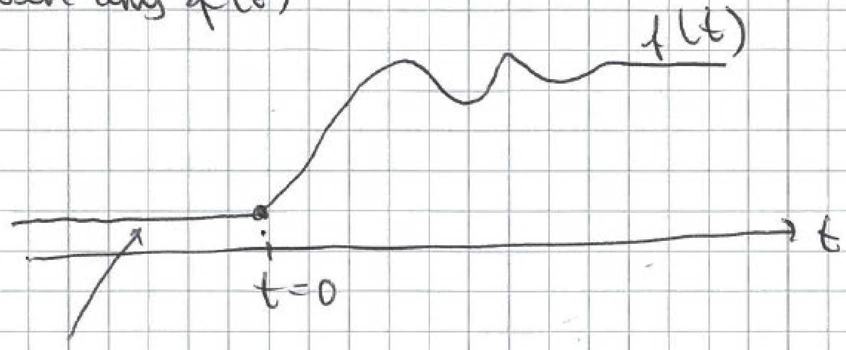
DEFINITION LAPLACE TRANSFORM

Laplace transforms

Given any $f(t)$



Usually at steady state for $t<0$

Laplace transform of $f(t)$

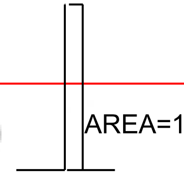
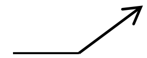
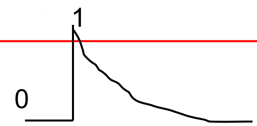
$$F(s) = \mathcal{L}(f(t)) \overset{\text{def.}}{=} \int_0^\infty f(t)e^{-st}\,dt$$

I usually misuse notation and write $f(s)$

"$f(t)$ weighted with e^{-st} and then integrated from $t=0$ to ∞ so that s becomes new independent variable instead of time t"

s [time^{-1}]: Complex number (new indep. variable instead of t)

Table 3.1 Laplace Transforms for Various Time-Domain Functions[a]

	$f(t)$			$F(s)$
1.	$\delta(t)$	(unit impulse)		1
2.	$S(t)$	(unit step)		$\frac{1}{s}$
3.	t	(ramp)		$\frac{1}{s^2}$
4.	t^{n-1}			$\frac{(n-1)!}{s^n}$
5.	e^{-bt}			$\frac{1}{s+b}$
6.	$\frac{1}{\tau} e^{-t/\tau}$			$\frac{1}{\tau s + 1}$
7.	$\frac{t^{n-1} e^{-bt}}{(n-1)!}$	$(n > 0)$		$\frac{1}{(s+b)^n}$
8.	$\frac{1}{\tau^n (n-1)!} t^{n-1} e^{-t/\tau}$			$\frac{1}{(\tau s + 1)^n}$
9.	$\frac{1}{b_1 - b_2} (e^{-b_2 t} - e^{-b_1 t})$			$\frac{1}{(s+b_1)(s+b_2)}$

Rule exponential (e^{-bt}): Opposite sign in Laplace

Table 3.1 Laplace Transforms for Various Time-Domain Functions[a]

$f(t)$	$F(s)$
10. $\frac{1}{\tau_1 - \tau_2}(e^{-t/\tau_1} - e^{-t/\tau_2})$	$\frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)}$
11. $\frac{b_3 - b_1}{b_2 - b_1} e^{-b_1 t} + \frac{b_3 - b_2}{b_1 - b_2} e^{-b_2 t}$	$\frac{s + b_3}{(s + b_1)(s + b_2)}$
12. $\frac{1}{\tau_1}\frac{\tau_1 - \tau_3}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{1}{\tau_2}\frac{\tau_2 - \tau_3}{\tau_2 - \tau_1} e^{-t/\tau_2}$	$\frac{\tau_3 s + 1}{(\tau_1 s + 1)(\tau_2 s + 1)}$
13. $1 - e^{-t/\tau}$	$\frac{1}{s(\tau s + 1)}$
14. $\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
15. $\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
16. $\sin(\omega t + \phi)$	$\frac{\omega \cos \phi + s \sin \phi}{s^2 + \omega^2}$
17. $e^{-bt} \sin \omega t$ (b, ω real)	$\frac{\omega}{(s + b)^2 + \omega^2}$
18. $e^{-bt} \cos \omega t$ (b, ω real)	$\frac{s + b}{(s + b)^2 + \omega^2}$
19. $\frac{1}{\tau\sqrt{1 - \zeta^2}} e^{-\zeta t/\tau} \sin(\sqrt{1 - \zeta^2}\, t/\tau)$ $(0 \le \lvert\zeta\rvert < 1)$	$\frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$

