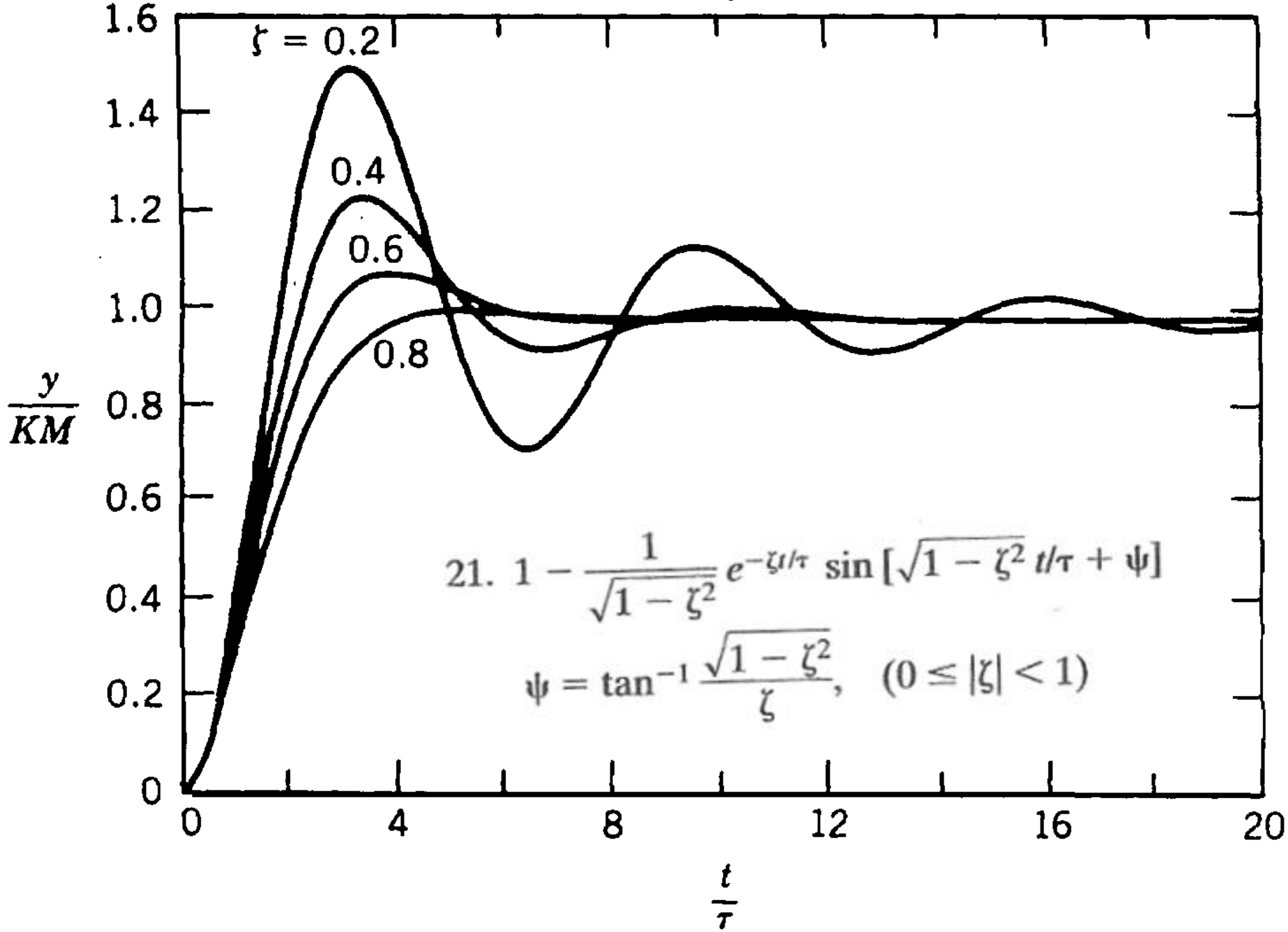


Figure 5.8. Step response of underdamped second-order processes.

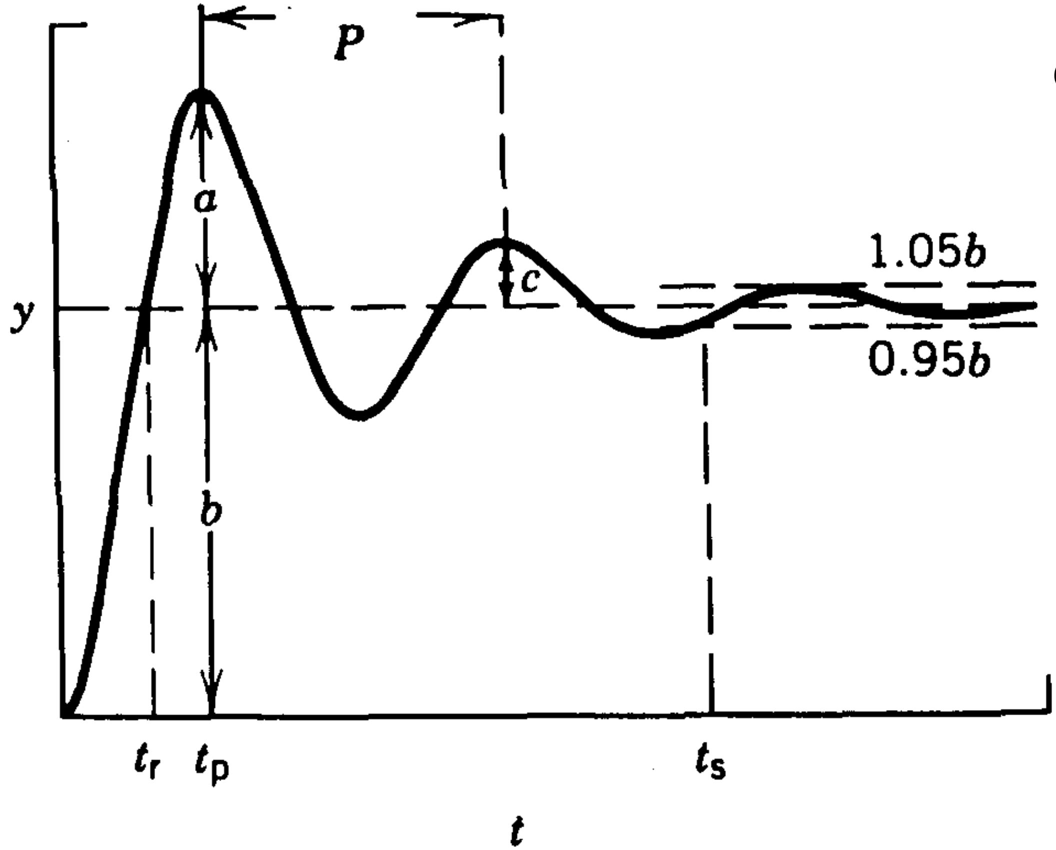
Table 3.1 Laplace Transforms for Various Time-Domain Functions[a] (continued)

$f(t)$	$F(s)$
20. $1 + \frac{1}{\tau_2 - \tau_1}(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2})$ $(\tau_1 \neq \tau_2)$	$\frac{1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
21. $1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin[\sqrt{1-\zeta^2}\, t/\tau + \psi]$ $\psi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}, \quad (0 \leq \lvert\zeta\rvert < 1)$	$\frac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
22. $1 - e^{-\zeta t/\tau}[\cos(\sqrt{1-\zeta^2}\, t/\tau)$ $+ \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2}\, t/\tau)]$ $(0 \leq \lvert\zeta\rvert < 1)$	$\frac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
23. $1 + \frac{\tau_3 - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_3 - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}$ $(\tau_1 \neq \tau_2)$	$\frac{\tau_3 s + 1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
24. $\frac{df}{dt}$	$sF(s) - f(0)$
25. $\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$
26. $f(t - t_0)S(t - t_0)$	$e^{-t_0 s} F(s)$

[a]Note that $f(t)$ and $F(s)$ are defined for $t \geq 0$ only.

Alternative forms of step response for 2nd order system

Chapter 5



$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$|\zeta|<1$

Time to first peak: $t_p = \pi\tau/\sqrt{1-\zeta^2}$ (5-50)

Overshoot: $OS = \exp\left(-\pi\zeta/\sqrt{1-\zeta^2}\right)$ (5-51)

Decay ratio: $DR = (OS)^2 = \exp\left(-2\pi\zeta/\sqrt{1-\zeta^2}\right)$ (5-52)

Period: $P = \frac{2\pi\tau}{\sqrt{1-\zeta^2}}$ (5-53)

Small ζ

$t_p = \pi\tau$

$OS = \exp(-\pi\zeta)$

$P = 2\pi\tau$

COMPLEX POLES IN PRACTISE

Underdamped (Oscillating) second-order systems ($|\zeta|<1$)

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Corresponds to complex poles, $\lambda_{1,2} = \sigma \pm i\omega$

Process systems:
Oscillations are usually caused by (too) aggressive control

Example: P-control of second-order process, $k/(\tau_1 s+1)(\tau_2 s+1)$

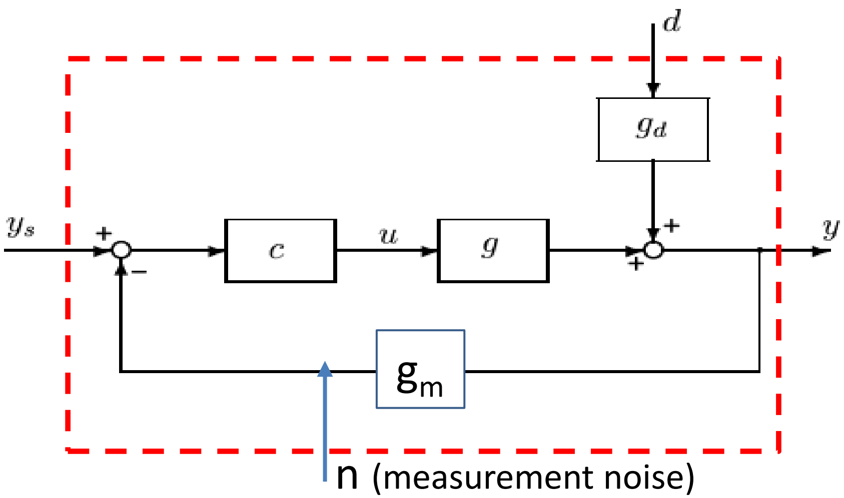
- Oscillates (ζ<1) if $K_c k$ is <u>large</u>

But there also cases where we need control to avoid oscillatioons:

Example 2: PI-control of integrating process, k'/s

- Need control to stabilize
- Oscillates (ζ<1) if $K_c k'$ is <u>small</u>

3. Closed-loop transfer function



(1) Process: $y = g(s)\ u + g_d(s)\ d$

(2) Controller: $u = c(s)\ (y_s - y_m)$

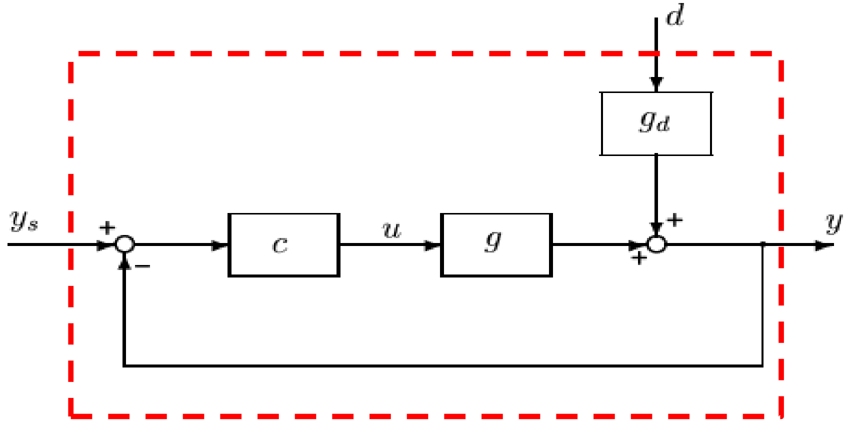
(3) Measurement: $y_m = g_m(s)\ y + n$

Closed-loop response: Want to find effect of y_s , d an n on output y.

Task: Eliminate u and y_m to find

$$y = T(s)\ y_s + T_d(s)\ d + T_n(s)\ n$$

Closed-loop transfer functions

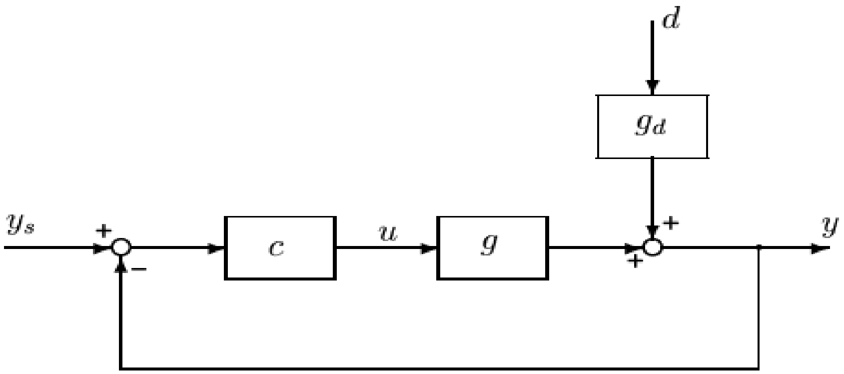


- Here: Perfect measurement ($n=0$, $g_m=1$):
- Closed-loop output response: $y = T\, y_s + T_d\, d$

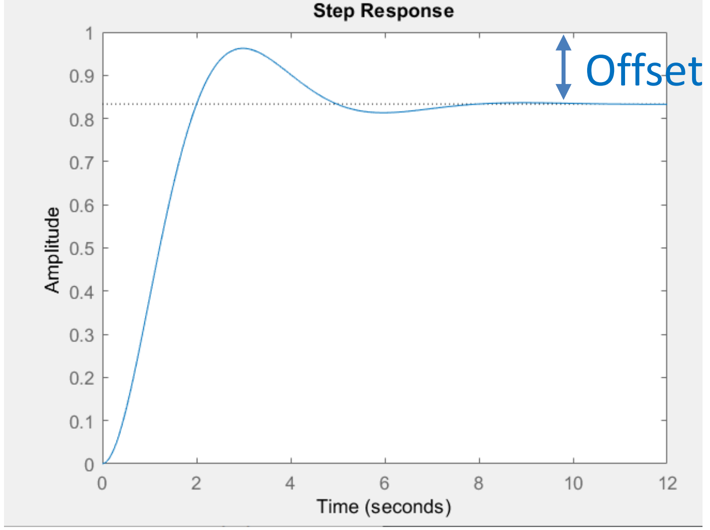
  $T = gc/(1+gc)$

  $T_d = g_d/(1+gc)$
- General rule for negative feedback:
  - Transfer function = «direct(s)» / (1 + «loop(s)»)
  - Here: Loop (L) = g c

