
The parts in red and most of the plots are not needed for a correct solution.

Problem 1 (35%). Controller tuning

(a) **Response from d2 to y**: $G_d(s) = G_1 G_2 = 20 \frac{e^{-0.3s}}{(3s+1)(8s+1)}$. The step response is second-order with a steady-state gain of $20 > 1$ (see blue curve). So we definitely need control (that is, we need to use $u$). To sketch it, one may apply the half rule as a starting point; get $G_d1 = 20 \frac{e^{-1.8s}}{(9.5s+1)}$; it reaches 63% at $t = 1.8 + 9.5 = 12.3s$ (see red curve; we note that if we make the red curve a bit more S-shaped then we will get close to the correct blue curve.).
(b) The transfer function for designing $c$ is $G_1 G_2 G_m = 20 e^{-(0.3 + 0.5)s}/(8s+1)(3s+1)(0.8s+1)$. Use half rule to find first-order process, $k=20$, $\tau = 8 + 3/2 = 9.5$, $\theta = 0.3 + 0.5 + 0.8 + 3/2 = 3.1$. **SIMC-PI with tight control.** Choose $\tau_{uc} = \theta = 3.1$. Get $K_c \approx \frac{1}{k} \cdot \frac{\tau}{\tau_{uc} + \theta} = 0.077$, $\tau_i = \min(\tau, 4 \cdot (\tau_{uc} + \theta)) = \min(9.5, 24.8) = 9.5$. Comment: I show below closed-loop simulations for a unit step disturbance in $d_2$. Left: $y(t)$. Right: $u(t)$. This is of course not expected at the exam.

(c) Use half rule to find second-order process, $k=20$, $\tau_1 = 8$, $\tau_2 = 3 + 0.8/2 = 3.4$, $\theta = 0.3 + 0.5 + 0.8/2 = 1.2$. **PID with tight control.** Choose $\tau_{uc} = \theta = 1.2$. PID-control (series form): $K_c = \frac{1}{k} \cdot \frac{\tau}{\tau_{uc} + \theta} = 0.167$, $\tau_i = \min(\tau_1, 4 \cdot (\tau_{uc} + \theta)) = \min(8, 9.6) = 8$, $\tau_d = \tau_2 = 3.4$. (In the simulations I added a 1st order filter for the measurement with $\tau_f = 0.34$). NOTE: We have $(1 + \tau_d/\tau_i) = 1.425$ so the tunings for the standard ideal-PID, $c(s) = K_c(1 + 1/\tau_i \cdot s + \tau_d \cdot s)$, are $K'_c = K_c \cdot 1.425 = 0.238$, $\tau_i' = \tau_i \cdot 1.425 = 11.4$, $\tau_d' = \tau_d / 1.425 = 2.386$.

(d) Will it be OK with PI- or PID-control? First, it is clear that PID is beneficial since $\tau_2 = 3.4 > \theta = 1.2$ (get 1.5 points for saying this). However, whether it will be good enough is difficult to tell without doing simulations (as I have done..). A good approach is to look at
the frequency \( \omega_d \) where \(|G_d(j\omega_d)|=1\). We know that we at least need \( \tau_a < 1/\omega_d \). At high frequencies (\( \omega > 1/3 = 0.33 \)) the gain asymptote becomes: \(|G_d(j\omega)|=20/(24\omega^2)\). Get \(|G_d(j\omega_d)|=1\) at \( \omega_d = \sqrt{20/24} = 0.913 \) [rad/s]. This means that we need \( \tau_a < 1/\omega_d < 1.1 \). This is certainly not satisfied with PI-control (\( \tau_a = 3.1 \)), but with PID-control we are almost OK (\( \tau_a = 1.2 \)). So maybe it is OK? No, the actual requirement is \(|S\cdot G_d(j\omega)|<1\) which, since \( S=1/(L+1) \), gives the approximation \(|L|>|G_d|\) at low frequencies. Since \(|G_d|\) has a slope of -2 (while \(|L|\) has about -1 at \( \omega_c = 1/\tau_a \)) this is difficult to satisfy at \( \omega<\omega_c \) even though the condition \( \tau_a < 1/\omega_d \) means that we have satisfied \(|L|>|G_d|\) at \( \omega=\omega_c \). Indeed, the simulation show that \( y \) goes almost up to 4 even with PID control (with PI it goes to 8).

(e) Block diagram feedforward (assuming perfect measurement of \( d_2 \)):

In this case \( G = G_1 \cdot G_2 = G_d \) (Note that the measurement dynamics for \( y \) don’t matter when we consider feedforward). Ideal feedforward then gives \( c_{FF} = -G_d/G = -1 \). Yes, FF is recommended as it ideally gives perfect control for disturbance \( d_2 \). Here “Ideally” = perfect model and perfect measurement of disturbance. (Comment: FF will also in theory be perfect for \( d_1 \)).

