Robust and Stochastic MPC; what is the correct formulation?

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Workshop on optimal control / real-time optimization related to the process industries

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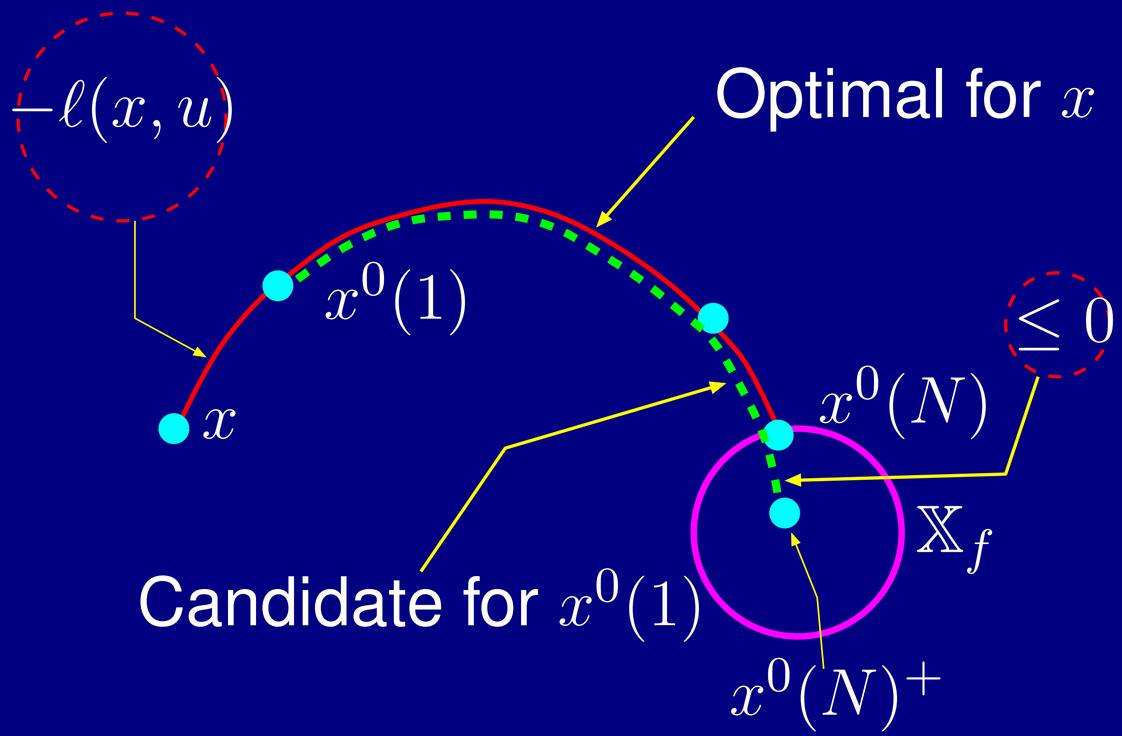
Objectives

- Present two contrasting approaches to robust and stochastic MPC
- First: simpler, practically and analytically
- Second: widely (universally) used
- Which is correct or better?
- Approaches presented first in context of linear MPC

Deterministic MPC

- System; $x^+ = f(x,u)$; $x(0) = x_0$
- Online OC Problem:
  $\mathbb{P}_N(x): \quad \min_{\mathbf{u}}\{V_N(x,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\}$
- $\mathbf{u} \triangleq (u_0, u_1, \ldots, u_{N-1})$, $\mathbf{x} \triangleq (x_0, x_1, \ldots, x_N)$
- If $x = x(t)$, x_i is prediction of $x(t+i)$ and u_i is prediction of $u(t+i)$
- and u_i is prediction of $u(t+i)$
- $V_N(x,\mathbf{u}) = \sum_{i=0}^{N-1} \ell(x_i,u_i) + V_f(x_N)$
- $\mathcal{U}_N(x)$ is set of control sequences $\mathbf{u}$ satisfying state, control and terminal constraints if initial state is x

Current and Candidate Trajectories



- Recursive feasibility plus cost change $\leq -\ell(x,u)$ imply asymptotic stability of origin.

