Robust and Stochastic MPC; what is the correct formulation?

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Workshop on optimal control / real-time optimization related to the process industries

Trondheim, Norway

November 8, 2019
Objectives

• Present two contrasting approaches to robust and stochastic MPC
• First: simpler, practically and analytically
• Second: widely (universally) used
• Which is correct or better?
• Approaches presented first in context of linear MPC
Deterministic MPC

- **System:** \( x^+ = f(x, u); \ x(0) = x_0 \)

- **Online OC Problem:**
  \[ P_N(x) : \min_u \{ V_N(x, u) \mid u \in \mathcal{U}_N(x) \} \]

- **\( u \triangleq (u_0, u_1, \ldots, u_{N-1}) \), \( x \triangleq (x_0, x_1, \ldots, x_N) \)**

- If \( x = x(t) \), \( x_i \) is prediction of \( x(t + i) \) and \( u_i \) is prediction of \( u(t + i) \)

- and \( u_i \) is prediction of \( u(t + i) \)

- \( V_N(x, u) = \sum_{i=0}^{N-1} \ell(x_i, u_i) + V_f(x_N) \)

- **\( \mathcal{U}_N(x) \)** is set of control sequences \( u \) satisfying state, control and terminal constraints if initial state is \( x \)
Current and Candidate Trajectories

- $\ell(x, u)$
- $x^0(1)$
- $x^0(N) + \mathbb{X}_f$
- $x^0(N)^+$

**Candidate for** $x^0(1)$

**Optimal for** $x$

• Recursive feasibility plus cost change $\leq -\ell(x, u)$ imply asymptotic stability of origin.
Linear Robust/Stochastic MPC

- Linear system is only seriously considered case for stochastic MPC
- System: $x^{+} = Ax + Bu + w$
- Constraints: $x \in X, u \in U$
- Disturbance $w$ takes values in $W$
- $X, U, W$ all convex, compact, and contain origin in their interiors.
- Robust and stochastic MPC require online determination of control policy $\pi$ i.e. sequence of control laws $\pi_i(\cdot)$
- In which $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}^m \ (u_i = \pi_i(x_i))$
Control parameterization

- Optimizing over $\pi$ with each law $\pi_i(\cdot)$ an arbitrary function is impossible.
- Hence parameterize $\pi(\cdot)$.
- Nominal system:
  
  \[
  \bar{x}^+ = A\bar{x} + B\bar{u}
  \]

  \[
  u = \bar{u} + Ke, \quad e \triangleq x - \bar{x}
  \]

- The control policy $\pi$ is now parameterized by the control sequence $\bar{u} = (\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1})$.
- $V_N(x, \pi)$ is replaced by $\bar{V}_N(x, \bar{u})$. 
Consequence of parameterization

• Can now optimize over control sequence \( \bar{u} \) (as in deterministic MPC) instead of control policy \( \pi \).

• Also \( e \triangleq x - \bar{x} \) satisfies

\[
e^+ = A_K x + w
\]

\[
A_K \triangleq A + BK
\]

• \( u \) chosen to ensure \( \bar{x}(t) \to 0 \)

• \( A_K \) stable and \( w \) random, zero mean, ensure \( e(t) = x(t) - \bar{x}(t) \) tends to its steady state distribution.

• This plus \( \bar{x}(t) \to 0 \) ensures \( x(t) \) tends to steady state distribution of \( e(t) \); best possible outcome
Evolution of state $x$ given $u = \bar{u} + K(x - \bar{x})$

- $\bar{u}(\cdot)$ solution of nominal OC Pb $\bar{\mathcal{P}}_N(x(0))$
- $u = \bar{u} + K(x - \bar{x})$, $K$ stabilizing linear controller
- Control $u$ keeps $x$ in Tube (initial state $x(0)$)
Variation of bounding set $S_K(t)$ ($e(t) \in S_K(t)$)

- $S_K(t) \triangleq \{e(t) | w \in \mathbb{W}^t\}$. Since:

$$e^+ = A_K e + w, \quad e(0) = 0$$

$$S^+_K = A_K S_K + \mathbb{W}, \quad S_K(0) = \{0\}$$

- So, the sequence ($S_K(t)$) is monotonically increasing if $e(0) = 0$. If $A_K$ is a stability matrix, $S_K(t)$ converges (as Kolmanovsky and Gilbert have shown) to a compact, convex set $S_K(\infty)$. 

