A) Energy balance:

\[ \frac{\partial (H \cdot m)}{\partial t} = H_{in} \cdot m_{in} - H_{out} \cdot m_{out} + Q \]

Assume \( T_{in} = 0 \) and well mixed tank (\( T_{in} = T_{out} = T \))

\[ H = C_r (T - T_{in}) = C_r T \]
\[ H_{in} = q \cdot T_{in} \cdot C_r \]
\[ H_{out} = q \cdot T \cdot C_r \]

Constant mass and \( C_r \):

\[ \frac{\partial (H \cdot m)}{\partial t} = \frac{2}{\delta t} (C_r \cdot T \cdot m) = m \cdot C_r \frac{2T}{\delta t} \]

\[ m \cdot C_r \frac{2T}{\delta t} = q \cdot C_r \cdot T_{in} + 4 \cdot C_r \cdot T + UA (T_0 - T) \]
At steady state:

\[ \frac{dT}{dt} = 0 \]

\[ Q = Q^* = 10 \text{ kg/s} \]
\[ m = 5000 \text{ kg} \]
\[ C_p = 4184 \text{ kJ/kg} \]
\[ Q_{in} = Q^*_{in} = 1 \text{ kg/s} \]
\[ T_{in} = T^*_{in} = 50^\circ C \]
\[ T_c = T^*_c = 10^\circ C \]
\[ UA = 42 \text{ kW/k} \]

\[ m \cdot C_p \frac{dT}{dt} = Q \cdot C_p \cdot T_{in} - Q^* \cdot C_p \cdot T + UA \left( T_c - T \right) \]

\[ 0 = Q^* \cdot C_p \cdot T^*_{in} - T^* \left( Q^* \cdot C_p + UA \right) + UA \cdot T^*_c \]

\[ \Rightarrow T^* = \frac{Q^* \cdot C_p \cdot T^*_{in} + UA \cdot T^*_c}{Q^* \cdot C_p + UA} \]

\[ = \frac{10 \text{ kg/s} \cdot 4184 \text{ kJ/kg} \cdot 50^\circ C + 42 \text{ kW/k} \cdot 10^\circ C}{10 \text{ kg/s} \cdot 4184 \text{ kJ/kg} + 42 \text{ kW/k}} \]

\[ = 30^\circ C \]
Original equation:

\[ mC_p \frac{\partial T}{\partial t} = qC_p T_{in} - qC_p T + UA(T_c - T) = k \]

Using a first order Taylor expansion:

\[ mC_p \frac{\partial k}{\partial t} = \frac{\partial k}{\partial T} \Delta T + \frac{\partial k}{\partial T_c} \Delta T_c + \frac{\partial k}{\partial q} \Delta q + \frac{\partial k}{\partial q_c} \Delta q_c \]

\[ \Delta T, \Delta T_{in}, \Delta T_c, \Delta q \text{ and } \Delta q_c \text{ are deviation variables.} \]

\[ \frac{\partial k}{\partial T_{in}} = qC_p \]

\[ \frac{\partial k}{\partial T_c} = UA \]

\[ \frac{\partial k}{\partial q} = C_p T_{in} - C_p T' = C_p (T_{in} - T') \]

\[ \frac{\partial k}{\partial q_c} = 0 \]

\[ \frac{\partial k}{\partial T} = -qC_p - UA \]