Linear Robust/Stochastic MPC

- Linear system is only seriously considered case for stochastic MPC
- System: $x^+ = Ax + Bu + w$
- Constraints: $x \in \mathbb{X}$, $u \in \mathbb{U}$
- Disturbance w takes values in $\mathbb{W}$
- $\mathbb{X}$, $\mathbb{U}$, $\mathbb{W}$ all convex, compact, and contain origin in their interiors.
- Robust and stochastic MPC require online determination of control policy $\boldsymbol{\pi}$ i.e. sequence of control laws $\pi_i(\cdot)$
- In which $\pi_i : \mathbb{R}^n \to \mathbb{R}^m$ ($u_i = \pi_i(x_i)$)

Control parameterization

- Optimizing over $\boldsymbol{\pi}$ with each law $\pi_i(\cdot)$ an arbitrary function is impossible
- Hence parameterize $\pi(\cdot)$
- Useful parameterization (Mayne:Langson:2001)
- Nominal system:

$$\bar{x}^+ = A\bar{x} + B\bar{u}$$

$$u = \bar{u} + Ke, \quad e \triangleq x - \bar{x}$$

- The control policy $\boldsymbol{\pi}$ is now parameterized by the control sequence $\bar{\mathbf{u}} = (\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1})$
- $V_N(x, \boldsymbol{\pi})$ is replaced by $\bar{V}_N(x, \bar{\mathbf{u}})$.

Consequence of parameterization

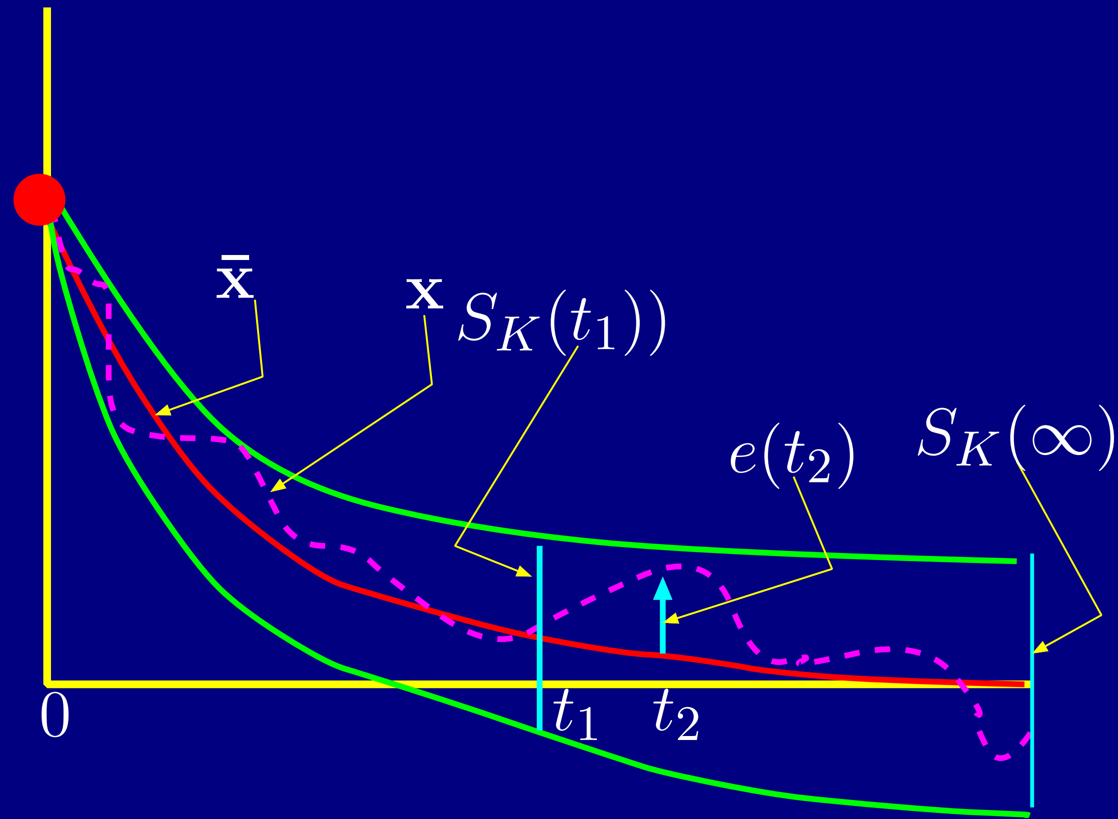
- Can now optimize over control sequence $\bar{\mathbf{u}}$ (as in deterministic MPC) instead of control policy $\boldsymbol{\pi}$.
- Also $e \triangleq x - \bar{x}$ satisfies