(f) Block diagram cascade (assuming here perfect measurement of \( y_2 \), that is, \( g_{2m}(s)=1 \)):
C2 is designed based on $G_2=2$ (so $\tau_2=0$, $\theta_2=0$). We use $\tau_{ac2}=0.6$. This value is reasonable if the effective delay in $G_2$ is less than about $0.6/2=0.3$. Get with SIMC: $K_{c2}=0$, $\tau_{ai}=\min(0,4*0.6)=0$ (!). The integral time is zero so this is actually an I-controller with $K_I=(1/2)*1/(\tau_{ac2}+\theta_2)=1/(2*0.6)=0.833$. So $c_2=0.833/s$. (if you want to approximate it s a PI-controller then select $K_c$ small and use $\tau_{ai}=K_c/0.833$). Yes, cascade control will be helpful for $d_2$ because of the dynamics and delay in both $G$ and the measurement (it will be helpful even with feedforward because there are always nonlinearity in $G_2$ and error in the measurement of $d_2$). Yes, we need to retune $c$, in particular, because the gain in $T_2$ is 1 and in $G_2$ it is 2.

(g) New tunings for $c$ with cascade. Replace $G_2$ by $T_2$, where $T_2=1/(\tau_{ac2}s+1)=1/(0.6s+1)$ (in this case this is exact; not an approximation). The half rule now gives the following 1st order with delay model for design of $c$: $k=10$, $\tau=8+3/2=9.5$, $\theta=0.3+0.5+3/2+0.8+0.6=3.7$. SIMC-PI with tight control gives $\tau_{ac}=\theta=3.7$. New PI-controller $c$ becomes: $K_c=(1/k)*\tau/(\tau_{ac}+\theta)=0.13$, $\tau_{ai}=\min(\tau,4*(\tau_{ac}+\theta)=\min(9.5,29.6)=9.5$.

**Simulations with cascade** (below, $y$ is left, $u$ is right) show that we (as expected) get good response for disturbance $d_2$ with $|y|<0.8<1$. We see that slave controller $c_2$ makes the input $u(t)$ drop quickly down to the “ideal” value of -1.
Problem 2 (35%). Closed-loop stability

Solution

\[ C(s) = \frac{Kc(\tau i s + 1)}{\tau i s} \]

With \( \tau i = 20 \) and \( 4/20 = 0.2 \) we get \( L(s) = G(s)C(s) = \frac{0.2Kc(-2s+1)}{s(6s+1)} \)

Closed-loop transfer functions have \( 1+L(s) \) in denominator. Multiplying to get a polynomial, we find that

\[ d(s) = s(6s+1) + 0.2Kc(-2s+1) = 6s^2 + (1 - 0.4Kc)s + 0.2Kc. \]

With \( Kc = 1 \) we get \( d(s) = 6s^2 + 0.6s + 0.2 \) (note that we may multiply \( d(s) \) by any constant). All coefficients are positive so it’s closed-loop stable. Yes, the condition is necessary and sufficient for a 2nd order system.

(b) Gain margin. The second coefficient in \( d(s) \) becomes zero for \( Kc = 1/0.4 = 2.5 \) (so this is the maximum gain to have stability). So with \( Kc = 1 \), we have that \( GM = 2.5 \) (20 log(2.5) = 7.96 dB).

(c) Bode plot with GM and PM. I here used Matlab (the phase plot is a bit strange; it should start from -90 from the integrator, but Matlab adds 360 so it starts from 270; mathematically it’s the same because a complex number comes back to the same value of you add any multiple of 360 degrees). For the solution it OK with an approximate plot based on asymptotes. There are three asymptotes for \( L \) (using \( Kc = 1 \)):

- low \( w \): \( L = 0.2/s \) (slope=-1, phase=-90).
- \( w > w_b1 = 1/6 \): \( L = 0.2/s*6s \) (slope=-2, phase=-180).
- \( w > w_b2 = 1/2 \): \( L = -0.2*2/6s \) (slope=-1, phase=-270).