This yields a linearized model equal to:

\[ mC_p \frac{\partial k}{\partial t} = qC_p \Delta T_{in} + UA \Delta T_c + C_p (T_{in} - T') \Delta q + 0 \cdot \Delta q_c - (qC_p + UA) \Delta T \]
Taking the Laplace Transform of both sides of the equation yields:

\[ m \cdot c_p \cdot s \cdot T(s) = \frac{q^*_c}{c_p} T_{in}(s) + U A T_c(s) \]

\[ + \frac{c_p (T_{in}^0 - T^0)}{m c_p \cdot s + (q^*_c + U A)} q(s) \]

\[ T(s) \left( m \cdot c_p \cdot s + (q^*_c + U A) \right) = \frac{q^*_c}{c_p} T_{in}(s) + U A T_c(s) \]

\[ + \frac{c_p (T_{in}^0 - T^0)}{m c_p \cdot s + (q^*_c + U A)} q(s) \]

\[ T(s) = \frac{\frac{q^*_c}{c_p} T_{in}(s) + U A}{m c_p \cdot s + (q^*_c + U A)} T_c(s) \]

\[ + \frac{c_p (T_{in}^0 - T^0)}{m c_p \cdot s + (q^*_c + U A)} q(s) \]

\[ \frac{q^*_c}{q^*_c + U A} T_{in}(s) + \frac{U A}{q^*_c + U A} T_c(s) \]

\[ = \frac{\frac{q^*_c}{c_p} \cdot s + 1}{c_p (T_{in}^0 - T^0)} \frac{q^*_c}{q^*_c + U A} \]

\[ + \frac{\frac{m c_p}{q^*_c + U A} \cdot s + 1}{q^*_c + U A} q(s) \]

\[ \Rightarrow \quad q_1 = \frac{\frac{q^*_c}{q^*_c + U A} \cdot s + 1}{q^*_c + U A} \]

\[ q_2 = \frac{U A}{q^*_c + U A} \cdot s + 1 \]

\[ q_3 = \frac{\left( c_p (T_{in}^0 - T^0) \right)}{q^*_c + U A} \]

\[ q_4 = 0 \]
After tuning, this makes the transfer functions equal to

\[ g_1 = \frac{0.5}{250 \cdot s + 1} \]

\[ g_2 = \frac{0.5}{250 \cdot s + 1} \]

\[ g_3 = \frac{1}{250 \cdot s + 1} \]

\[ g_4 = 0 \]
\[ V_1 = 1 \text{ m}^3 \]
\[ V_2 = 4 \text{ m}^3 \]
\[ q = 0.2 \text{ m}^3/\text{s} \]

a) \[ T_2(s) = g_2(s) T_1(s) \]
\[ T_1(s) = g_1(s) T_0(s) \]

The tanks are well mixed, so \( g_1 \) is given as

\[ g_1(s) = \frac{k}{c_1 s + 1} \]

\( c_1 \) is equal to the residence time in tank 1:

\[ c_1 = \frac{V_1}{q} = \frac{1 \text{ m}^3}{0.2 \text{ m}^3/\text{s}} = 5 \text{ s} \]

\( k \) is the gain, which is equal to 1, since \( T_1 \) will equal \( T_0 \) at steady state

\[ \Rightarrow \quad g_1(s) = \frac{1}{5 s + 1} \]

The same reasoning can be used for \( g_2 \):

\[ g_2(s) = \frac{k}{c_2 s + 1} = \frac{1}{4/c_2 s + 1} \]

\[ g_2(s) = \frac{1}{20 s + 1} \]
\[ T_0(s) = \frac{1}{s} \]

\[ T_1(s) = 9_1(s) \cdot T_0(s) = \frac{1}{(5s+1) \cdot s} \]

\[ T_2(s) = 9_2(s) \cdot T_1(s) = \frac{1}{(20s+1)(5s+1) \cdot s} \]

**Transfroming to time domain:**

\[ T_0(t) = 5(s) \text{ (unit step at } t=0) \]

\[ T_1(t) = 1 - e^{-t/5} = 1 - e^{-0.2t} \]

\[ T_2(t) = 1 + \frac{1}{5 - 20} \left( 20e^{-t/20} - 5e^{-t/5} \right) \]

\[ = 1 - \frac{1}{5} \left( 4e^{-0.2t} - e^{-0.4t} \right) \]

\[ = 1 - \frac{1}{3} \left( 4e^{-0.2t} - e^{-0.4t} \right) \]

The transformations were done using formulas for Laplace transformations in the lecture notes.