$$e^+ = A_K x + w$$

$$A_K \triangleq A + BK$$

- $\mathbf{u}$ chosen to ensure $\bar{x}(t) \to 0$
- A_K stable and w random, zero mean, ensure $e(t) = x(t) - \bar{x}(t)$ tends to its steady state distribution.
- This plus $\bar{x}(t) \to 0$ ensures $x(t)$ tends to steady state distribution of $e(t)$; best possible outcome

Evolution of state x given $u = \bar{u} + K(x - \bar{x})$



- $\bar{u}(\cdot)$ solution of nominal OC Pb $\bar{\mathbb{P}}_N(x(0))$
- $u = \bar{u} + K(x - \bar{x})$, K stabilizing linear controller
- Control u keeps x in Tube (initial state $x(0)$)

Variation of bounding set $S_K(t)$ ($e(t) \in S_K(t)$)

- $S_K(t) \triangleq \{e(t) \mid \mathbf{w} \in \mathbb{W}^t\}$. Since:

$$e^+ = A_K e + w, \qquad e(0) = 0$$

$$S_K^+ = A_K S_K + \mathbb{W}, \quad S_K(0) = \{0\}$$

- So, the sequence $(S_K(t))$ is monotonically increasing if $e(0) = 0$. If A_K is a stability matrix, $S_K(t)$ converges (as Kolmanovsky and Gilbert have shown) to a compact, convex set $S_K(\infty)$.

Robust and stochastic MPC

- System, nominal system, control parmeterization, constraints on $x \in \mathbb{X}$,$u \in \mathbb{U}$, bound $w \in \mathbb{W}$ as above for both robust and stochastic MPC
- In addition, both robust and stochastic MPC have a terminal constraint $x_N \in \mathbb{X}_f$ in the optimal control problem solved online
- For simplicity, we assume w is random, zero mean and that the optimal control problem $\mathbb{P}_N(x)$ solved on line at state x is the same for both problems.

Optimal Control Problem $\mathbb{P}_N(x)$

- The optimal control problem $\mathbb{P}_N(x)$ is

$$\min_{\boldsymbol{\pi}}\{V_N(x,\boldsymbol{\pi}) \mid \boldsymbol{\pi} \in \Pi_N(x)\}$$

- in which $\Pi_N(x)$ is set of policies $\boldsymbol{\pi}$ satisfying the state, control and terminal constraints, and

$$V_N(x,\boldsymbol{\pi}) \triangleq E_{|x} \sum_{i=0}^{N-1} \ell(x_i,\pi_i(x_i)) + V_f(x_N)$$

with $\ell(x,u) \triangleq x'Qx + u'Ru$; also $E_{|x}$ denotes expectation conditional on $x_0 = x$.

Equivalent Problem $\bar{\mathbb{P}}_N(\bar{x})$

- Because $x = \bar{x} + e$, $u = \pi(x) = \bar{u} + Ke$, e is random, zero mean:

$$V_N(x, \boldsymbol{\pi}) = \bar{V}(\bar{x}, \bar{u}) + c, \quad \bar{V}(\bar{x}, \bar{u}) \triangleq \sum_{i=0}^{N-1} \ell(\bar{x}_i, \bar{u}_i) + V_f(\bar{x}_N)$$

  The constant c arises from the variance of e.