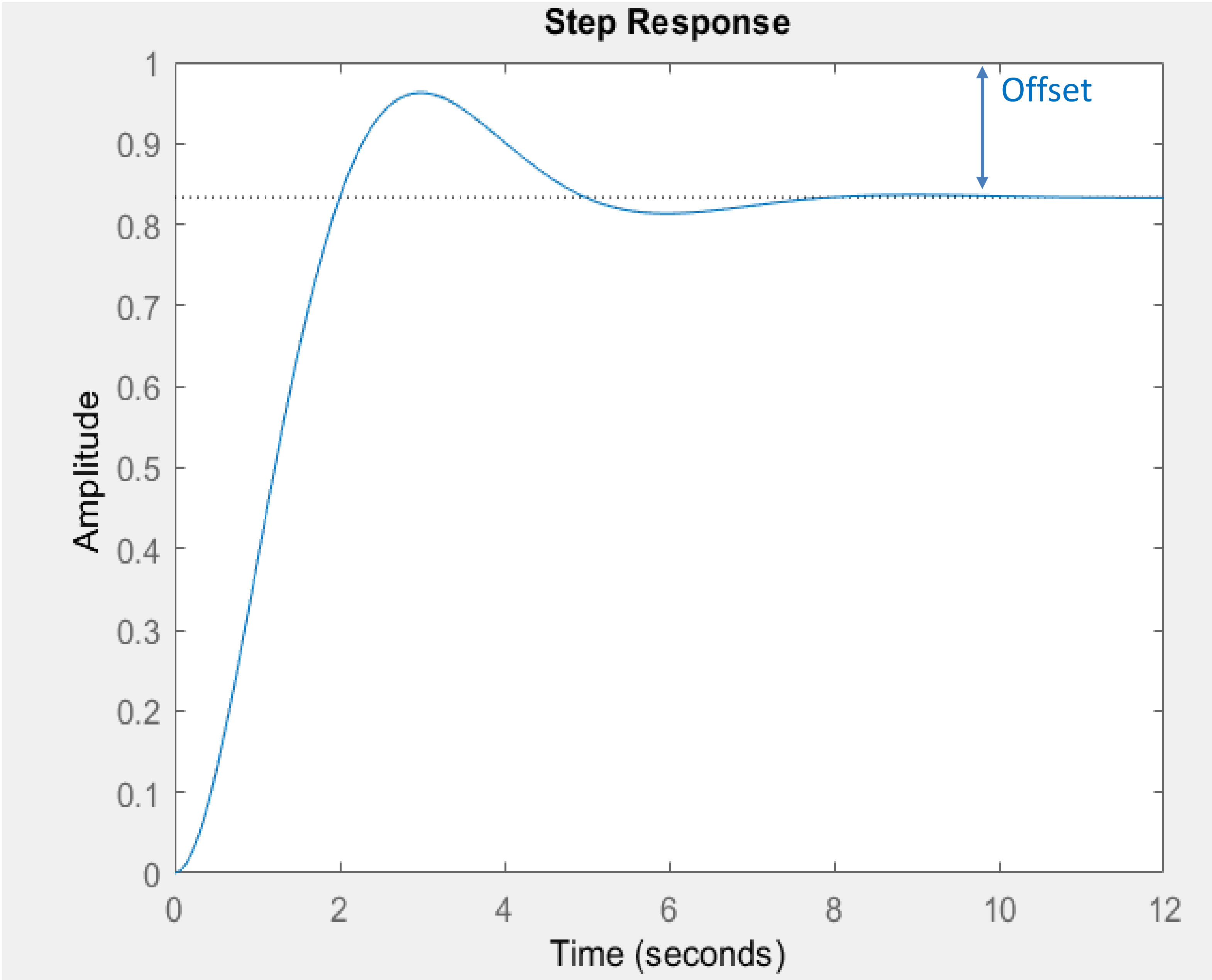
Example 1. Setpoint response for P-control of 2nd order process



g(s) = 1/[(4s+1)(s+1)]
P-controller: c(s) = K_c =4.5
Derive T(s)

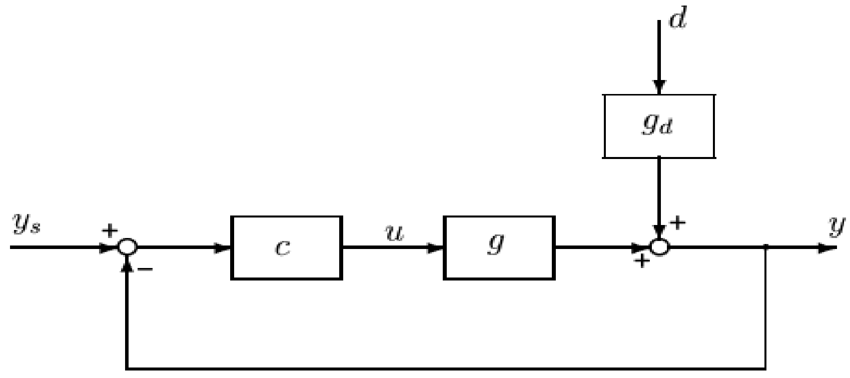


Note: Larger value of K_c gives less offset but more oscillations



Step Response
Offset
Amplitude
Time (seconds)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
12

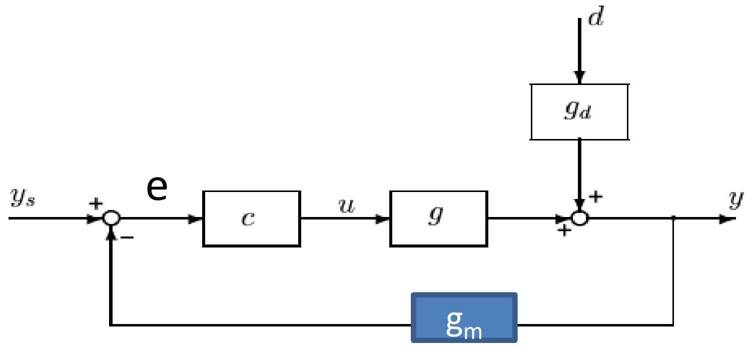
Sensitivity function S



- Consider control error: $e = y - y_s$
- No control (c=0): $e = g_d d$ - ys
- With control (closed-loop): $e = S\,(g_d d - y_s)$
  - S gives the effect of feedback. S = 1/(1+loop)
    - No control: S=I
    - Perfect control (infinite c): S=0

Steady-state offset with P-control

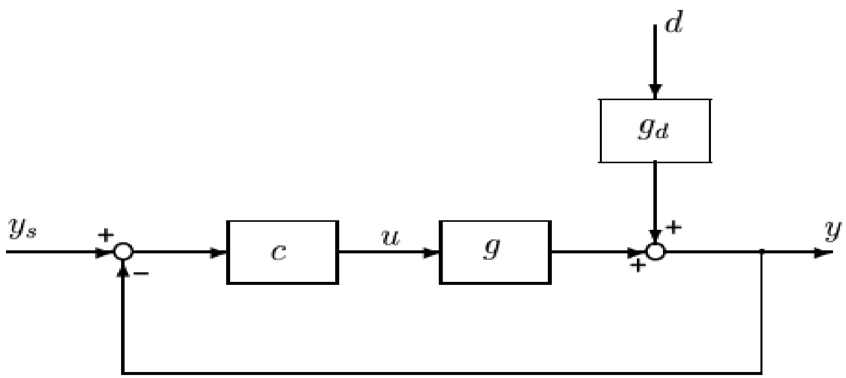
(k=process gain, K_c= controller gain)



Define sensitivity fuction: $S(s) = 1/(1+L)$ where $L(s)$=loop = $g\ c\ g_m$.
S is transfer function from y_s to control error e: $e = S(s)\ y_s$
Steady-state offset to step change in setpoint: $e = S(0)\ y_s$ where $S(0)= 1/(1+K_c k)$

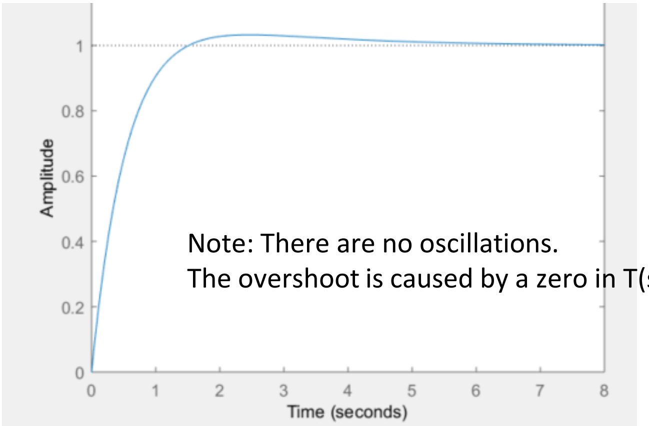
Example. k=1, K_c = 4.5. Relative Steady-state offset e/y_s is $S(0)=1/(1+K_c k)$ = 1/5.5= 0.18 (18%)

Example 2. PI-control of 1st order process



Setpoint response with K_c=3:

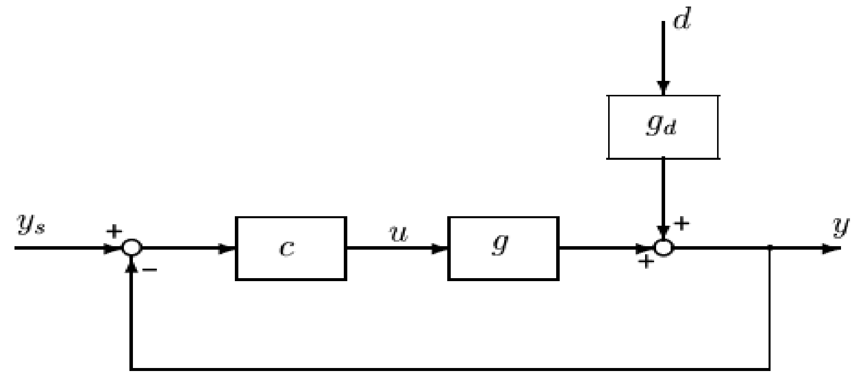
Example. $g(s) = g_d(s) = 2/(3s+1)$
PI-controller: $c(s) = K_c*(2s+1)/2s$



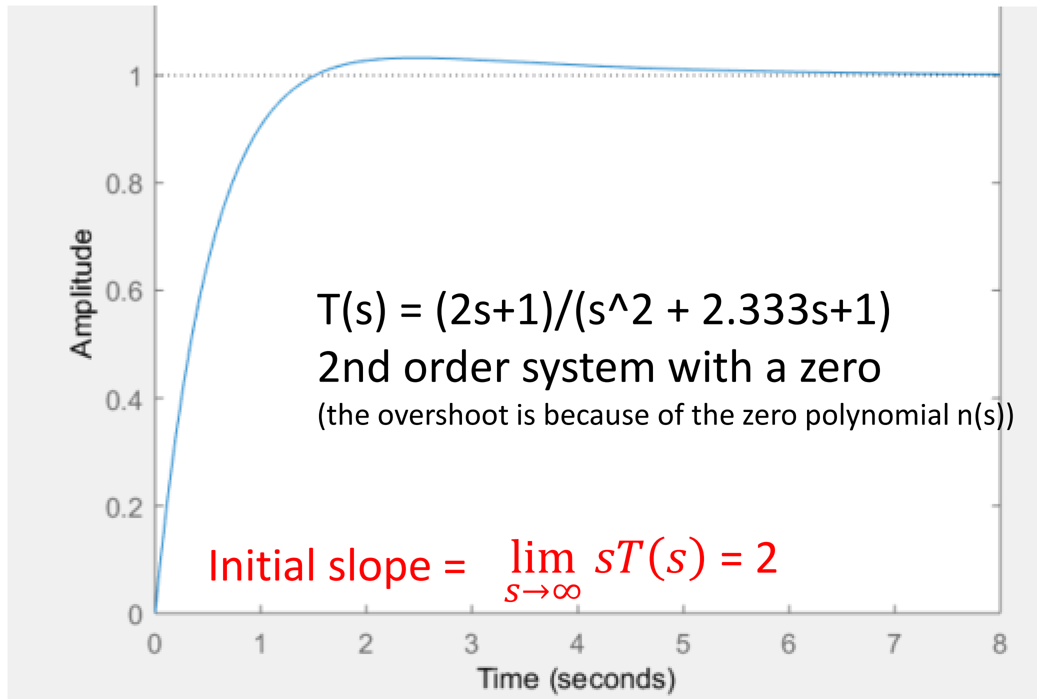
Note: Larger value of K_c gives faster response offset but less robustness to delay

Setpoint response. y_s=1, d=0

$y = T(s)\, y_s$




```
c>> s = tf('s')
>> g = 2/(3*s+1)
g =
    2
  -------
  3 s + 1
 >> c = 3*(1+1/(2*s))
c =
  6 s + 3
  -------
    2 s
>> T = g*c/(1+g*c)
T =
     72 s^3 + 60 s^2 + 12 s
  ------------------------------
  36 s^4 + 96 s^3 + 64 s^2 + 12 s
>> T1 = minreal(T)
T1 =
      2 s + 1
  -----------------
  s^2 + 2.333 s + 1

>> step(T1)
```



T(s) = n(s)/d(s)

n(s) = 2s + 1

d(s) = s^2 + 2.333s + 1 = $\tau^2 + 2\tau\zeta s + 1$,   with τ=1, ζ= 1.167 (no oscillations since $\zeta > 1$))