(d) DM=PM/wc. PM is evaluated at frequency \( w_{180} \) where phase of \( L \) is -180.

\[ GM \text{ is evaluated at frequency } w_{180} \text{ where phase of } L \text{ is } -180 \]

\[ s = tf('s') \]
\[ L = 0.2*(-2*s+1)/(s*(6*s+1)) \]
\[ Margin(L) \]

To find PM analytically we first find \( wc \) by solving

\[ |L(jwc)| = 0.2*\sqrt{4*wc^2+1}/(wc*\sqrt{36*wc^2+1}) = 1. \]
This gives $wc = 0.154$. At this frequency the phase is:

$$\text{Phase } L = -\pi/2 - \text{atan}(2*wc) - \text{atan}(6*wc) = -2.6155 \text{ rad},$$

so the phase margin is $3.14 - 2.615 = 0.527 \text{ rad. QED,}$

Similarly, to find $GM$ analytically, we find $w_{180}$ as the frequency where phase$(L) = -\pi \text{ rad}$ and then we evaluate $|L|$ at this frequency. We iterate on $w$ until the phase is $-180$. This gives $w = 0.289$ and

$$|L(jw_{180})| = 0.2*\sqrt{ (2*0.289)^2+1} / 0.289* \sqrt{(6*0.289)^2+1} = 0.40,$$

so $GM = 1/|L(jw_{180})| = 2.5.$

(e) P-control. $L(s) = G(s)C(s) = \frac{4}{Kc(-2s+1)(20s+1)(6s+1)}$.

Pole polynomial: $d(s) = (20s+1)(6s+1) + 4 Kc(-2s+1) = 120 s^2 + (26-8Kc)s + (1+4Kc)$.

2nd coeff. Is zero when $Ku = 26/8 = 3.25$.

At this point $d(s) = 120 s^2 + (1+4Ku) = 120 s^2 + 14$

Poles are solutions to $d(s) = 0$: Get $s = \pm j\omega_u$ where $\omega_u = \sqrt{14/120} = 0.3416 \text{ rad/s}$. So $Pu = 2\pi/\omega_u = 18.4 \text{ s}$.

ZN-tunings: $Kc = 0.45$ $Ku = 1.46$ and $taui = Pu/1.2 = 15.3 \text{ s}$.

(f) SIMC: Half rule gives $\theta = 6/2+2 = 5$, $\tau = 20+6/2 = 23$. With $\tau_a = \theta$ we get:

$Kc = (1/4)*23/(5+5) = 0.575$. $taui = 23$ (which is much less aggressive than ZN).

NOT EXPECTED ON EXAM: Below is a comparison of the response $y(t)$ with the three PI-controllers (Blue; $Kc=1$, $taui=20$, Brown: ZN, Yellow: SIMC). “As usual”, the response $y(t)$ is for a step setpoint change (at $t=0$) followed by a step input disturbance (as $t=100$). I think SIMC is the best although the disturbance response is a bit slow; it has the best setpoint response and is by far the most robust (see box). ZN will easily go unstable since the gain margin is only $1.54.$
Problem 3 (30%). Modelling and control of mixing process

Solution:

(a) Linear valve equation: \( F_1 = C_v z_1 \sqrt{(p_1-p_2)/\rho} \). Linearized: \( \Delta F_1 = k \Delta z_1 \) where \( k = C_v \sqrt{(p_1-p_2)/\rho} \). We note that process gain \( k \) varies with the square root of the pressure difference \( DP=p_1-p_2 \). But \( k \) is constant if we control \( p_1 \) and \( p_2 \).

(b) (i) Total mass balance (assuming constant density):
\[
\frac{dV}{dt} = F_1 + F_d - F_2
\]  
(1)

Component balance:
\[
\frac{d(c_2 V)}{dt} = F_1 c_1 + F_d c_d - F_2 c_2 
\]  
(2)

(ii) At steady state, the last balance gives (with \(c_1=0\)): 

\[
0 = F_d c_d - F_2 c_2. 
\]

Thus, \(F_d = \frac{F_2 c_2}{c_d} = \frac{2*20}{700} = 0.0571 \text{ m}^3/\text{min}\).

(iii) Assuming \(V\) constant and using \(c_1=0\), the balances become:

\[
F_1 + F_d = F_2 \\
V \frac{dc_2}{dt} = F_d c_d - F_2 c_2 
\]

Linearize and deviation variables (assuming \(F_d\) and \(c_d\) constant):

\[
\Delta F_1 = \Delta F_2 \\\nV \frac{d\Delta c_2}{dt} = -F_2^* \Delta c_2 - c_2^* \Delta F_2 
\]

Combining

\[
V \frac{d\Delta c_2}{dt} = -F_2^* \Delta c_2 - c_2^* \Delta F_1 
\]

Taking Laplace (in deviation variables):

\[
V s c_2(s) + F_2^* c_2(s) = -c_2^* F_1(s) 
\]

So

\[
c_2(s) = -\frac{c_2^*}{F_2^*} F_1(s) = -\frac{10}{\tau s + 1} F_1(s) 
\]

where \(\tau = \frac{V}{F_2^*} = 0.36/2 = 0.18 \text{ min} = 11 \text{ s}\).