- Resultant deterministic O.C. problem $\bar{\mathbb{P}}_N(\bar{x})$ is;

$$\min_{\bar{\mathbf{u}}} \{\bar{V}_N(\bar{x}, \bar{u}) \mid \bar{\mathbf{u}} \in \bar{\mathcal{U}}_N(\bar{x})\}$$

  $\bar{\mathcal{U}}_N(\bar{x})$ is the set of control sequences that satisfy the tightened state, control and terminal constraints for the nominal system

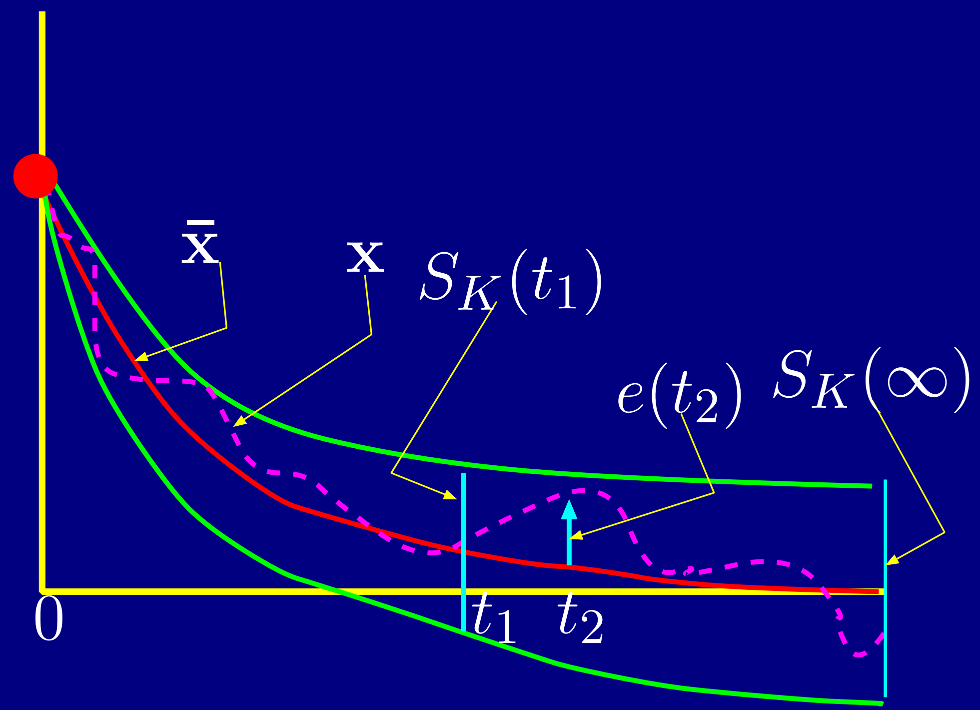
Tightened constraints: Robust MPC

- Want e.g. $\bar{\mathbb{X}}$ such that
  $\bar{x}(t) \in \bar{\mathbb{X}} \implies x(t) = \bar{x}(t) + e(t) \in \mathbb{X} \; \forall e(t) \in S_K(t)$
- Tightened constraints are easily computed if the constraint sets $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{X}_f$ are polyhedral
- Let $c'x \leq d$ be one inequality defining $\mathbb{X}$
- $c'x(t) = c'\bar{x}(t) + c'e(t)$ so $c'x(t) \leq d \iff$
- $c'\bar{x}(t) + c'e(t) \leq d$ for all $e(t) \in S_K(t)$
- So $c'x(t) \leq d$ if $c'\bar{x}(t) \leq \bar{d}(t)$ and
- $\max_{e(t)}\{c'e(t) \mid e(t) \in S_K(t)\} \leq d - \bar{d}(t)$
- maximize over $\mathbf{w} \in \mathbb{W}^t$ instead of $e \in S_K(t)$