Note: This is step response of first-order system

Table 3.1 Laplace Transforms for Various Time-Domain Functions[a] (continued)

$f(t)$	$F(s)$
20. $1 + \frac{1}{\tau_2 - \tau_1}(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2})$ $(\tau_1 \neq \tau_2)$	$\frac{1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
21. $1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin[\sqrt{1-\zeta^2}\, t/\tau + \psi]$ $\psi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}, \quad (0 \leq \lvert\zeta\rvert < 1)$	$\frac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
22. $1 - e^{-\zeta t/\tau}[\cos(\sqrt{1-\zeta^2}\, t/\tau)$ $+ \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2}\, t/\tau)]$ $(0 \leq \lvert\zeta\rvert < 1)$	$\frac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
23. $1 + \frac{\tau_3 - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_3 - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}$ $(\tau_1 \neq \tau_2)$	$\frac{\tau_3 s + 1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
24. $\frac{df}{dt}$	$sF(s) - f(0)$
25. $\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots$ $- sf^{(n-2)}(0) - f^{(n-1)}(0)$
26. $f(t - t_0)S(t - t_0)$	$e^{-t_0 s}F(s)$

[a]Note that $f(t)$ and $F(s)$ are defined for $t \geq 0$ only.

Alternative forms of step response for 2nd order system (entries 21–22)

Laplace of derivatives (entries 24–25)

Transfer function for time delay μ is $e^{-\mu s}$

Laplace Transform

Definition*: $F(s) = L(f(t)) = \int_0^\infty f(t)e^{-st}dt$

Usually f(t) is in deviation variables so f(t=0) = 0

Examples

1. f(t)=$\delta(t)$ $(impulse)$. $f(s) = 1$

2. f(t)=a (step change)

a/s = Laplace of step a

$$f(s) = \int_0^\infty ae^{-st}dt = -\frac{a}{s}\left[e^{-st}\right]_0^\infty = 0 - \left[-\frac{a}{s}\right] = \frac{a}{s}$$

5. f(t)=e^{-bt}

$$f(s) = \int_0^\infty e^{-bt}e^{-st}dt = \int_0^\infty e^{-(b+s)t}dt = \frac{1}{b+s}\left[-e^{-(b+s)t}\right]_0^\infty = \frac{1}{s+b}$$

*Will often misuse notation and write f(s) instead of F(s)

Most important property for us.
Laplace of derivative.

$$L\left(\frac{df}{dt}\right) = \int_0^\infty \frac{df}{dt} e^{-st} dt = sL(f) - f(0) = s\,F(s) - f(0)$$

Differentiation: replaced by multiplication with s
Integration: replaced by multiplication with 1/s

Proof:

$$\int_0^\infty \frac{df}{dt} e^{-st} dt = \int_0^\infty e^{-st} df$$

Set $v=e^{-st}$ and $du=df$ and use integration by parts

Other properties of Laplace transform:

A. Final value theorem

$$y(t=\infty) = \lim_{s \to 0} sY(s)$$

"steady-state value"

Example: $$Y(s) = \frac{1}{\tau s + 1}\frac{a}{s}$$

$$y(\infty) = \lim_{s \to 0} \frac{a}{\tau s + 1} = a$$

Comment (relevant to step responses):
If Y(s) = g(s)/s then y(t=∞)=g(0)

B. Initial value theorem

$$\lim_{t \to 0} y(t) = \lim_{s \to \infty} sY(s)$$

Example

$$For \quad Y(s) = \frac{4s+2}{s(s+1)}$$

$$y(0) = 4$$

by initial value theorem
(multiply Y(s) by s and set s=1)

$$y(\infty) = 2$$

by final value theorem
(multiply Y(s) by s and set s=0)

C. Initial slope property

$$\lim_{t \to 0} y'(t) = \lim_{s \to \infty} s^2 Y(s)$$

Comment on notation

	Sigurd	Book (Seborg)
	y(t)	y(t)
Steady-state value	y*	$\bar{y}$
Deviation	Δy = y – y*	y' = y - $\bar{y}$
Laplace of Δy(t)	y(s)	Y(s)

$$f(t) \underset{\text{Inverse, } \mathcal{L}^{-1}}{\overset{\text{Laplace}}{\rightleftharpoons}} f(s)$$

1. Complex contour integration
2. Split into sum of simple terms where $f(t)$ is known (partial fraction expansion)

Example 3.2:

1 in new edition

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 4$$

$$y(0) = y'(0) = y''(0) = 0$$

y is deviation variable, system initially at rest (s.s.)