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_0 )</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( T_1 )</td>
<td>0.86</td>
<td>0.98</td>
<td>0.99</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( T_2 )</td>
<td>0.24</td>
<td>0.52</td>
<td>0.7</td>
<td>0.82</td>
<td>0.89</td>
<td></td>
</tr>
</tbody>
</table>

![Graph showing the response of systems \( T_0(t) \) and \( T_2(t) \)]
\[ T_0 = \sin(0.2t) \quad \Rightarrow \quad \omega = 0.2 \]

Can find \( T_1(t) \) by calculating amplitude ratio \( AR \) and phase shift \( \phi \):

\[
AR = \left| g_1(\omega j) \right| = \left| g_1(0.2j) \right| = \frac{1}{\sqrt{(0.2)^2 + 1}} = \frac{1}{\sqrt{1.24}} \approx 0.97
\]

\[
\phi = \angle g_1(\omega j) = \angle g_1(0.2j) = -\arctan(0.2) = -\frac{\pi}{2} \approx -0.78 \text{ rad}
\]

\[
T_1(t) = \frac{1}{\sqrt{2}} \sin(0.2t - \frac{\pi}{2})
\]

Same procedure can be used for \( T_2(t) \):

Transfer function from \( T_0 \) to \( T_2 \) is equal to \( g = g_1 \cdot g_2 = \frac{1}{(20.5+1)(5.5+1)} \)

\[
AR = \left| g_2(\omega j) \right| = \frac{1}{\sqrt{(20.5)^2 + 1}} = 0.17
\]

\[
\phi = -\arctan(20.5) - \arctan(5.5) = -2.11 \text{ rad}
\]

\[
T_2(t) = 0.17 \cdot \sin(0.2t - 2.11)
\]
<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>T₀</td>
<td>0</td>
<td>0.14</td>
<td>0.24</td>
<td>0.34</td>
<td>0.44</td>
<td>0.54</td>
<td>0.64</td>
<td>0.74</td>
<td>0.84</td>
<td>0.94</td>
<td>1.04</td>
</tr>
<tr>
<td>T₁</td>
<td>-0.5</td>
<td>0.15</td>
<td>0.25</td>
<td>0.35</td>
<td>0.45</td>
<td>0.55</td>
<td>0.65</td>
<td>0.75</td>
<td>0.85</td>
<td>0.95</td>
<td>1.05</td>
</tr>
<tr>
<td>T₂</td>
<td>-0.15</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
<td>1.0</td>
</tr>
</tbody>
</table>

The diagrams show three functions labeled T₀, T₁, and T₂, plotted against a horizontal axis from 0 to 50.
a) \[ u = \bar{z} \]
\[ y_1 = T \]
\[ y_2 = T_1, F_1, F_2, T_3 \]
\[ \alpha = T_2 \]

b)

Cascade would be good around the heat exchanger, as there are dynamics and nonlinearity which could be handled by inner loop

can measure \( T_1 \), and use it to adjust \( z_1 \). Setpoint for \( T_1 \) can be provided by master controller for \( T \)

Feedforward could be used by measuring \( T_2 \) and using this to adjust \( z \). This would make it possible to increase coolant flow quickly if \( T_2 \) increases, which would reduce the effect of the disturbance.
Cascading would also be good around the valve $z$. Values usually contain nonlinearity, and a first slave controller would allow other controllers to adjust $F_2$ directly, making the response better. The slave controller could be implemented by measuring $F_2$ and using a flow controller to adjust $z$. Setpoint for $F_2$ would be provided by a master controller.