Tightened constraints: Stochastic MPC

- Want e.g. tightened constraint set $\mathbb{X}$ such that $\bar{x}(t) \in \bar{\mathbb{X}} \implies \mathbb{P}r[x(t) = \bar{x}(t) + e(t) \in \mathbb{X}] \geq 1 - \varepsilon$
- Let $c'x \leq d$ be one inequality defining $\mathbb{X})$
- Then $\mathbb{P}r[c'x(t) \leq d] \geq 1 - \varepsilon$
- if $c'\bar{x}(t) \leq \bar{d}(t)$ and $\mathbb{P}r[c'e(t) \leq d - \bar{d}(t)] \geq 1 - \varepsilon$
- So probabilistic constraint sets $\bar{\mathbb{X}}(t)$, $\bar{\mathbb{U}}(t)$ and $\bar{\mathbb{X}}_f$ can be easily computed if they are polyhedral.
- $c'e(t) > d - \bar{d}(t) \implies$ constraint transgressed.
  Comparing:
  Robust: $c'e(t) \leq d - \bar{d}(t)$ with probability 1
  Stochastic: $c'e(t) \leq d - \bar{d}(t)$ with probability $1 - \varepsilon$

First approach to robust and stochastic MPC



- See Mayne, Langson (Electronic Letters 2001)
- Compute $(\bar{\mathbf{u}}, \bar{\mathbf{x}})$. Deterministic Pb. MPC on (nominal model) satisfying tightened constraints.
- Compute $u = \bar{u} + K(x - \bar{x})$.
- Very simple to compute and to analyse

Analysis of 1'st approach

- $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ obtained via determinstic MPC
- Under standard conditions, origin is asymptotically (exponentially) stable for MPC controlled system

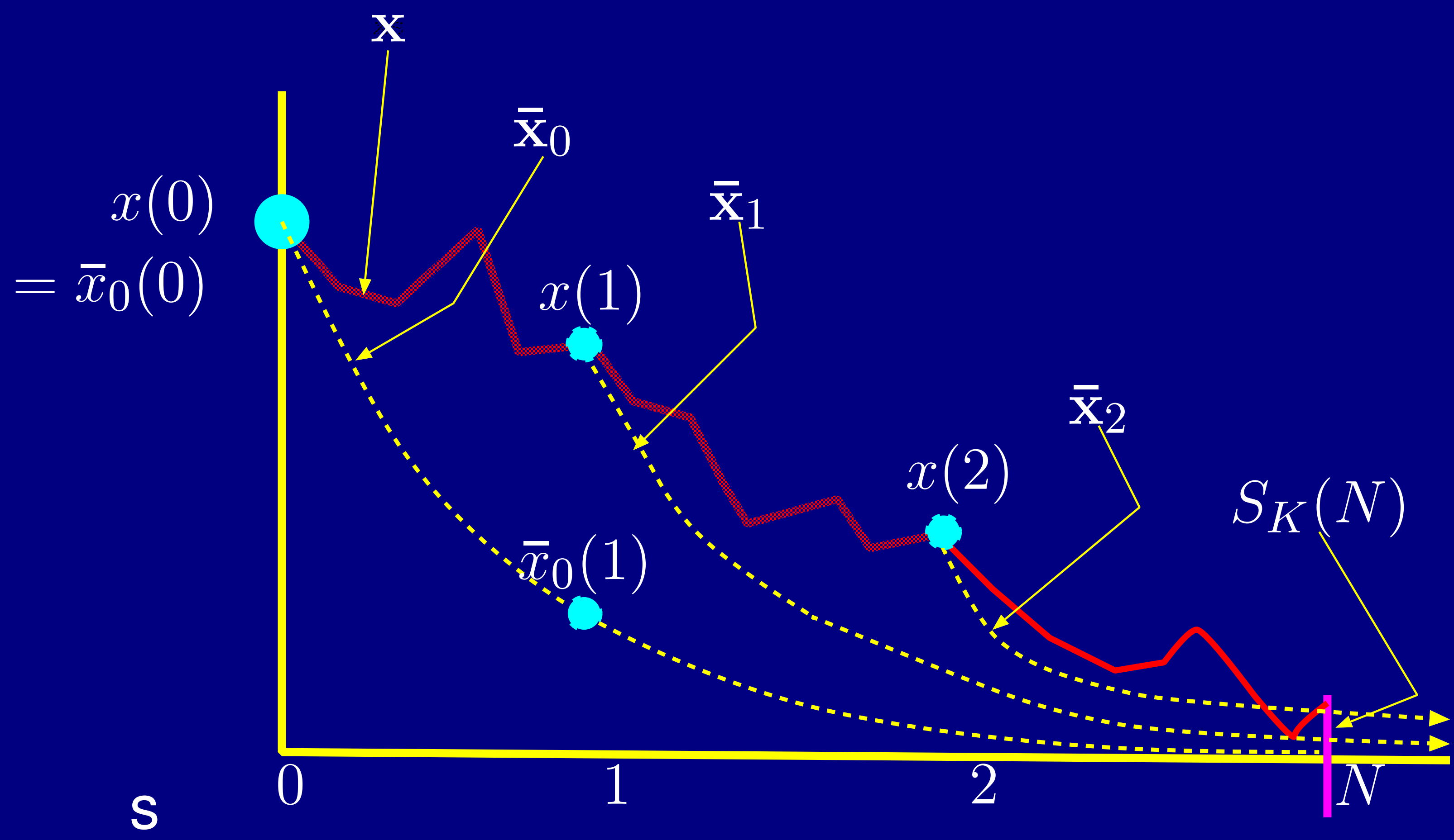
$$\bar{x}^+ = A\bar{x} + \kappa_N(\bar{x}), \quad \bar{x}(0) = x(0)$$