To find transient response for $y(t)$

1. Take Laplace Transform (L.T.)
2. Factor, use partial fraction decomposition
3. Take inverse L.T. Use table

Step 1 Take L.T. (note zero initial conditions)

$$s^3Y(s) + 6s^2Y(s) + 11sY(s) + 6Y(s) = \frac{4}{s}$$

Rearranging,

$$Y(s) = \frac{4}{(s^3 + 6s^2 + 11s + 6)s}$$

Step 2a. Factor denominator of Y(s)

$$s(s^3+6s^2+11s+6) = s(s+1)(s+2)(s+3)$$

Step 2b. Use partial fraction decomposition

$$\frac{4}{s(s+1)(s+2)(s+3)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+1} + \frac{\alpha_3}{s+2} + \frac{\alpha_4}{s+3}$$

Multiply by s, set s = 0

$$\left.\frac{4}{(s+1)(s+2)(s+3)}\right|_{s=0} = \alpha_1 + s\left[\frac{\alpha_2}{s+1} + \frac{\alpha_3}{s+2} + \frac{\alpha_4}{s+3}\right]\Bigg|_{s=0}$$

$$\frac{4}{1 \cdot 2 \cdot 3} = \alpha_1 = \frac{2}{3}$$

For α_2, multiply by (s+1), set s=-1 (same procedure for α_3, α_4)

$$\alpha_2 = -2,\ \alpha_3 = 2,\ \ \alpha_4 = -\frac{2}{3}$$

Step 3. Take inverse of L.T. $\left(Y(s) = \frac{2}{3s} - \frac{2}{s+1} + \frac{2}{s+2} - \frac{2/3}{s+3}\right)$

Use table:

$$y(t) = \frac{2}{3} - 2e^{-t} + 2e^{-2t} - \frac{2}{3}e^{-3t}$$

$t \to \infty$: $y(t) \to \frac{2}{3}$ $t = 0$: $y(0) = 0$. (check original ODE)

You can use this method on any order of ODE, limited only by factoring of denominator polynomial (characteristic equation)

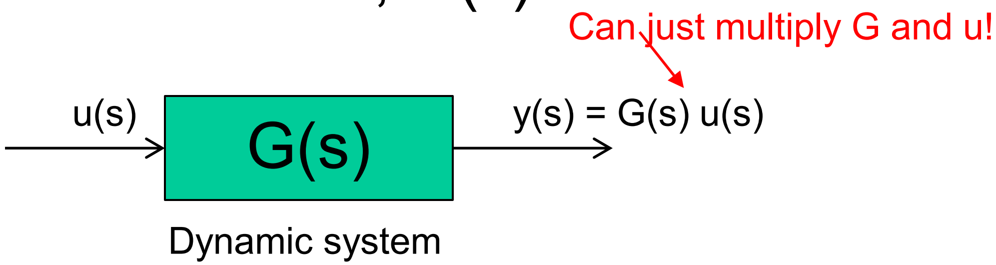
Must use modified procedure for repeated roots, imaginary roots

Use of Laplace Transforms in control

1. Standard notation in dynamics and control for linear systems (shorthand notation)
   - Independent variable: Change from t (time) to s (complex variable; inverse time)
   - Just a mathematical change in variables: Like going from x to y=log(x)

2. Converts differential equations to algebraic operations

3. Advantageous for block diagram analysis.
   Transfer function, G(s):



General procedure in this course

1. Nonlinear dynamic model
2. Steady state model: Use to find missing data
3. Introduce deviation variables and linearize*
4. Laplace of both sides of linear model (t ! s)
5. Algebra ! Transfer function, G(s)
6. Block diagram
7. Controller design

*Note: We will only use Laplace for linear systems!