Comment: It is also possible to linearize everything without making any assumptions. It gives the same result, and we don’t need to assume \(V\) constant: Combining with (1), the left hand side of the component balance (2) becomes

\[
\frac{d(c_2 V)}{dt} = V \frac{dc_2}{dt} + c_2 \frac{dV}{dt} = V \frac{dc_2}{dt} + c_2 (F_1 + F_d - F_2) 
\]

and we get (exactly!)

\[
V \frac{d(c_2)}{dt} = F_1 (c_1 - c_2) + F_d (c_d - c_2) 
\]

Note that the outflow \(F_2\) drops out of the component balance (as usual). Linearize and deviation variables:

\[
V^* \frac{d(\Delta c_2)}{dt} = F_1^* (\Delta c_1 - \Delta c_2) + F_d^* (\Delta c_d - \Delta c_2) + (c_1^* - c_2^*) \Delta F_{1-d} + (c_d^* - c_2^*) \Delta F_{d} 
\]

Assuming pure water in stream 1 (\(c_1=0, \Delta c_1 = 0\)), and assuming \(F_d\) and \(c_d\) constant (\(\Delta F_d = 0, \Delta c_d = 0\)) and using \(F_2^* = F_1^* + F_d^*\) gives as before equation (*):

\[
V^* \frac{d\Delta c_2}{dt} = -F_2^* \Delta c_2 - c_2^* \Delta F_1 
\]
Comments for control loops:

- Ratio control from $F_d$ to $F_1$ (we almost always have ratio of feeds to a mixer). It would be possible to have ratio control from $F_d$ to $F_2$, but it is most common to use it on the feeds.
- The inventory control (involving $p_1$ and $p_2$) is radiating around the given flow ($F_d$, $F_1$).
- The pressure control on $p_2$ in the tank may be replaced by a level controller on $V$. This would assume that there is gas inside the mixer so that there is a level, and the pressure $p_2$ inside the tank would then need to be set some other way, for example, by a vent.
- The ratio setpoint is set by a slow composition controller (CC)
- The slave flow controller for $F_1$ is needed for implementing ratio control, but it also give disturbance rejection for $p_1$ and $p_2$ (they are withing the slave loop) and it linearizes the nonlinear valve.
- Order of tuning: First implement FC for $F_{12}$ (fast), then the ratio controller ($X$), then the two pressure controllers and finally the composition controller.

Comment: The flowsheet in Problem 3 may correspond to parts of the block diagram in Problem 1 (with time in seconds and excluding $z_2$ and $ω$). We have $d_2=p_1$, $d_1=F_d$, $u=z_1$, $y=c$ and $y_2=F_1$. $G_2=2$ represents the valve. For the mixer ($G$), the two time constants of 8s and 3s (rather than a single time constant of 11s) may represent some zones in the mixer (mixing is not immediate). The delay in $G$ of 0.3s may be due to the pipeline out of the mixer. The measurement delay is 0.5s and there is also a time constant of 0.8s for the measurement of $y=c_2$. 
COMMENTS MADE AFTER CORRECTING THE EXAMS (from Lucas Cammann)

Problem 1

1) As you already saw in the exams that you corrected, not many people drew the second order response correctly. There were some however, and others drew the response obtained from the half-rule and wrote next to it that it should have an S-shape.
2) Many people forgot to include Gm when doing the tunings in exercise 1 b) and c). I deducted three points in b) for this, and another point in c).
3) I noted that almost all the students seemed to understand cascade well, but were struggling with the concept of feedforward. Here I was also a bit strict with the sketch, most of the students did not manage to put c_ff into the block diagram correctly. But this was only one point anyways.

Problem 2

4) I was surprised that quite some students managed to get more points in the second problem than in the first because this seems to have more “easy points” with the two tuning exercises.
5) The most common mistake I saw was a sign error in the phase calculation, where students neglected the minus of the RHP zero.
6) Another common mistake, sadly, was that some students constructed the Bode plots for G instead of L.

Problem 3

7) Problem 3 gave the least points, relatively speaking, potentially also because there were no bonus points to be obtained.
8) Very little people got rewarded the entire 5 points of a), because almost nobody reported the gain.
9) Problem b) worked relatively well, but some people did very weird things in the linearization. Something a lot of people did was to only set up the steady-state mass and component balances, linearize them and then have an “s” magically appear out of nowhere during the Laplace transform.
10) Exercise c) was hit or miss. There were some weird suggestions, and often times I had the impression that the students remembered something vaguely and then tried to make up the rest to make it fit. But this might also be because this was the last exercise, if the students worked through the exam from start to finish.