- Since $x(t) = \bar{x}(t) + e(t)$, $\bar{x}(t) \to 0$, $e(t) \in S_K(t)$ and $S_K(t) \to \bar{\mathbb{X}}_\infty \triangleq S_K(\infty)$, it follows that

$$x(t) \to \bar{\mathbb{X}}_\infty, \ \& \ x(t_1) \in \bar{\mathbb{X}}_\infty \implies x(t_2) \in \bar{\mathbb{X}}_\infty \ \forall t_2 \geq t_1$$

for all realizations $\mathbf{w}$ of the random disturbance w

Second approach to robust and stochastic MPC



Analysis of 2'nd approach; recursive feasibility

- Given tightened constraints for $\bar{\mathrm{x}}_0$, tightened constraint for $\bar{\mathrm{x}}_1$ cannot be determined as in deterministic MP (because $\bar{x}_1(1) = x(1) \neq \bar{x}_0(1)$ as in version 1)
- But can be determined using $x(1) = \bar{x}_0(1) + w(0)$, so that $e(0) = w(0)$ AND $e^+ = A_K e + w$ yielding the sequence e and, hence, stricter, tightened affine constraints for $\bar{\mathrm{x}}_1$ (using $x(\tau) = \bar{x}_1(\tau) + e(\tau)$).
- This yields a candidate for recursive feasibility
- Lorenzen et al also impose a computationally expensive 'first-step constraint' that guarantees recursive feasibility.

Analysis of 2'nd approach; stochastic convergence

- In Approach 2, the nominal state $\bar{x}$ is random
- Because $\bar{x}$ is reset to x at each time (and x is random)
- It follows that $\bar{V}_N$ is random
- Stochastic analysis is required to establish convergence (eg convergence in probability) of the state to a set enclosing the origin in its interior.
- Lorenzen et al incorrectly use a result of Chisci et al (2001) (viz, $u(k) - Kx(k) \to 0$) to establish convergence in probability to $\mathbb{X}$, the minimal positive invariant set for $x^+ = A_K x + w$; the state x moves randomly to $\mathbb{X}$ and then randomly in $\mathbb{X}$.
- Convergence can be established using appropriate stabilizing conditions.

Comparison: 1. Ease of Implementation

- Implementation of first approach simple: almost standard deterrministic MPC
- Main complication: continual tightening of constraint (due to increasing effect of disturbance)
- Mitigated; by replacing $S_K(t)$ by $S_K(\infty)$ for $t \geq T_I$ say.
- Implementation of seond approach considerably more complex due to complication caused by randomness of ‘nominal’ control $\bar{u}(t)$.
- Caused by resetting $\bar{x}(t)$ to $x(t)$ at each t (in obtaining solution to $\mathbb{P}_N(x(t))$.