The cascade around the heat exchanger would also allow for adjustment of $T_1$ without waiting for measurement of $T$. This would improve performance, as there is a large delay in measuring $T_1$. Figure showing the two cascade loops and the feedforward controller.
\[
\begin{align*}
    y_2 &= \frac{3}{2s+1} \quad u = g_1 u \\
    y_1 &= \frac{g_2 + 2d}{8s+1} = \frac{y_2}{8s+1} + \frac{2}{8s+1} d \\
    g_m &= \frac{q}{8s+1} \\
    &= g_2 \cdot y_2 + g_d \cdot d \\
\end{align*}
\]

\(\text{a) Open loop:}\)

\(\text{I: Feedback: } u = \frac{G(s)}{K(s)} (y_{is} - y_{im})\)
II: Cascade control

\[ u = C_2 \cdot (y_{2s} - y_{2m}) \]

\[ y_{2s} = C_1 \cdot (y_{1s} - y_{1m}) \]

III: Feedforward control

\[ u = C_{FF} \cdot d \]

\[ d \]

\[ d \]

\[ C_{FF} \]

\[ y_{2s} \]

\[ y_{2m} \]

\[ y_1 \]
Tuning C(s)

\[ g = g_1 \cdot g_2 \cdot g_m = \frac{3}{s+1} \cdot \frac{1}{8s+1} \cdot \frac{1}{s+5+1} \]

\[ = \frac{3}{(s+1)(2s+1)(s+5+1)} \]

**Rule:**
\[ \hat{\theta} = \theta + \frac{\tau}{2} + \frac{\tau}{3} = \theta + \frac{2}{2} + 0.5 = 1.5 \]
\[ \hat{\tau}_1 = \tau_1 + \frac{\tau}{2} = 8 \cdot \frac{2}{2} = 9 \]
\[ = 3 \cdot \frac{3 - e^{-1.5s}}{s+5+1} = \frac{3}{s+5} \]

**S I M C rules for PD (with \( \tau_c = 0 \) for right control):**

\[ K_c = \frac{1}{k} \cdot \frac{\tau_c + 0.5}{2} = \frac{3}{2} \cdot \frac{9}{1.5s+1.5} = 1 \]

\[ \tau_c = \min(\tau_c, 4(\tau_c + 0.5)) = \min(9, 12) = 9 \]

\[ s \cdot C(s) = 1 \cdot \frac{9s+1}{9s} \]
Tuning $C_1$ and $C_2$:

1. Tune slave controller first:

$$g = g_1 \cdot g_m = \frac{3}{2.5 + 1} \cdot \frac{1}{0.5 \cdot 5 + 1}$$

Half rule:

$$\Theta = \theta + \frac{\xi}{2} = 0 + \frac{0.5}{2} = 0.25$$

$$\zeta = \gamma \cdot \frac{\xi}{2} = 2 \cdot \frac{0.5}{2} = 0.5$$

$$\Rightarrow \zeta = \frac{3 - \Theta}{2.5 \cdot 5 + 1} = \frac{3 - 0.25}{2.5 \cdot 5 + 1} \Rightarrow \Theta = 0.25, \ z = 3, \ \gamma = 2, 2.5$$

S1MC with $\gamma = \Theta$:

$$K_c = \frac{1}{R} \cdot \frac{2.5}{2.5 + 2.5} = 1, 5$$

$$C_D = \min (2, \gamma \cdot (\gamma + \Theta)) = \min (2, 2.5 \cdot 2) = 2$$

$$C_2 = \frac{1.5}{2.5 + 1}$$
Need to estimate closed loop response in inner loop to tune $C_1$:

$$T_{inner} = \frac{C_2 \cdot g_i \cdot g_m}{1 + C_2 \cdot g_i \cdot g_m} \approx \frac{e^{-0.25s}}{0.25s + 1}$$

$$G_{outer} = T_{inner} \cdot g_2 \cdot g_m = \frac{e^{-0.25s}}{0.25s + 1} \cdot \frac{1}{8s + 1} \cdot \frac{1}{8s + 1}$$