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Robust and stochastic MPC

- System, nominal system, control parameterization, constraints on $x \in \mathcal{X}, u \in \mathcal{U}$, bound $w \in \mathcal{W}$ as above for both robust and stochastic MPC
- In addition, both robust and stochastic MPC have a terminal constraint $x_N \in \mathcal{X}_f$ in the optimal control problem solved online
- For simplicity, we assume $w$ is random, zero mean and that the optimal control problem $P_N(x)$ solved on line at state $x$ is the same for both problems.
Optimal Control Problem $\mathbb{P}_N(x)$

- The optimal control problem $\mathbb{P}_N(x)$ is
  \[
  \min_{\pi} \{ V_N(x, \pi) \mid \pi \in \Pi_N(x) \}
  \]

- in which $\Pi_N(x)$ is set of policies $\pi$ satisfying the state, control and terminal constraints, and
  \[
  V_N(x, \pi) \triangleq E_{|x} \sum_{i=0}^{N-1} \ell(x_i, \pi_i(x_i)) + V_f(x_N)
  \]

with $\ell(x, u) \triangleq x'Qx + u'Ru$; also $E_{|x}$ denotes expectation conditional on $x_0 = x$. 
Equivalent Problem $\overline{P}_N(\bar{x})$

- Because $x = \bar{x} + e$, $u = \pi(x) = \bar{u} + Ke$, $e$ is random, zero mean:

$$V_N(x, \pi) = \bar{V}(\bar{x}, \bar{u}) + c, \quad \bar{V}(\bar{x}, \bar{u}) \triangleq \sum_{i=0}^{N-1} \ell(\bar{x}_i, \bar{u}_i) + V_f(\bar{x}_N)$$

The constant $c$ arises from the variance of $e$.

- Resultant deterministic O.C. problem $\overline{P}_N(\bar{x})$ is:

$$\min_{\bar{u}} \{\bar{V}_N(\bar{x}, \bar{u}) \mid \bar{u} \in \overline{U}_N(\bar{x})\}$$

$\overline{U}_N(\bar{x})$ is the set of control sequences that satisfy the tightened state, control and terminal constraints for the nominal system.
Tightened constraints: Robust MPC

- Want e.g. $\overline{X}$ such that $\overline{x}(t) \in \overline{X} \implies x(t) = \overline{x}(t) + e(t) \in X \forall e(t) \in S_K(t)$

- Tightened constraints are easily computed if the constraint sets $X$, $U$ and $X_f$ are polyhedral

- Let $c'x \leq d$ be one inequality defining $X$

- $c'x(t) = c'\overline{x}(t) + c'e(t)$ so $c'x(t) \leq d \iff c'\overline{x}(t) + c'e(t) \leq d$ for all $e(t) \in S_K(t)$

- So $c'x(t) \leq d$ if $c'\overline{x}(t) \leq \overline{d}(t)$ and

- $\max_{e(t)}\{c'e(t) \mid e(t) \in S_K(t)\} \leq d - \overline{d}(t)$

- maximize over $w \in W^t$ instead of $e \in S_K(t)$
Tightened constraints: Stochastic MPC

- Want e.g. tightened constraint set $\widetilde{X}$ such that $\bar{x}(t) \in \widetilde{X} \implies \Pr[x(t) = \bar{x}(t) + e(t) \in X] \geq 1 - \varepsilon$
- Let $c'x \leq d$ be one inequality defining $X$)
- Then $\Pr[c'x(t) \leq d] \geq 1 - \varepsilon$
- If $c'\bar{x}(t) \leq \bar{d}(t)$ and $\Pr[c'e(t) \leq d - \bar{d}(t)] \geq 1 - \varepsilon$
- So probabilistic constraint sets $\widetilde{X}(t)$, $\mathcal{U}(t)$ and $\widetilde{X}_f$ can be easily computed if they are polyhedral.
- $c'e(t) > d - \bar{d}(t) \implies$ constraint transgressed.
  
Comparing:

Robust: $c'e(t) \leq d - \bar{d}(t)$ with probability $1$

Stochastic: $c'e(t) \leq d - \bar{d}(t)$ with probability $1 - \varepsilon$
First approach to robust and stochastic MPC

- See Mayne, Langson (Electronic Letters 2001)
- Compute \((\overline{u}, \overline{x})\). Deterministic Pb. MPC on (nominal model) satisfying tightened constraints.
- Compute \(u = \overline{u} + K(x - \overline{x})\).
- Very simple to compute and to analyse.
Analysis of 1’st approach

• \((\bar{x}, \bar{u})\) obtained via deterministic MPC
• Under standard conditions, origin is asymptotically (exponentially) stable for MPC controlled system

\[
\bar{x}^+ = A\bar{x} + \kappa_N(\bar{x}), \quad \bar{x}(0) = x(0)
\]