Comparison 2. Performance

- Depends of definition
- IF Performance is minimization of $V_N(x, \boldsymbol{\pi})$, in which x is the initial state $x(0)$ of the system, then

$$V_N(x, \boldsymbol{\pi}) = \bar{V}_N(\bar{x}, \bar{\mathbf{u}}) + c, \quad \bar{x} = x = x(0)$$

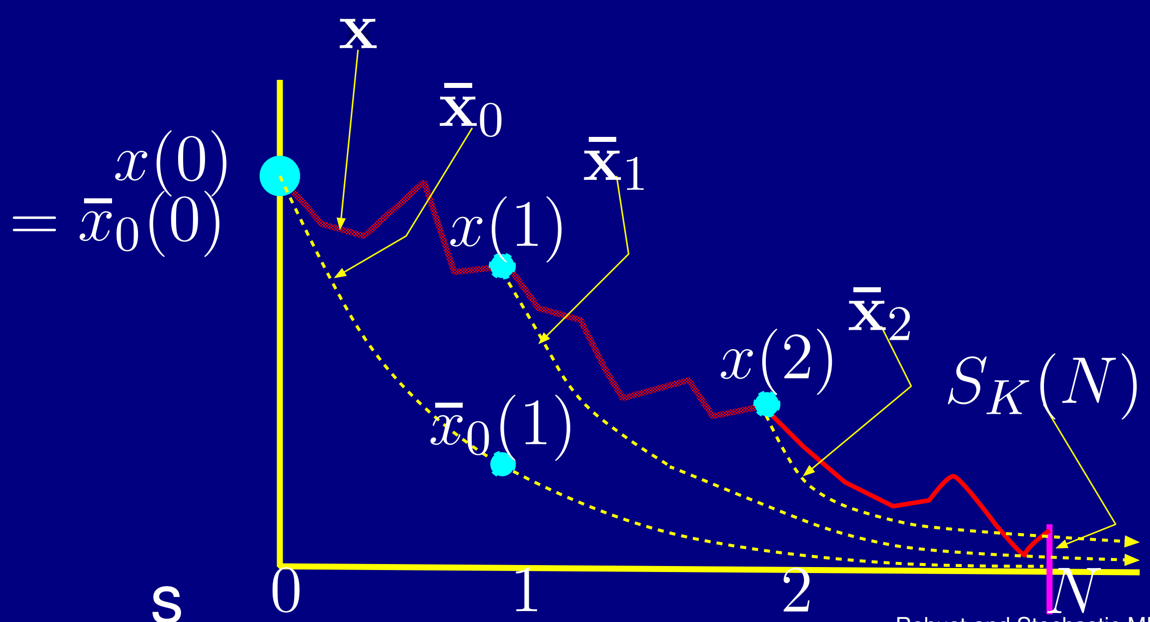
- Approach 1 is optimal (minimizes $\bar{V}_N(x(0), \bar{\mathbf{u}})$)
- Stated objective (e.g. Lorenzen et al), Chatterjee and Lygeros, 2015) is often $\limsup(1/N)V_N(x, \pi)$ that is:

$$\limsup(1/N)E_{|x}\sum_{i=0}^{N-1}\ell(x_i, \pi_i(x_i)) + V_f(x_N)$$

This objective also implies Approach 1 is optimal; $E_{|x}$ denotes expectation conditional on $x(0) = x$.

Comparison

- Approach 2 (minimizing $\bar{\mathbb{P}}_N(x)$ at each state x) may yield lower cost for actual realization of disturbance sequence (rather than average cost)
- But confuses role of $\bar{\mathbf{u}}$ (controlling mean) and Ke (controlling variance).



CONCLUSION

- Is proposing solving $\bar{\mathbb{P}}_N(\bar{x})$ rather than $\bar{\mathbb{P}}_N(x)$ absurd?
- Proposal limited to robust/stochastic MPC of linear systems using control parameterization $u = \bar{u} + K(x - \bar{x})$
- And cost is cost from initial state $x(0)$
- Not cost to go from current state $x(t)$
- Possible new avenue of MPC research
- Control strategies that address (more or less separately) controlling mean (eg $\bar{\mathbf{u}}$) and variance (eg $u = \bar{u} + Ke$ OR MPC controlling e)
- As in nonlinear robust MPC (mayne et al: IJRNLC 2011)

THANK YOU!