= $(\tau_1 s + 1)(\tau_2 s + 1)$ with τ_1=1.178, τ_2=0.566

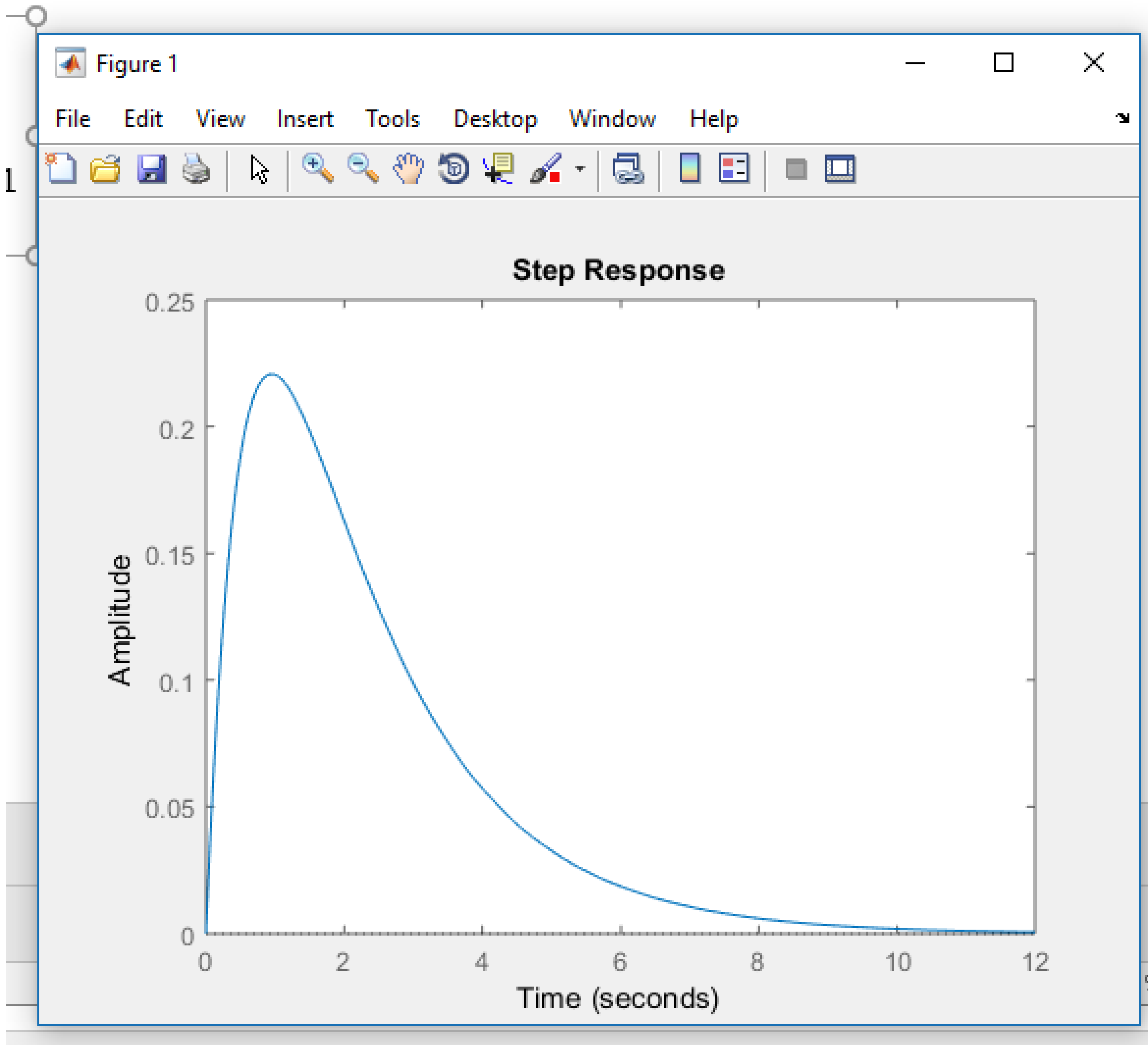
Input Disturbance response (g_d=g)

Td =

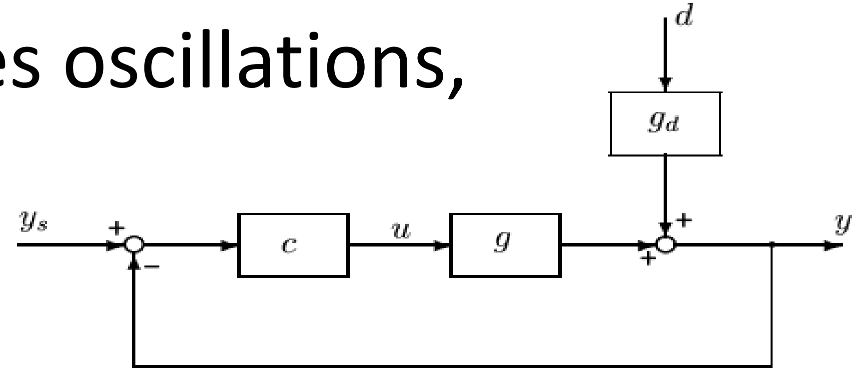
0.6667 s

----

s^2 + 2.333 s + 1

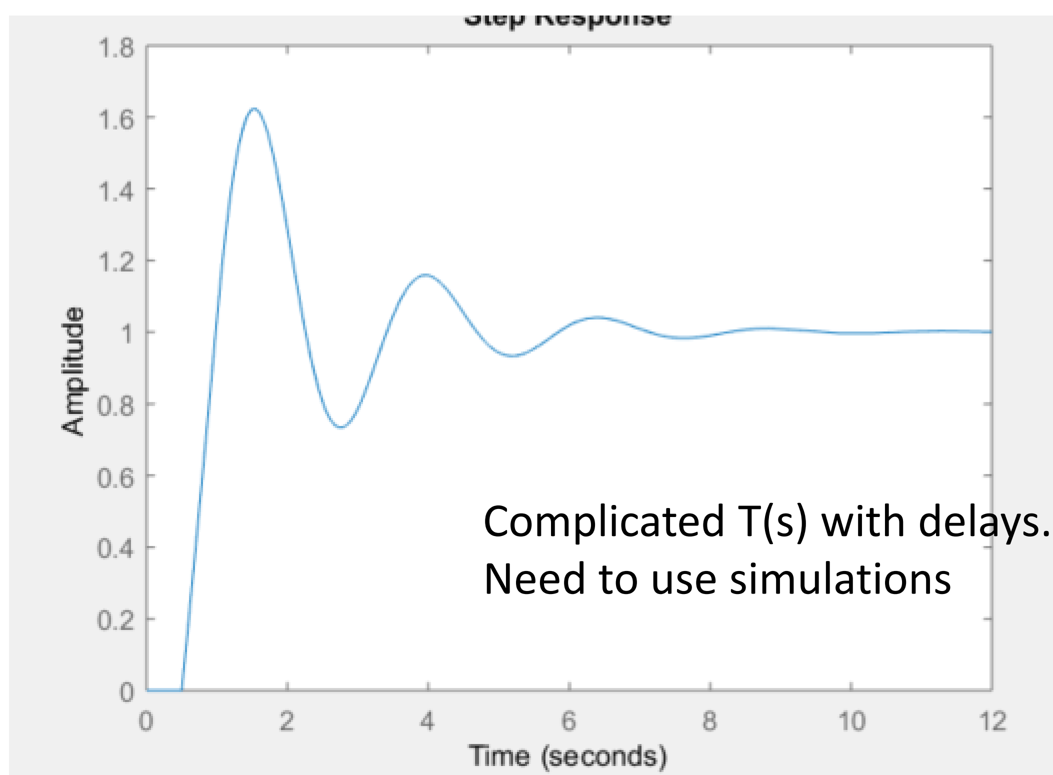


Comment: Adding delay gives oscillations, $\theta = 0.5$ s



Setpoint response. y_s=1, d=0

```
>> delay = exp(-0.5*s)
delay =
  exp(-0.5*s) * (1)
>> L =g*c*delay

L =
                    12 s + 6
  exp(-0.5*s) * -----------
                    6 s^2 + 2 s

 >> T2 = L/(1+L)
 Internal delays (seconds): 0.5  0.5

>> step(T2)
```



Complicated T(s) with delays…
Need to use simulations

Unstable with $\theta = 1$ s

Example 3. PI-control of level

$$y = \Delta h, u = -\Delta q_{out}, d = \Delta q_{in}$$

$$y(s) = \frac{k'}{s}(u(s) + d(s)), \quad k' = 1/A$$

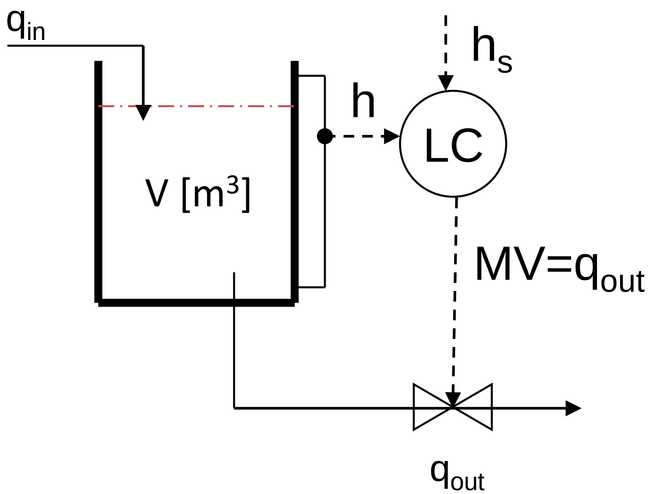
- $g(s) = k'/s$
- $c(s) = K_c(1+1/\tau_I s)$

- Derive condition to avoid «slow» ocillations that may occur when K_c is too small*

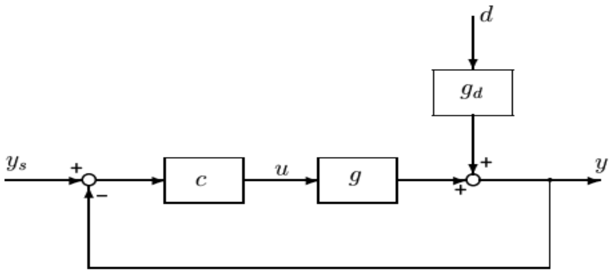
*Yes, this may seem a bit strange, but for PI-control of integrating process you may get oscillatons when Kc is too small!
In addition, you may of course get the more common «fast» oscillations if Kc is too large because of «overreaction» with time delay.

Model for PI-control of integrating process (level)

FLOWSHEET:



BLOCK DIAGRAM:



$y = \Delta h$
$u = -\Delta q_{out}$
$d = \Delta q_{in}$

Mass balance with constant density (V=Ah):

$$dV/dt = q_{in} - q_{out}$$

Deviation variables + linearize (well, it's already linear!)

$$A\, d\Delta h/dt = \Delta q_{in}(t) - \Delta q_{out}(t)$$

Laplace

$$\Delta h(s) = \frac{\Delta q_{in}(s) - \Delta q_{out}(s)}{As} = (k'/s)\,(u+d)$$

$$g(s) = \frac{k'}{s}$$

$$c(s) = K_c\left(1 + \frac{1}{\tau_I s}\right)$$

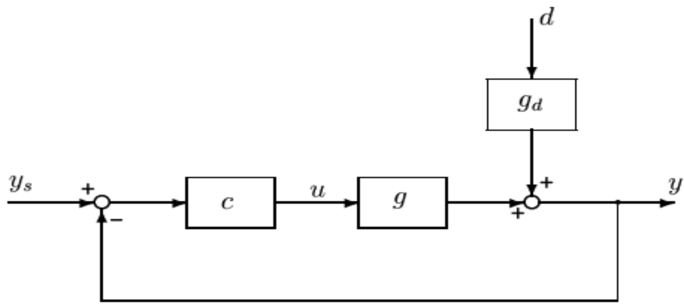
$k' = 1/A$

Task: Derive condition to avoid «slow» ocillations that may occur when K_c is too small

Integrating process with PI-control:

$$G(s) = \frac{k'}{s}$$

$$C(s) = K_c(1 + \frac{1}{\tau_I s})$$



General rule to avoid slow oscillations ($\zeta \geq 1$) :

$$k' K_C \tau_I \geq 4$$

Need large controller gain and/or large integral time (!)

Alternative Proof:

$$G(s) = k\frac{e^{-\theta s}}{\tau_1 s+1} \approx \frac{k'}{s} \text{ where } k' = \frac{k}{\tau_1};\ C(s) = K_c\left(1 + \frac{1}{\tau_I s}\right)$$

Closed-loop poles:

$$1 + GC = 0 \Rightarrow 1 + \frac{k'}{s}K_c\left(1 + \frac{1}{\tau_I s}\right) = 0 \Rightarrow \tau_I s^2 + k' K_c \tau_I s + k' K_c = 0$$

To avoid oscillations we must not have complex poles:

$$B^2 - 4AC \geq 0 \Rightarrow k'^2 K_c^2 \tau_I^2 - 4k' K_c \tau_I \geq 0 \Rightarrow k' K_c \tau_I \geq 4$$

Closed-loop responses

Closed-loop response to disturbance d at input and setpoint change

$$y = \frac{g}{1+gc}d + \frac{gc}{1+gc}y_s$$

PI-control of integrator:

$$g(s) = \frac{1}{s}; \quad c(s) = K_c\frac{\tau_I s+1}{\tau_I s}$$

Get

$$y = \frac{\tau_I s}{\tau_I s^2+K_c\tau_I s+K_c}d + \frac{K_c(\tau_I s+1)}{\tau_I s^2+K_c\tau_I s+K_c}y_s$$

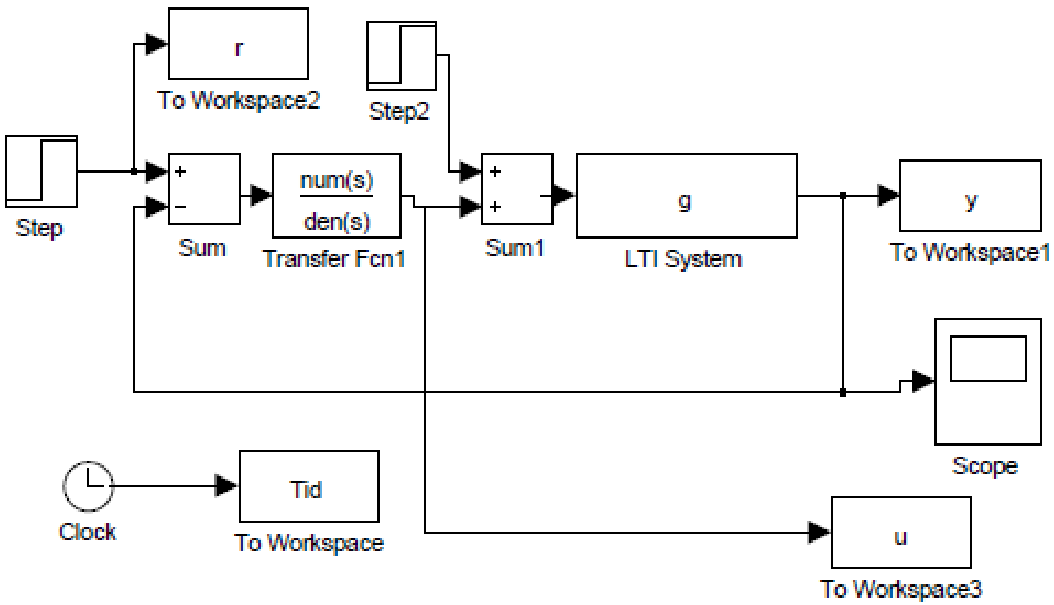
With $\tau_I = 1, K_c = 0.25$:

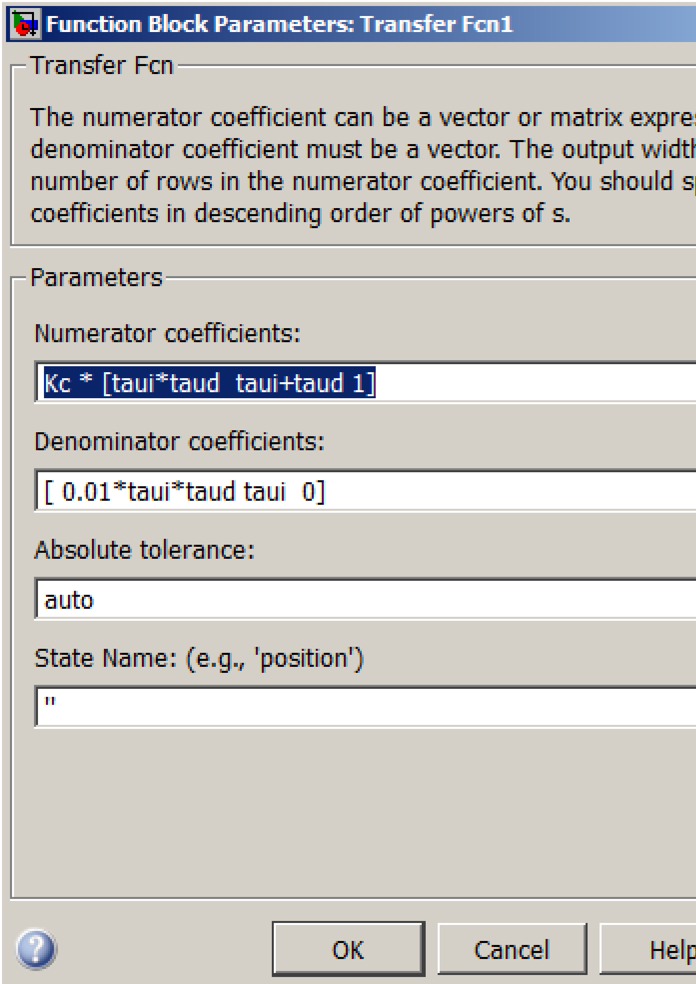
$$y = \frac{s}{s^2+0.25s+0.25}d + \frac{0.25(s+1)}{s^2+0.25s+0.25}y_s = \underbrace{\frac{4s}{4s^2+s+1}}_{T_d = \; h(s)}d + \underbrace{\frac{(s+1)}{4s^2+s+1}}_{T(s)}y_s$$

Notes:

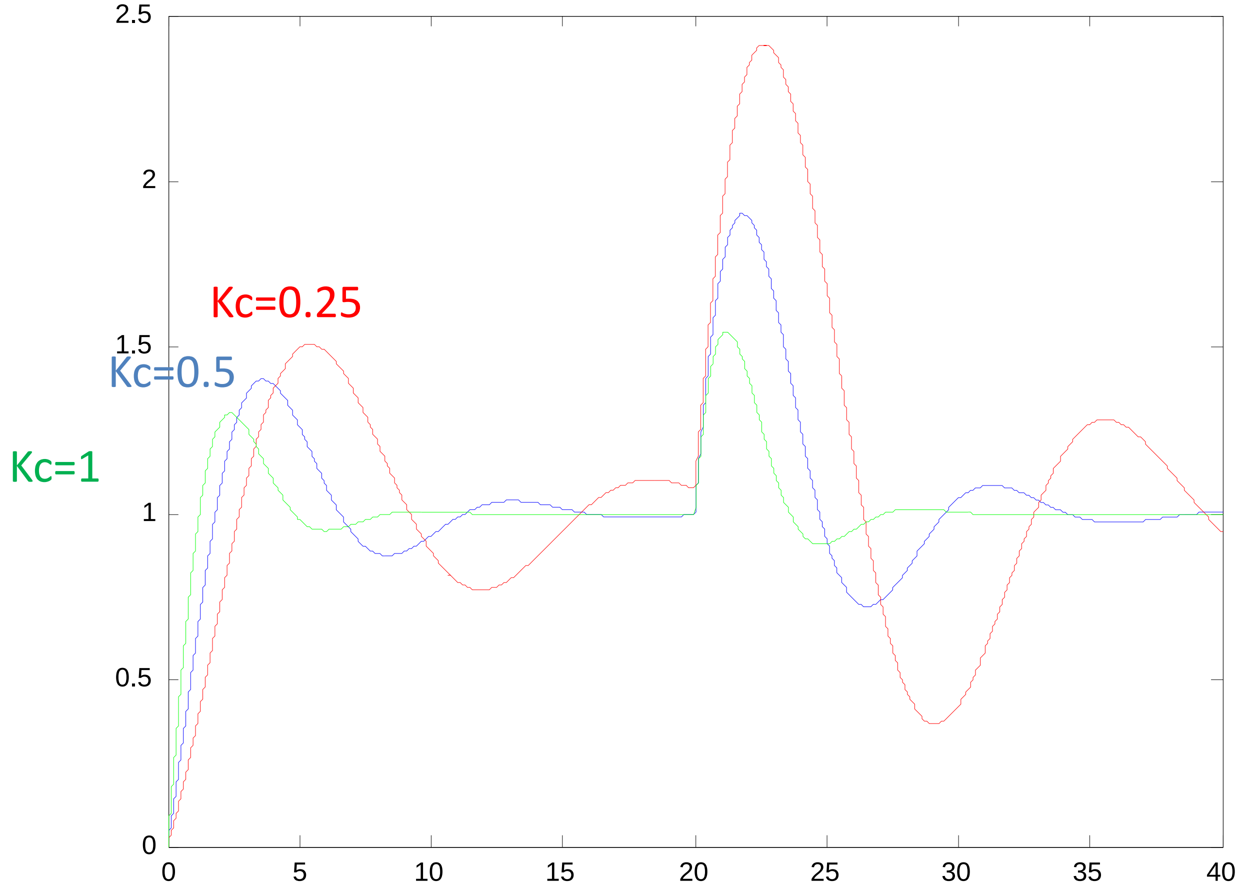
- Steady-state gain $h(0)$ for disturbance transfer function $h(s)$ is zero (because controller has integral action)
- Steady-state gain $T(0)$ for setpoint transfer function $T(s)$ is 1 (because controller has integral action)
- Denominator is on form $\tau^2 s^2 + 2\tau\zeta s + 1$ with $\tau = 2$ and $\zeta = 0.25 < 1$, so there will be oscillations with period $P \approx 2\pi\tau$
- Initial response ($t \rightarrow 0$) to disturbance is the same as with no control ($h(s) = \frac{g}{1+gc} \rightarrow g(s)$ when $s \rightarrow \infty$ since $g(s)c(s) \rightarrow 0$ (which is the case for all real systems))

Simulink, tunepid4





PI-control of integrator (level control). G = 1/s, taui=1. **VARY Kc**




```
%tunepid4
s=tf('s')
theta=0
g=(1/s)*exp(-theta*s) % integrating
taud=0
taui=1
Kc=0.5 % oscillations (Kc*k'*taui = 0.5 < 4)
sim tunepid4; plot(Tid,y); hold on %

Kc=0.25 % more oscillations (Kc*k'*taui = 0.25
sim tunepid4; plot(Tid,y,'red');

Kc=1   % less oscillations (Kc*k'*taui = 1)
sim tunepid4; plot(Tid,y,'green');
hold off
```

Setpoint change at t=0

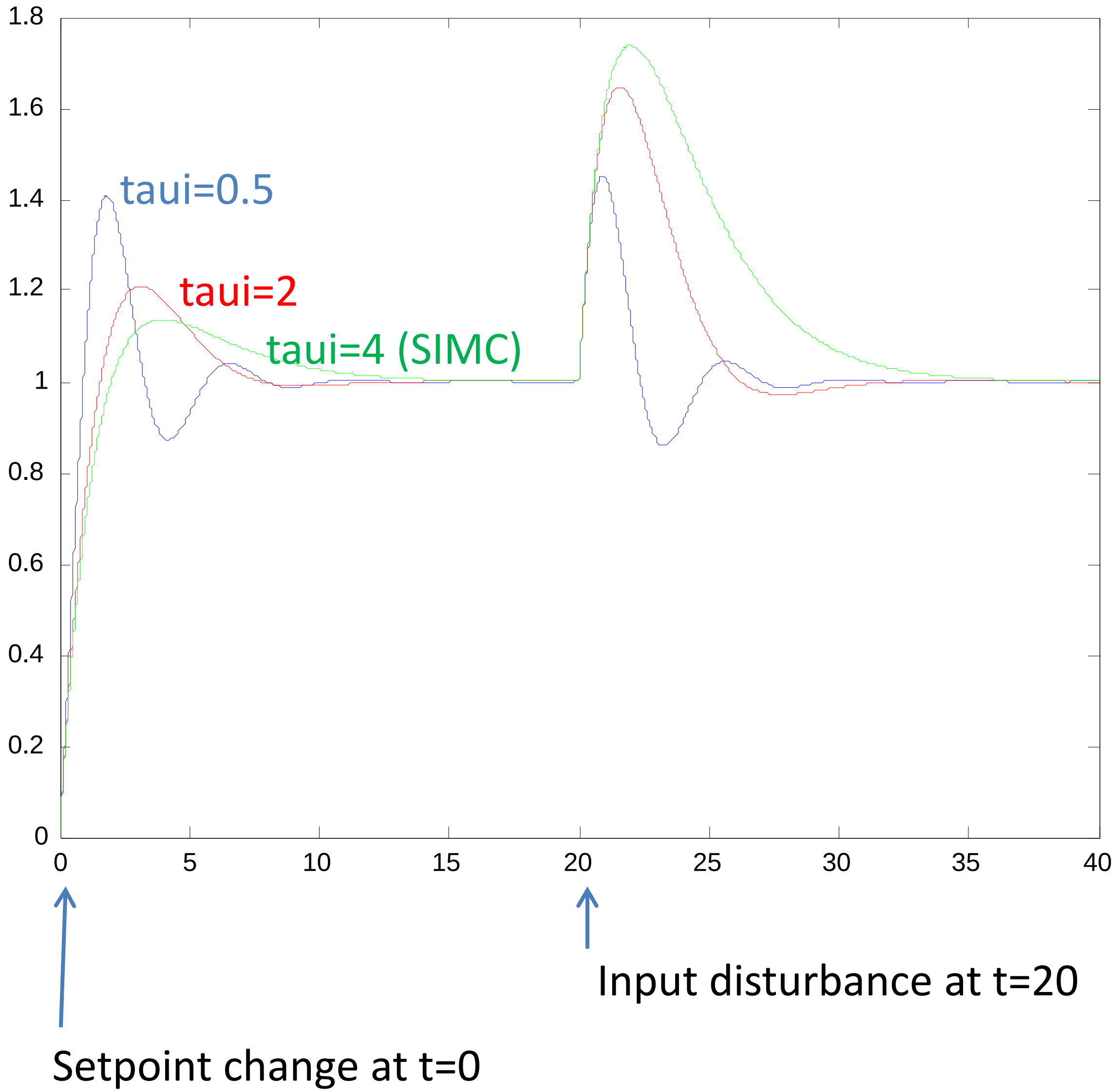
Input disturbance at t=20

Note: Need higher controller gain to reduce "slow" oscillations!

$$Avoid\ slow\ oscillations:\ k' K_C \tau_I \geq 4$$

So would need to increase Kc to 4 in this case

PI-control of integrator (level control). G = 1/s, Kc=1. **VARY taui**



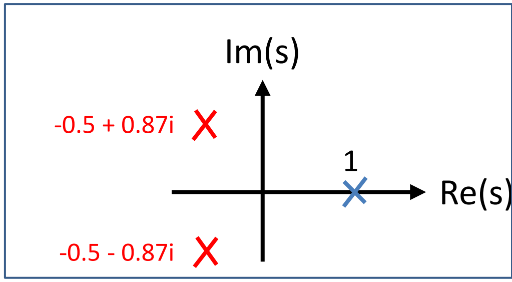
Note: Need larger integral time reduce "slow" oscillations

$Avoid\ slow\ oscillations: k' K_C \tau_I \geq 4$

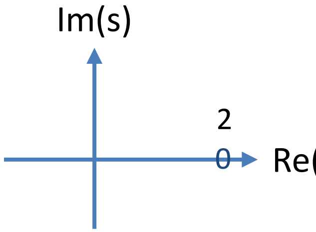
So need to use taui=4 to have no oscillations (SIMC-rule).
Get T(s)=(4s+1)/(2s+1)^2

4. Back to Poles and zeros

- Transfer functions *G(s)* of linear, time-invariant networks of first-order systems are ratios of two polynomials in *s* (Laplace variable)
  - G(s) = n(s)/d(s)
- Polynomials have roots.
  root in denominator, d(s)=0: $G(s) \to \infty$ "pole" (x)
  root in numerator, n(s)=0: $G(s) \to 0$ "zero" (0)
- Effect on dynamics:
  - Poles determine stability and fast or slow dynamics
    - Poles in right half plane (RHP): Unstable .
      - Example: g(s)=1/(s-1). Has RHP-pole at s=1
    - Complex poles (=eigenvalues of A): Oscillations
      - Example: g(s) = 1/(s^2 + s + 1). Solve d(s)= s^2 + s + 1 = 0. Get poles s1 = -0.5 + 0.87*i, s2 = -0.5 – 0.87*i



  - Zeros are responsible for shape of response
    - Zeros in left half plane (LHP): «Lifts» the response and often give overshoot
    - Zeros in right half plane (RHP): always gives inverse response
      - Inverse response makes problems for feedback control
      - Example: g(s)=(s-2) / (10s^2+11s+1). So n(s)=s-2. Has RHP-zero at s=2



Example transfer function

$$g(s) = \frac{4s+2}{5s^2+5.5s+0.5}$$

Time constant form:

$$g(s) = k\frac{Ts+1}{(\tau_1 s+1)(\tau_2 s+1)} \text{ with } k = 4, T = 2, \tau_1 = 10, \tau_2 = 1$$

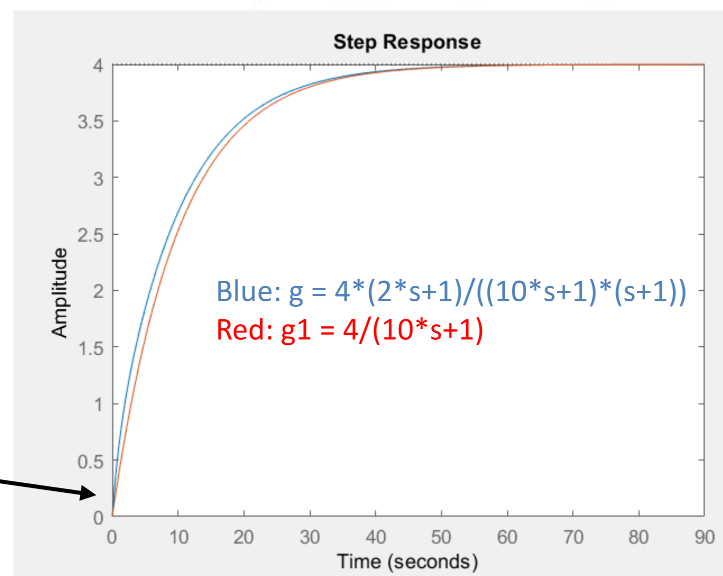
Pole-zero form:

$$g(s) = \frac{4}{5}\frac{s+0.5}{(s+0.1)(s+1)} = k'\frac{s-z}{(s-p_1)(s-p_2)}$$

with $k' = 4/5$,

zero $z = -1/T = -0.5$,

poles (or eigenvalues): $p_1 = \lambda_1 = -1/\tau_1 = -0.1$, $p_2 = \lambda_2 = -1/\tau_2 = -1$



Initial slopes are different.