• Since \(x(t) = \bar{x}(t) + e(t), \bar{x}(t) \to 0, e(t) \in S_K(t)\) and \(S_K(t) \to \bar{X}_\infty \triangleq S_K(\infty)\), it follows that

\[
x(t) \to \bar{X}_\infty, \ & x(t_1) \in \bar{X}_\infty \implies x(t_2) \in \bar{X}_\infty \ \forall t_2 \geq t_1
\]

for all realizations \(w\) of the random disturbance \(w\)
Second approach to robust and stochastic MPC

\[ x(0) = \overline{x}_0(0) \]

\[ S_K(N) \]
Analysis of 2’nd approach; recursive feasibility

- Given tightened constraints for $\overline{x}_0$, tightened constraint for $\overline{x}_1$ cannot be determined as in deterministic MP (because $\overline{x}_1(1) = x(1) \neq \overline{x}_0(1)$ as in version 1)
- But can be determined using $x(1) = \overline{x}_0(1) + w(0)$, so that $e(0) = w(0)$ AND $e^{+} = A_K e + w$ yielding the sequence $e$ and, hence, stricter, tightened affine constraints for $\overline{x}_1$ (using $x(\tau) = \overline{x}_1(\tau) + e(\tau)$).
- This yields a candidate for recursive feasibility
- Lorenzen et al also impose a computationally expensive ‘first-step constraint’ that guarantees recursive feasibility.
Analysis of 2’nd approach; stochastic convergence

- In Approach 2, the nominal state $\bar{x}$ is random.
- Because $\bar{x}$ is reset to $x$ at each time (and $x$ is random).
- It follows that $\bar{V}_N$ is random.
- Stochastic analysis is required to establish convergence (e.g., convergence in probability) of the state to a set enclosing the origin in its interior.
- Lorenzen et al incorrectly use a result of Chisci et al (2001) (viz, $u(k) - K x(k) \to 0$) to establish convergence in probability to $X$, the minimal positive invariant set for $x^+ = A_K x + w$; the state $x$ moves randomly to $X$ and then randomly in $X$.
- Convergence can be established using appropriate stabilizing conditions.
Comparison: 1. Ease of Implementation

- Implementation of first approach simple: almost standard deterministic MPC
- Main complication: continual tightening of constraint (due to increasing effect of disturbance)
- Mitigated; by replacing $S_K(t)$ by $S_K(\infty)$ for $t \geq T_I$ say.
- Implementation of second approach considerably more complex due to complication caused by randomness of ‘nominal’ control $\bar{u}(t)$.
- Caused by resetting $\bar{x}(t)$ to $x(t)$ at each $t$ (in obtaining solution to $\mathbb{P}_N(x(t))$).
Comparison 2. Performance

• Depends of definition
• IF Performance is minimization of $V_N(x, \pi)$, in which $x$ is the initial state $x(0)$ of the system, then

\[ V_N(x, \pi) = \bar{V}_N(x, \bar{u}) + c, \quad \bar{x} = x = x(0) \]

• Approach 1 is optimal (minimizes $\bar{V}_N(x(0), \bar{u})$)
• Stated objective (e.g. Lorenzen et al), Chatterjee and Lygeros, 2015) is often $\lim \sup (1/N) V_N(x, \pi)$ that is:

\[ \lim \sup (1/N) E_{x} \sum_{i=0}^{N-1} \ell(x_i, \pi_i(x_i)) + V_f(x_N) \]

This objective also implies Approach 1 is optimal; $E_{x}$ denotes expectation conditional on $x(0) = x$. 

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Comparison

• Approach 2 (minimizing $\overline{P}_N(x)$ at each state $x$) may yield lower cost for actual realization of disturbance sequence (rather than average cost)

• But confuses role of $\bar{u}$ (controlling mean) and $K_e$ (controlling variance).
CONCLUSION

• Is proposing solving $\overline{P}_N(x)$ rather than $\overline{P}_N(x)$ absurd?
• Proposal limited to robust/stochastic MPC of linear systems using control parameterization
  
  $$u = \overline{u} + K(x - \overline{x})$$

• And cost is cost from initial state $x(0)$
• Not cost to go from current state $x(t)$
• Possible new avenue of MPC research
• Control strategies that address (more or less separately) controlling mean (eg $\overline{u}$) and variance (eg $u = \overline{u} + Ke$) OR MPC controlling $e$
• As in nonlinear robust MPC (mayne et al: IJRNLC 2011)

THANK YOU!