Half rule:

$$\Theta = \Theta + \gamma_2/2 + \gamma_3 = 0.25 + \frac{0.5}{2} + 0.25 = 0.75$$

$$\gamma_1 = \gamma_1 + \gamma_2/2 = 8 + \frac{0.5}{2} = 8.25$$

$$G_{outer} = \frac{e^{-0.75s}}{8.25s + 1} \Rightarrow k = 0.75$$

SIMC with $\tau_c = \theta$:

$$K_C = \frac{k}{\theta} = \frac{0.25}{0.75 + 0.75} = 5.5$$

$$\gamma_1 = \min(\gamma_1, 4(\tau_c + \theta)) = \min(8.25, 6) = 6$$

$$C_1(s) = 5.5 \cdot \frac{6s + 1}{6s + 1}$$
II: Tuning CPP

- Ideal controller is given as
  \[ \text{ideal CPP} = -\frac{9d}{9m \cdot 9i \cdot 9a} \]
  \[ = -\frac{1}{\left(\frac{2}{8 \cdot s + 1}\right) \cdot \frac{3}{2 \cdot s + 1} \cdot \frac{1}{8 \cdot s + 1}} = -\frac{2}{3} \cdot \frac{2 \cdot s + 1}{(2 \cdot s + 1)(8 \cdot s + 1)} \]

  This is not realizable, as it has more zeros than poles. It can be made realizable by removing one of the zeros and adding a tunable pole:

  \[ \text{CPP} = -\frac{2}{3} \cdot \frac{2 \cdot s + 1}{8 \cdot s + 1} \]

  \( \zeta \) adjusts how aggressive the controller is. Small \( \zeta \) gives fast response. Could, for example, choose \( \zeta = 0.1 \).
Closed loop response for feedback is approximately:

\[
T_F \approx \frac{e^{-0.5s}}{e^{5s} + 1} = \frac{e^{-1.5s}}{1.5s + 1}
\]

For cascade this is:

\[
T_{\text{inter}} \approx \frac{e^{-0.75s}}{0.75s + 1}
\]

Cascade control therefore gives faster control than feedback.

Would not prefer feedforward, as the ideal controller is not realizable, due to \(g_m \cdot g_2\) being slower than \(g_2\).

\[\Rightarrow \text{Cascade control is therefore preferable}\]

For \(C\) and \(C_1\), the closed loop is dominated by \(C_1 = 8\), which is much larger than the other time constants. As \(C_2\) therefore has little effect, I would not suggest using PID here.

For \(C_2\), \(\tau_i\) and \(\tau_d\) are a bit closer, and PID could be used as \(\tau_d\) is also larger than the time delay, which is zero. This would therefore improve the performance of the slave controller.
(a) Bode stability condition:

\[ |L(\omega_{no})| < 1 \]

As shown in the figure,

\[ |L(\omega_{no})| \approx 0.3 \]

for this system. It is therefore stable.

(b) \[ GM = \frac{1}{|L(\omega_{no})|} \approx \frac{1}{0.3} \approx 3.3 \]

\[ PM = \angle L(\omega_{no}) + 180^\circ \approx -135^\circ + 180^\circ \]

\[ = 45^\circ = \frac{\pi}{4} \text{ rad} \]

Time delay margin = \[ DM = \frac{PM (\text{rad})}{\omega_c} \]

\[ = \frac{\pi/4 \text{ rad}}{0.763 \text{ rad/s}} \approx 1.25 \text{ s} \]

The figure is shown on the next page.
Problem 5 (10 %)

The frequency response of a loop transfer function \( L(s) = g(s)c(s)g_m(s) \) is shown in the Bode diagram below.

(a) (2%) Formulate the Bode stability condition. Is the system stable?
(b) (8%) What is the gain margin, phase margin (show on the figure) and what is the allowed extra time delay in the loop to remain stable?

Comment: You may write on this paper and use it as your solution.