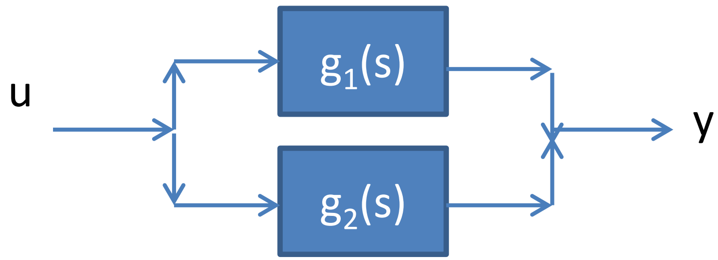
g: slope= 4*2/10 = 0.8

g1: slope= 4/10 = 0.4

```
>> s=tf('s')

s =

  s

Continuous-time transfer function.

>> g = 4*(2*s+1)/((10*s+1)*(s+1))

g =

       8 s + 4
  -----------------
  10 s^2 + 11 s + 1

Continuous-time transfer function.

>> g1 = 4 /(10*s+1)

g1 =

     4
  --------
  10 s + 1

Continuous-time transfer function.

>> step(g,g1)

>>
```

Zeros

- Zeros are common in practise
- Occur when there are several «paths» to the output.
- **RHP zero: «competing effects where slow wins (has largest gain)»**



- Example 1

$$g_1(s) = \frac{2}{10s+1}, \quad g_2(s) = \frac{0.3}{s+1}$$

$$g(s) = g_1 + g_2 = \frac{2(s+1)+0.3(10s+1)}{(10s+1)(s+1)} = 2.3\frac{2.17s+1}{(10s+1)(s+1)}$$

All coefficients positive: LHP zero

- Example 2

$$g_1(s) = \frac{2}{10s+1}, \quad g_2(s) = -\frac{0.3}{s+1}$$

$$g(s) = g_1 + g_2 = \frac{2(s+1)-0.3(10s+1)}{(10s+1)(s+1)} = 1.7\frac{-0.59s+1}{(10s+1)(s+1)}$$

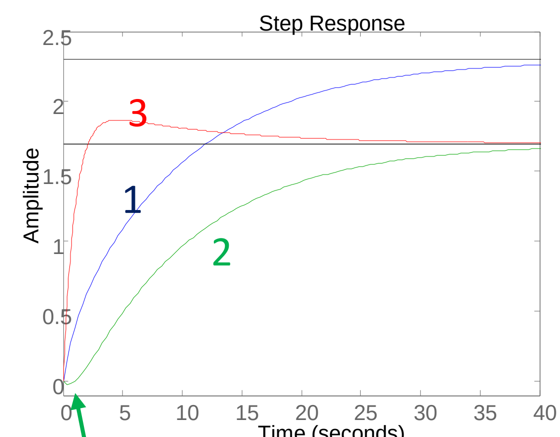
Sign change: RHP zero $\Rightarrow$ Inverse response

- Example 3

$$g_1(s) = -\frac{0.3}{10s+1}, \quad g_2(s) = \frac{2}{s+1}$$

$$g(s) = g_1 + g_2 = \frac{2(s+1)-0.3(10s+1)}{(10s+1)(s+1)} = 1.7\frac{11.3s+1}{(10s+1)(s+1)}$$

Note; Overshoot since 11.3>10
(overshoot: competing effects where fast wins)



Example 2: RHP-zero with «time constant» -0.59: Similar to delay of 0.59.

Zeros

$$G(s) = \frac{K(\tau_a s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} \tag{5-14}$$

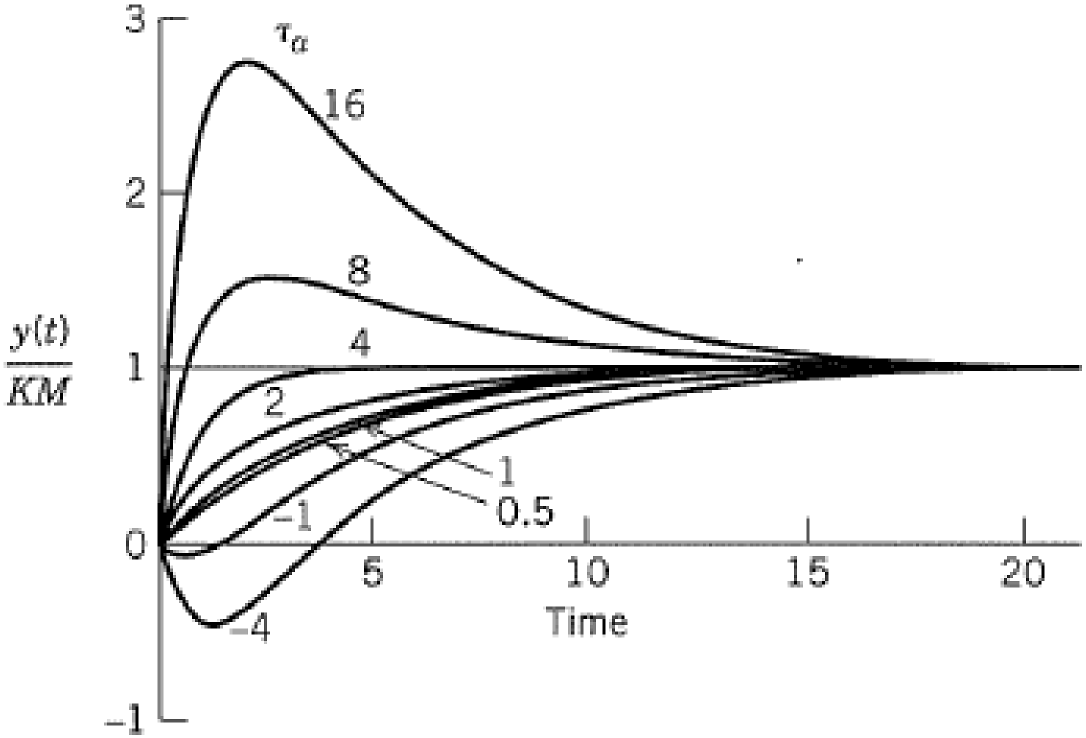
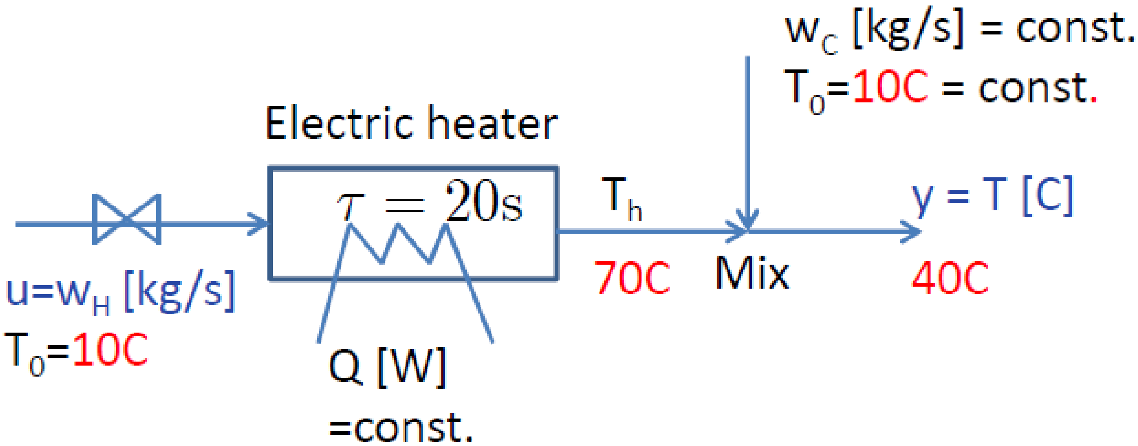


Figure 5.3 Step response of an overdamped second-order system (Eq. 5-14) for different values of τ_a ($\tau_1 = 4$, $\tau_2 = 1$).

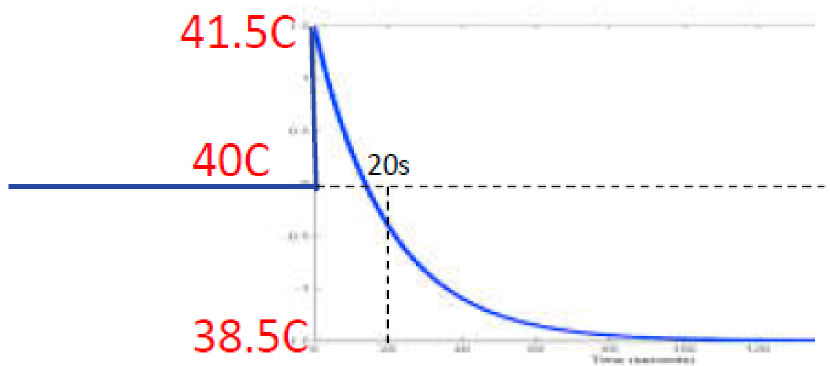
Examples of dynamic model structures

How do we get zeros?

RHP-zero (inverse response)



Response in y=T to a 10% step increase in u=w_H =0.1:

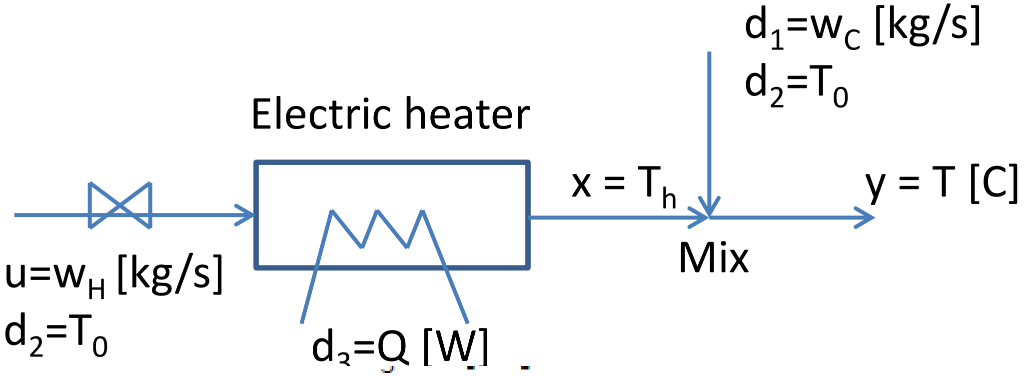


Two effects: 1) Direct effect of mixing: $g_1(s)=15$

2) Indirect effect of changed T_h: $g_2(s) = -30/(20s+1)$

$$g(s) = g_1 + g_2 = -15\frac{-20s+1}{20s+1}$$

Model derivation



1. Model. Assume:
   Mass m [kg] in heater constant
   c_P constant

Energy balance heater + mixer:

$$\frac{d(mc_P T_h)}{dt} = w_h c_P (T_0 - T_h) + Q$$

$$T = \frac{w_h T_h + w_c T_c}{w_c + w_h}$$

2. Linearize:

$y = \Delta T, x = \Delta T_h, u = \Delta w_h$

$$\tau \frac{dx}{dt} = -x + ku$$

$$y = Cx + Du$$

$$k = \frac{T_o^* - T_h^*}{w_h^*}$$

$$\tau = m/w_h^*$$

$$C = \frac{w_h^*}{w_c^* + w_h^*}$$

$$D = \frac{T_h^* - T^*}{w_c + w_h^*}$$

3. Nominal steady-state data:

$T_0 = 10C, T_h = 70C, T = 40C$

$w_h = w_c = 1kg/s, m = 20kg$

Gives:

$$k = \frac{T_o^* - T_h^*}{w_h^*} = \frac{10-70}{1} = -60$$

$$\tau = m/w_h^* = 20/1 = 20$$

$$C = \frac{w_h^*}{w_c^* + w_h^*} = 0.5$$

$$D = \frac{T_h^* - T^*}{w_c + w_h^*} = \frac{70-40}{2} = 15$$

4. Transfer function:

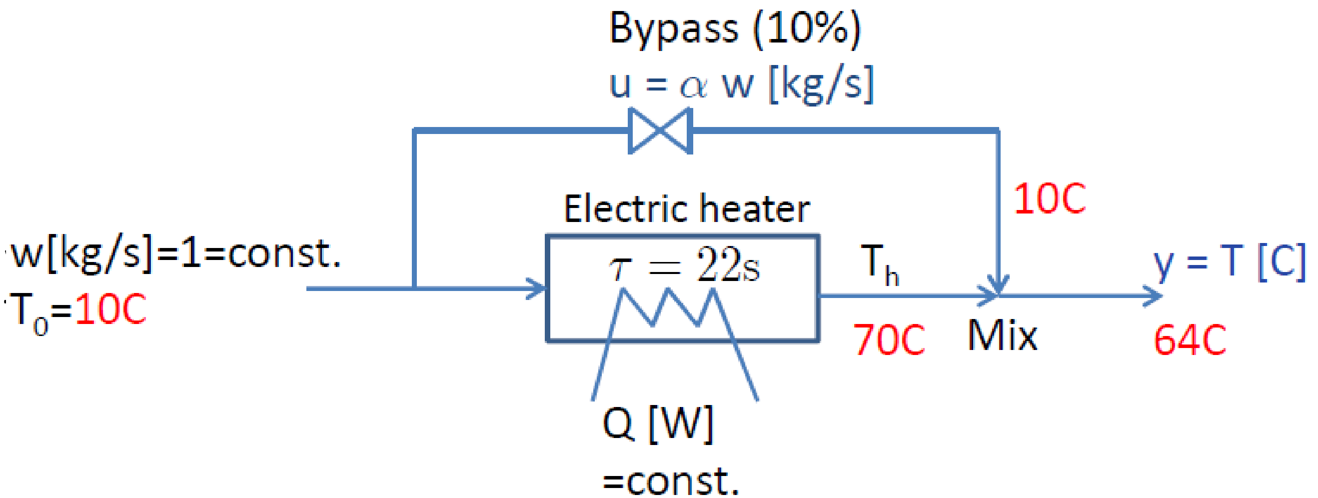
$y(s) = G(s)u(s)$

$$G(s) = C\frac{k}{\tau s + 1} + D$$

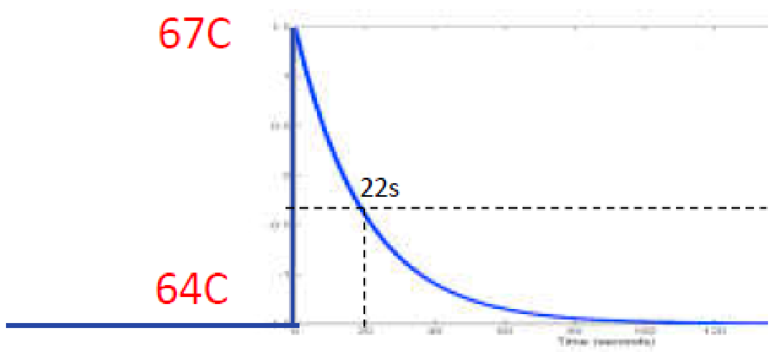
$$= 0.5\frac{-60}{20s+1} + 15$$

$$= -15\frac{-20s+1}{20s+1}$$

Zero at 0 (no steady-state effect)



Response in y=T to a step decrease in bypass fraction from 0.1 to 0.05:

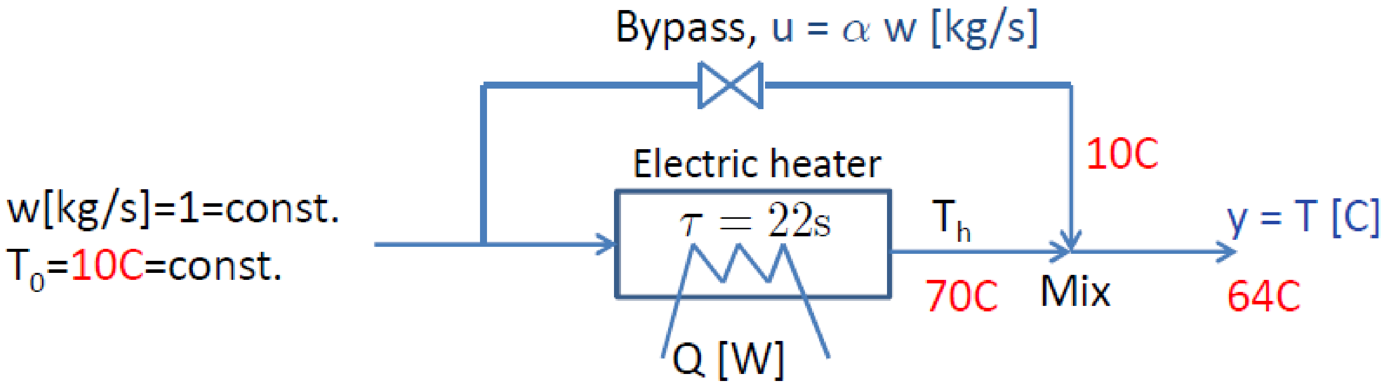


Two effects: 1) Direct effect of mixing: $g_1(s)=-60$

2) Indirect effect of changed T_h: $g_2(s) = 60/(22s+1)$

$$g(s) = g_1 + g_2 = -60\frac{22s}{22s+1}$$

Model derivation



1. Model. Assume:
   Mass m [kg] in heater constant
   c_P constant

Energy balance heater + mixer:

$\frac{d(mc_P T_h)}{dt} = (1-\alpha)wc_P(T_0 - T_h) + Q$

$T = (1-\alpha)T_h + \alpha T_c$

2. Linearize:

$y = \Delta T, x = \Delta T, u = \alpha$

$\tau\frac{dx}{dt} = -x + ku$

$y = Cx + Du$

$k = -\frac{T_o^* - T_h^*}{(1-\alpha^*)}$

$\tau = m/w_h^*$

$C = (1-\alpha^*)$

$D = (T_o^* - T_h^*)$

3. Nominal steady-state data:

$T_0 = 10C, T_h = 70C, T = 64C$

$w = 1kg/s, \alpha = 0.1, m = 20kg$

Gives:

$k = -\frac{T_o^* - T_h^*}{(1-\alpha^*)} = -\frac{10-70}{0.9} = 66.67$

$\tau = m/w_h^* = 20/0.9 = 22$

$C = (1-\alpha^*) = 0.9$

$D = (T_o^* - T_h^*) = -60$

4. Transfer function:

$y(s) = G(s)u(s)$

$G(s) = C\frac{k}{\tau s+1} + D$

$= 0.9\frac{66.67}{22s+1} - 60$

$= 60(\frac{1}{22s+1} - 1) = -60\frac{22s}{22s+1}$

Summary poles and zeros

- $G(s) = n(s) / d(s) = k'(s-z_1) / (s-p_1)(s-p_2)$..
- Example: $G(s) = 4\,(3s-1)/(s^2+s-2)$,

  Get: $k'=12$, $z_1=1/3$, $p_1=-2$ $p_2=1$

- Poles p (=eigenvalues of A)
  - Determine speed of response, exp(p*t)
  - Negative sign in d(s) $\Rightarrow$ p_2 in RHP: unstable, $\exp(p_2 {*} t) \rightarrow \infty$ (NEED control)
  - Pole p complex: oscillating response
- Zeros z
  - Determine shape of response
  - Negative sign in n(s) $\Rightarrow$ z_1 in RHP: inverse response (BAD for control)
  - LHP-zero may give overshoot

5. Approximations of transfer functions

- Skogestad half rule (get effective delay). IMPORTANT!
- Approximation of zeros (you are not expected to remember this)
- Approximation of delay as n(s)/d(s). «Going the other way»
  - Pade approximation

Skogestad Half Rule*

OBTAINING THE EFFECTIVE DELAY θ

Basis (Taylor approximation):

$$e^{-\theta s} \approx 1 - \theta s \quad \text{and} \quad e^{-\theta s} = \frac{1}{e^{\theta s}} \approx \frac{1}{1 + \theta s}$$

Effective delay =

"true" delay

+ inverse reponse time constant(s)

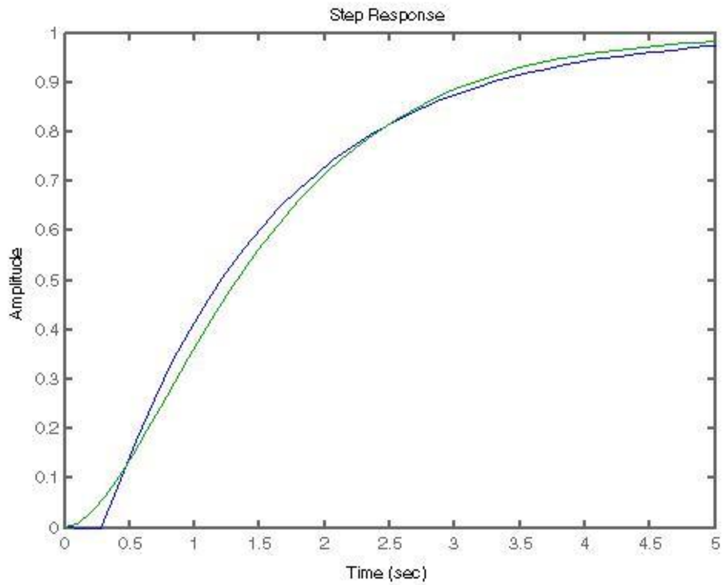
+ half of the largest neglected time constant (the "half rule")
(this is to avoid being too conservative)

+ all smaller high-order time constants

The "other half" of the largest neglected time constant is added to τ_1 (or to τ_2 if use second-order model).

* S. Skogestad, "Simple analytic rules for model reduction and PID controller design", *J.Proc.Control*, Vol. 13, 291-309, 2003 (Also reprinted in MIC)

Example 1



The second-order process

$$g_0(s) = \frac{1}{(1s+1)(0.6s+1)}$$

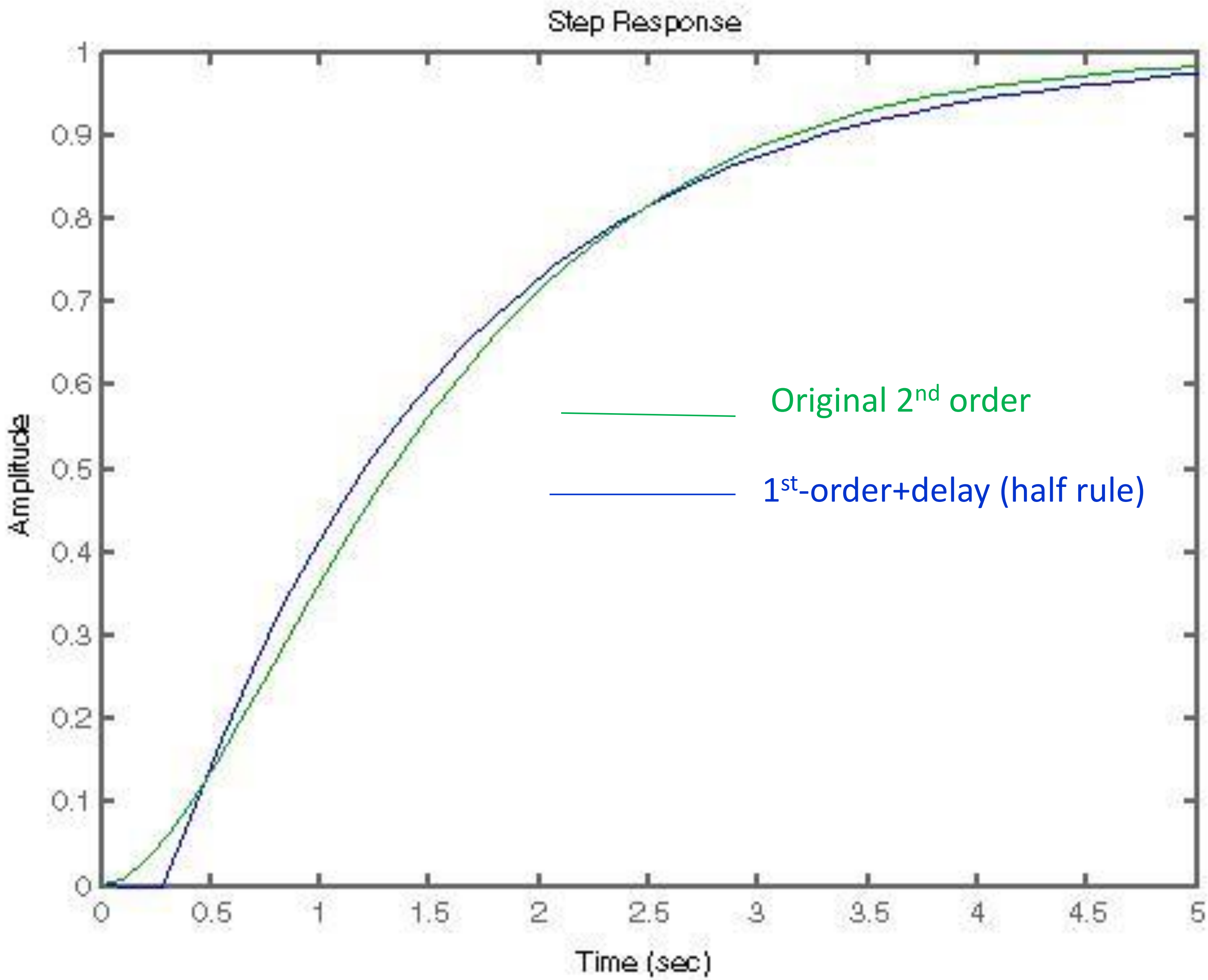
Half rule

is approximated as a first-order with delay process

$$g(s) = k\frac{e^{-\theta s}}{\tau_1 s + 1}$$

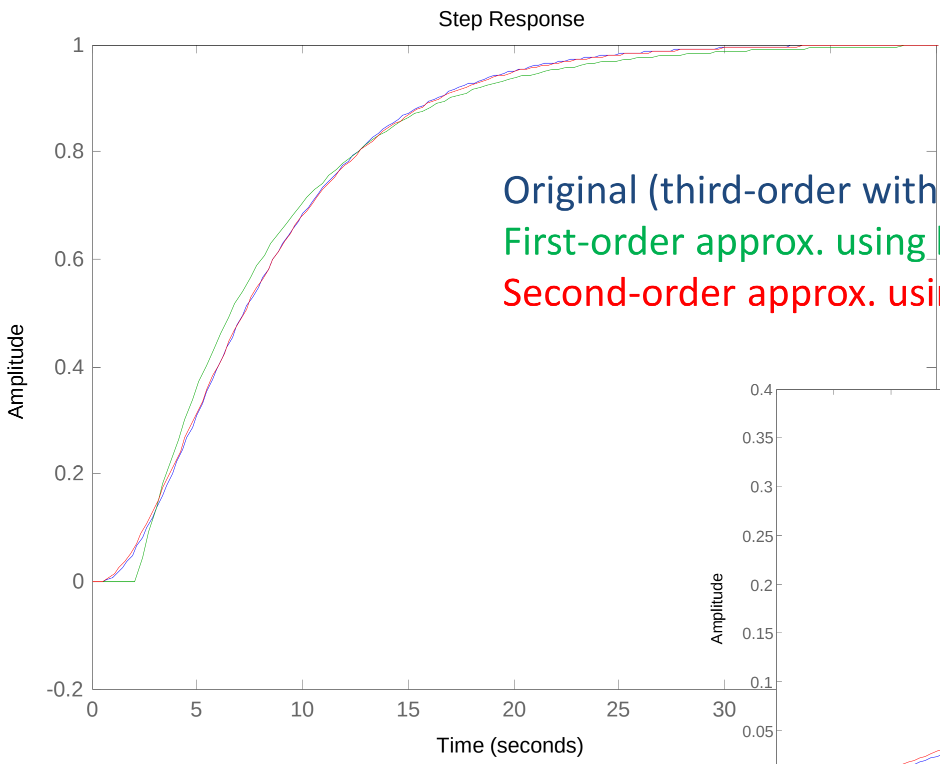
with

$$k = 1; \quad \tau_1 = 1 + 0.6/2 = 1.3; \quad \theta = 0.6/2 = 0.3;$$


Step Response
Amplitude
Time (sec)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Original 2nd order
1st-order+delay (half rule)

Example 2

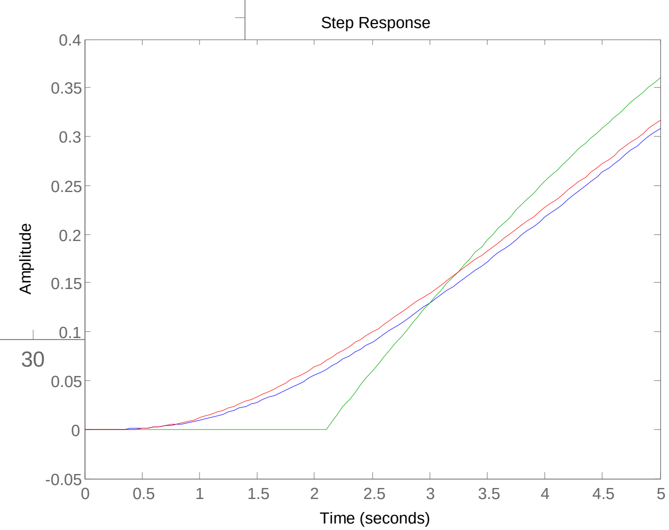
```
s=tf('s')
g=(-0.1*s+1)/[(5*s+1)*(3*s+1)*(0.5*s+1)]
g1 = exp(-2.1*s)/(6.5*s+1)
g2 = exp(-0.35*s)/[(5*s+1)*(3.25*s+1)]
step(g,g1,g2)
```



Original (third-order with inverse response)

First-order approx. using half rule

Second-order approx. using half rule



Example 3. Integrating process

$$g_0(s) = \frac{k'}{s(\tau_{20}s+1)}$$

Half rule gives

$$g(s) = \frac{k' e^{-\theta s}}{s} \text{ with } \theta = \frac{\tau_{20}}{2}$$

Proof:

Note that integrating process corresponds to an infinite time constant

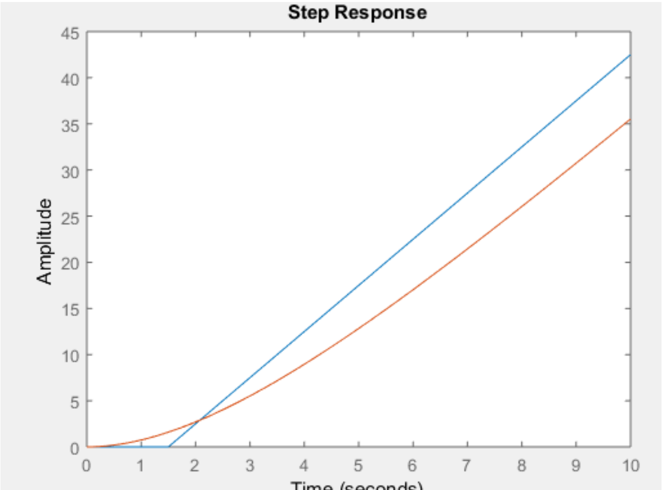
Write

$$g_0(s) = \frac{k'\tau_1}{\tau_1 s(\tau_{20}s+1)} = \frac{k'\tau_1}{(\tau_1 s+1)(\tau_{20}s+1)} \text{ where } \tau_1 \to \infty$$

and then apply half rule as normal, noting that $\tau_1 + \frac{\tau_{20}}{2} \approx \tau_1$:

$$g(s) \approx \frac{k'\tau_1 e^{-\frac{\tau_{20}}{2}s}}{(\tau_1 + \frac{\tau_{20}}{2})s} = k' \frac{e^{-\frac{\tau_{20}}{2}s}}{s}$$

Example. g0 = 5/(s*(3*s+1)),
g = 5*exp(-1.5*s)/s,
step(g,g0,10)



Approximation of LHP-zeros

$$\frac{T_0 s+1}{\tau_0 s+1} \approx \begin{cases} T_0/\tau_0 & \text{for } T_0 \geq \tau_0 \geq \cancel{\theta}\,\tau_c & \text{(Rule T1)} \\ T_0/\cancel{\theta}\,\tau_c & \text{for } T_0 \geq \cancel{\theta}\,\tau_c \geq \tau_0 & \text{(Rule T1a)} \\ 1 & \text{for } \cancel{\theta}\,\tau_c \geq T_0 \geq \tau_0 & \text{(Rule T1b)} \\ T_0/\tau_0 & \text{for } \tau_0 \geq T_0 \geq 5\cancel{\theta}\,\tau_c & \text{(Rule T2)} \\ \frac{(\tilde{\tau}_0/\tau_0)}{(\tilde{\tau}_0-T_0)s+1} & \text{for } \tilde{\tau}_0 \overset{\text{def}}{=} \min(\tau_0, 5\cancel{\theta}\,\tau_c) \geq T_0 & \text{(Rule T3)} \end{cases}$$

To make these rules more general (and not only applicable to the choice $\tau_c=\theta$): Replace θ (time delay) by τ_c (desired closed-loop response time). (6 places)

Example E3. *For the process (Example 4 in (Astrom* et al. *1998))*

$$g_0(s) = \frac{2(15s+1)}{(20s+1)(s+1)(0.1s+1)^2} \tag{13}$$

we first introduce from Rule T2 the approximation

$$\frac{15s+1}{20s+1} \approx \frac{15s}{20s} = 0.75$$

(Rule T2 applies since $T_0 = 15$ is larger than 5θ, where θ is computed below). Using the half rule, the process may then be approximated as a first-order time delay model with

$$k = 2 \cdot 0.75 = 1.5; \quad \theta = 0.1 + \frac{0.1}{2} = 0.15; \quad \tau_1 = 1 + \frac{0.1}{2} = 1.05$$

or as a second-order time delay model with

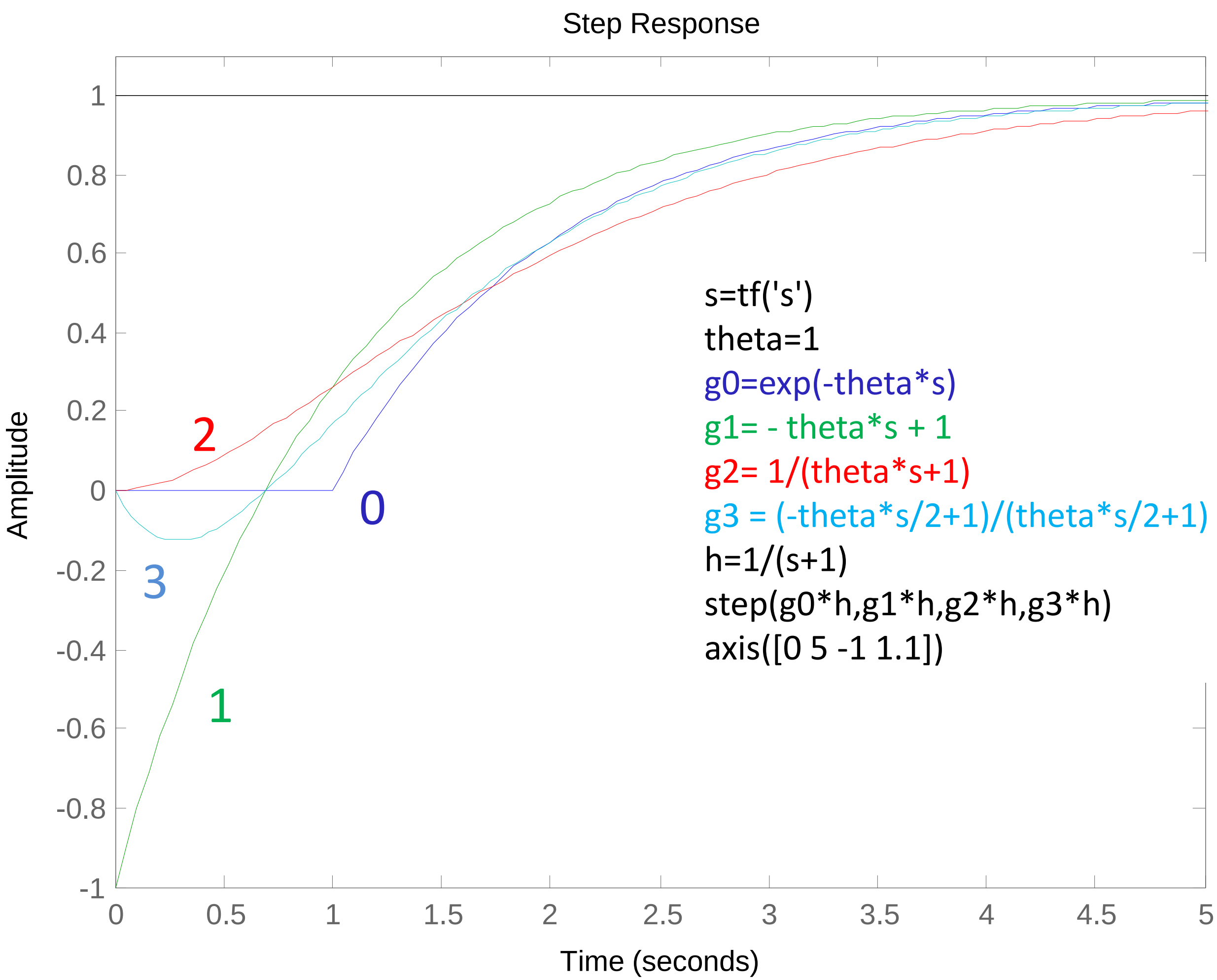
$$k = 1.5; \quad \theta = \frac{0.1}{2} = 0.05; \quad \tau_1 = 1; \quad \tau_2 = 0.1 + \frac{0.1}{2} = 0.15$$

τ_c = desired closed-loop time constant

“Going the other way”

Approximations of time delay

Example: Step response of first-order system plus delay




```
s=tf('s')
theta=1
g0=exp(-theta*s)                        % 0. Original time delay. theta
g1= - theta*s + 1                       % 1. As RHP-zero. –Ts+1, T=theta
g2= 1/(theta*s+1)                       % 2. As First order, tau=theta
g3 = (-theta*s/2+1)/(theta*s/2+1)       % 3. Combined: Pade-approx
h=1/(s+1)
step(g0*h,g1*h,g2*h,g3*h)
axis([0 5 -1 1.1])
```

n'th order Pade approximation of time delay

- Accurate for large n

$$e^{-\theta s} \approx \frac{(-\frac{\theta}{2n}s+1)^n}{(\frac{\theta}{2n}s+1)^n}$$

Note: Number of RHP-zeros
= number of 0-crossings of step response

```
s=tf('s')
theta=1
g0 =exp(-theta*s)                          % Original time delay
g1 = (-theta*s/2+1)/(theta*s/2+1)          % 1st-order Pade-approxima
g2 = (-theta*s/4+1)^2/(theta*s/4+1)^2     % 2nd-order Pade-approxin
g3 = (-theta*s/6+1)^3/(theta*s/6+1)^3     % 3rd-order Pade-approxim
h=1/(s+1)
step(g0*h,g1*h,g2*h,g3*h)
axis([0 5 -0.2 1.1])
```

Why use Pade?
To get model on state space form, dx/dt=Ax+Bu

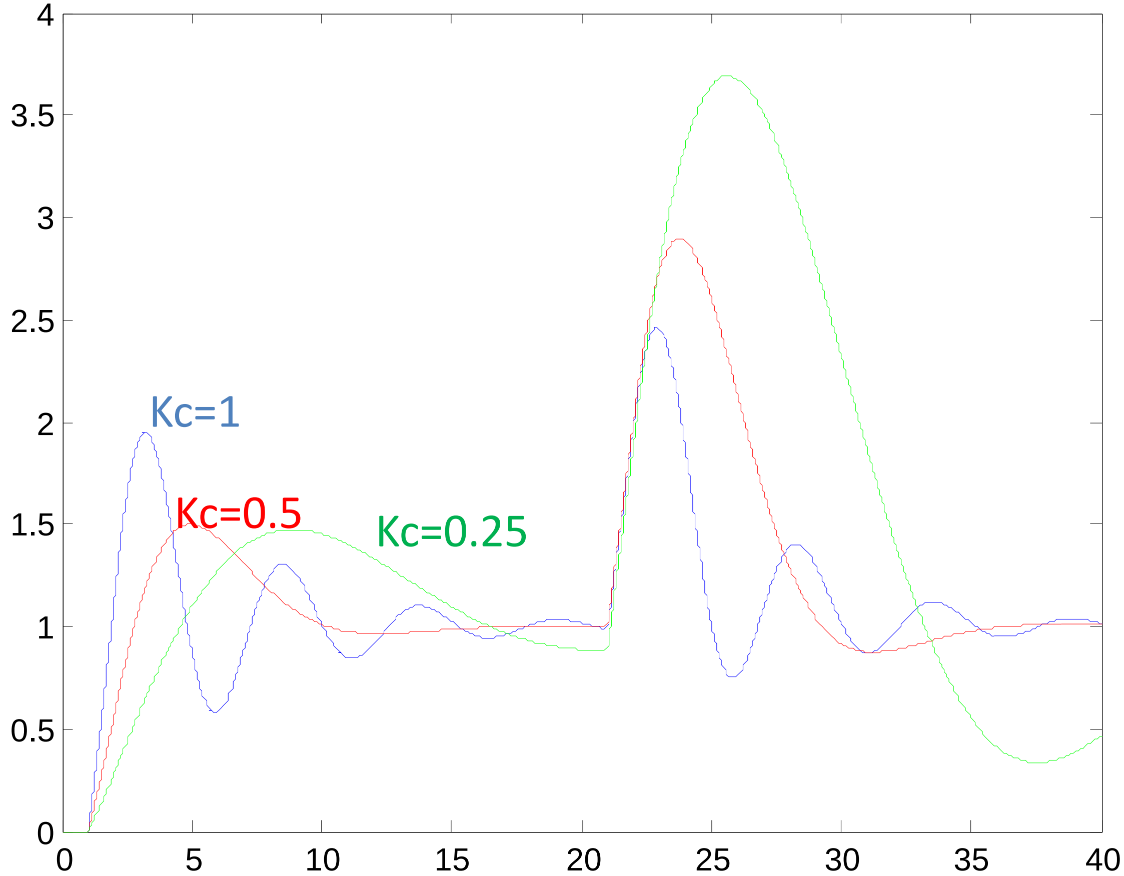
Extra slides

BUT more common case is:
Get oscillation if we have time delay
and use large K_c

PI-control of integrator (level control). G = 1/s, taui=4

ADD DELAY, theta = 1

$$g(s) = \frac{1}{s}e^{-\theta s}$$
$$c(s) = K_c(1 + \frac{1}{\tau_I s})$$




```
%tunepid4
s=tf('s')
theta=1
g=(1/s)*exp(-theta*s) % integrating with delay (lev
taud=0
taui=4
Kc=1 % Too high Kc.
% -> “fast” oscillations because of delay!!
sim tunepid4; plot(Tid,y); hold on %

Kc=0.5 % OK
sim tunepid4; plot(Tid,y,'red');

Kc=0.25 % Too low Kc.
% -> “slow” oscillations from integrator
sim tunepid4; plot(Tid,y,'green');
hold off
```

Setpoint change at t=0

Input disturbance at t=20

CONCLUSION

Kc too small (Kc=0.25): “Slow” oscillations (integrator not stabilize

Kc too large (Kc=1): “Fast” oscillations (because of time delay)

Summary: PI-control of integrating process (level)

$$g(s) = \frac{k'}{s}$$
$$c(s) = K_c(1 + \frac{1}{\tau_I s})$$

1. **Two <u>low</u> controller gain K_c** (combined with too much integral action, i.e. τ_I small):
   Can get «slow» oscillations
   $$Avoid\ slow\ oscillations:\ k' K_C \tau_I \geq 4$$

2. **Too <u>high</u> controller gain K_c** (combined with time delay in the loop):
   Can get the «normal» faster oscillations (and even instability)

Avoid fast oscillations (SIMC): $k' K_c \theta < 0.5$

Case study: k'=1, θ=1, taui=4.
1. Avoid slow oscillations : $K_c > 1$
2. Avoid fast oscillations: $K_c < 0.5$

Both not possible….. The best was K_c=0.5 (see simulation)

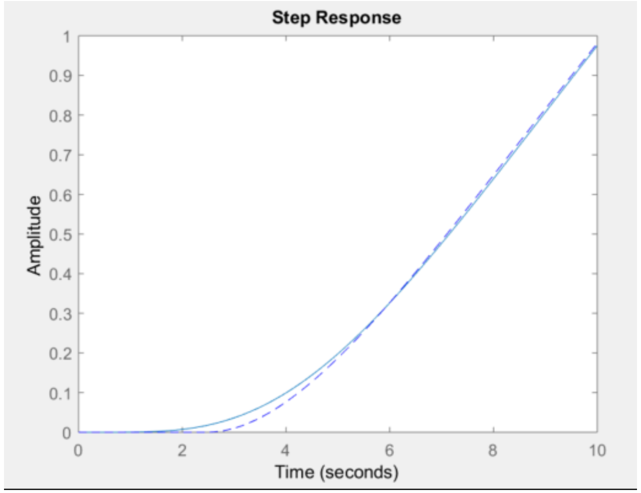
Comment. SIMC-rule would give, Kc=0.5, taui=8

Maybe useful later: Obtaining a model from data using procest (matlab)

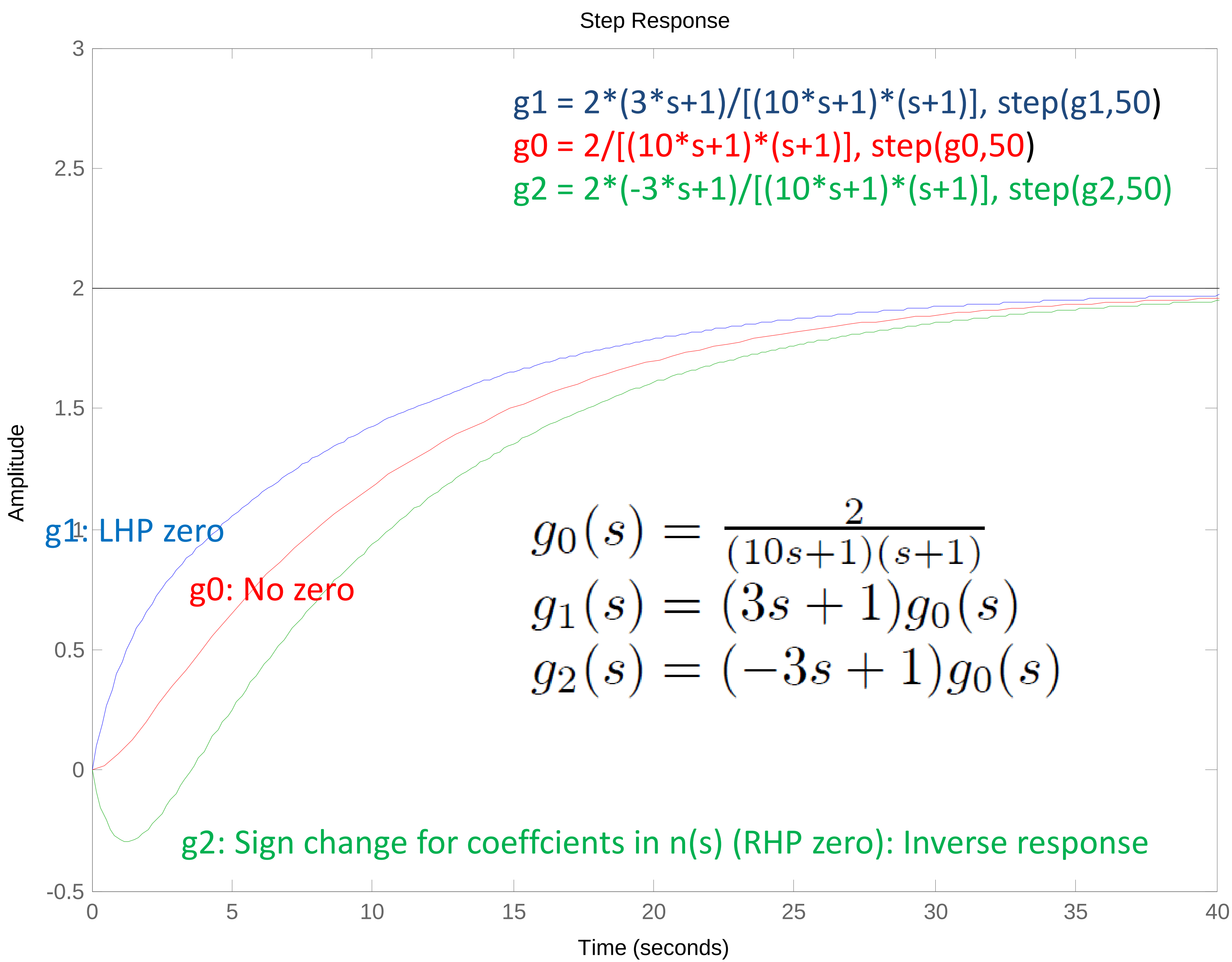
```
% We generate some artifical data from a high-order model
s=tf('s')
G = 3*(1-0.1*s)/((10*s+1)*(3*s+1)*((s+1)^3))

Ts=1;    % sampling time 1 s  (Comment: This may be too long; could make shorter to fit only initial response)
t = Ts*[0:109]';
u = [zeros(10,1); ones(100,1)];  % Step response
y = lsim(G,u,t);

% Now fit it to a second-order plus delay model using Matlab
data=iddata(y,u,Ts);  %
type=('P2D')        % P2D =  2nd order model + delay
sys = procest(data,type)

% Compare the two models

k=sys.Kp; tau1=sys.Tp1; tau2=sys.Tp2; Td=sys.Td;
Gfit = k*exp(-Td*s)/((tau1*s+1)*(tau2*s+1))
step(G,Gfit,'--')
figure(2),step(G,Gfit,'--',10)
```

OUTPUT FROM procest (MATLAB):

```
Process model with transfer function:
              Kp
 G(s) = ---------------- * exp(-Td*s)
        (1+Tp1*s)(1+Tp2*s)

        Kp = 2.9986
       Tp1 = 3.9211
       Tp2 = 9.6838
        Td = 2.478
```



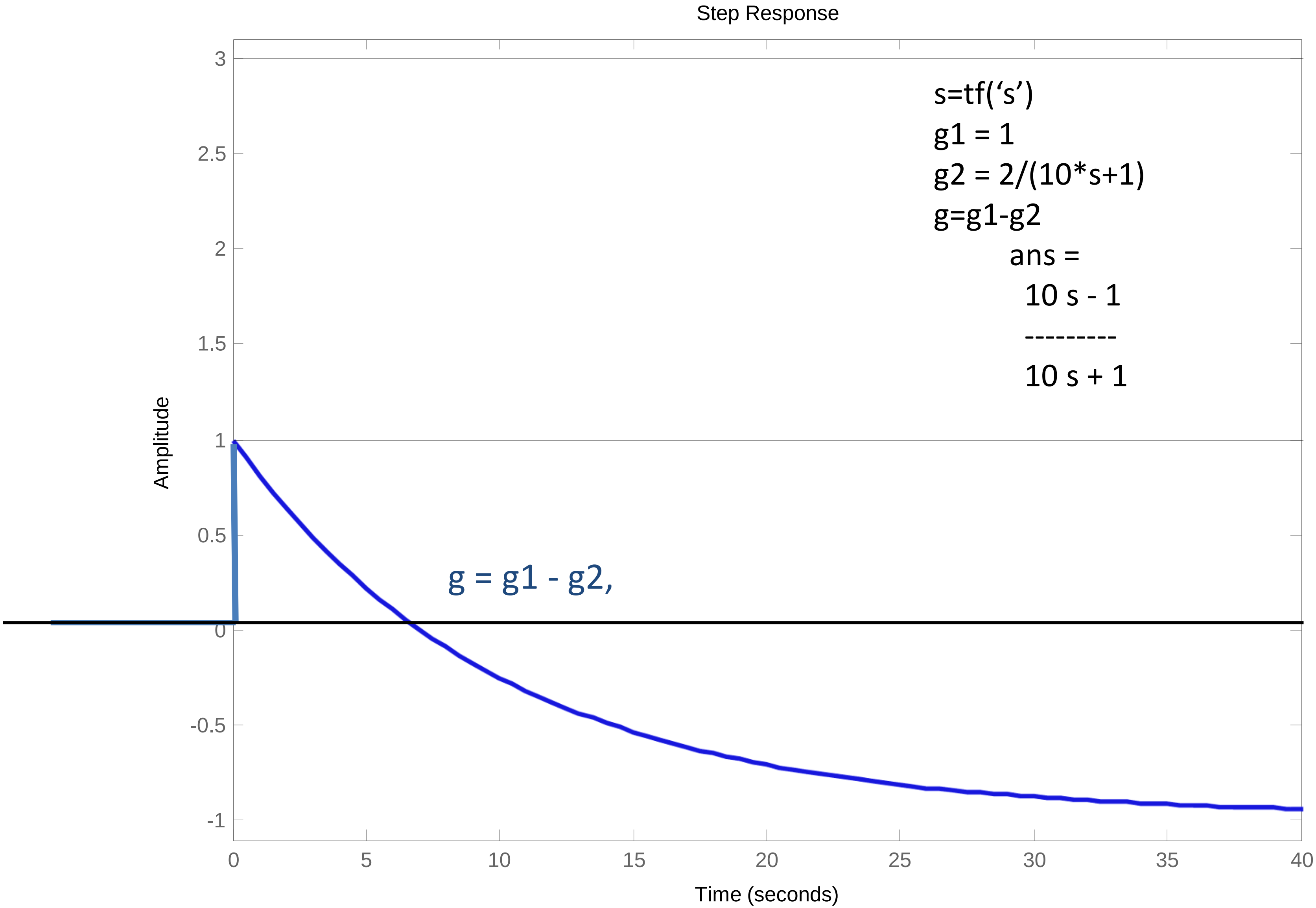
Zeros



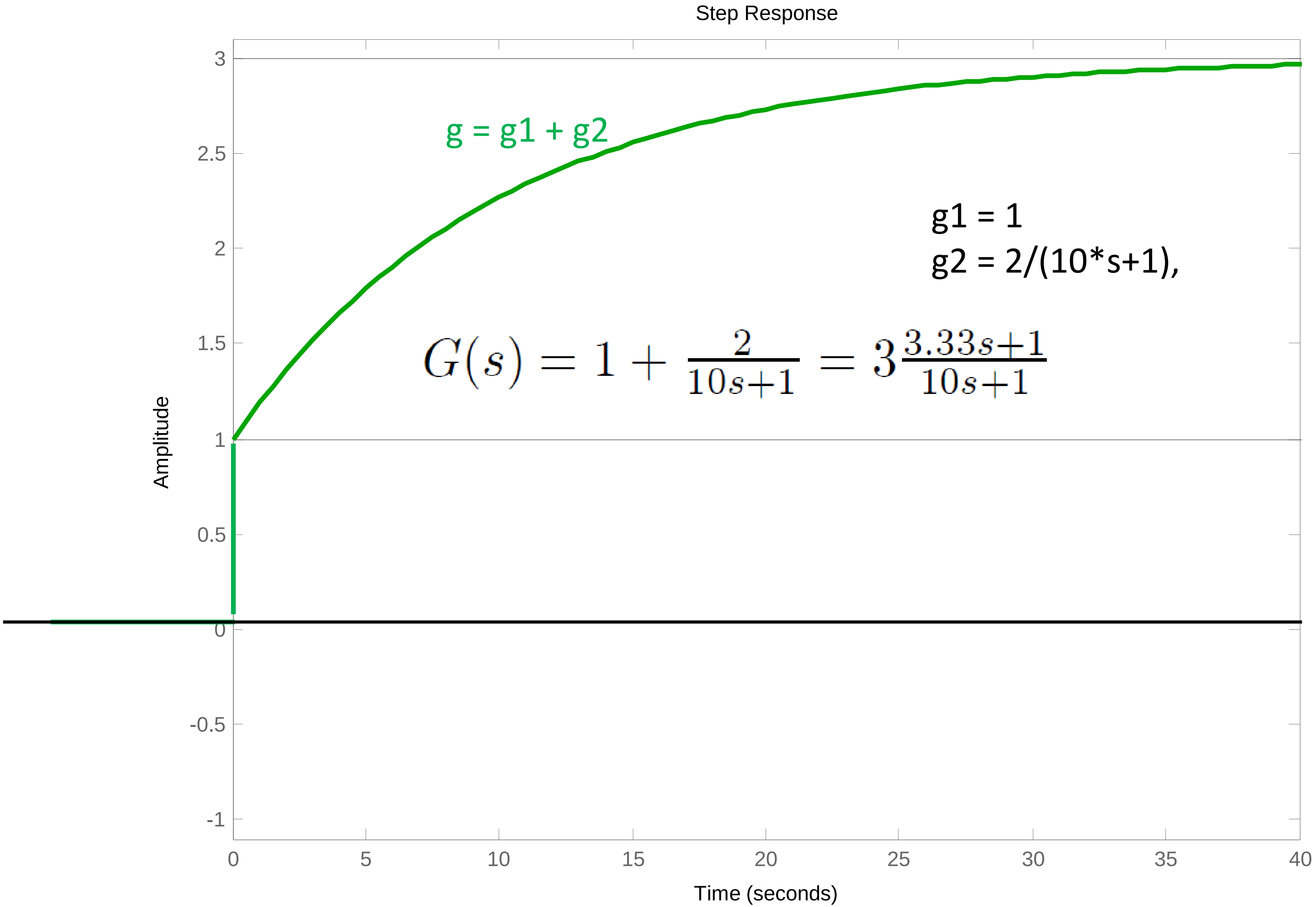
$$g_0(s) = \frac{2}{(10s+1)(s+1)}$$
$$g_1(s) = (3s+1)g_0(s)$$
$$g_2(s) = (-3s+1)g_0(s)$$

Example RHP zero: («competing effects where slow wins»).

Physical example electric heater: Increase hot water flow when Q is constant. $u = q_h$, $y = T$ (see below)



Example LHP zero: Note no overshoot here (since T=3.33 < τ=10)



$$G(s) = 1 + \frac{2}{10s+1} = 3\frac{3.33s+1}{10s+1}$$