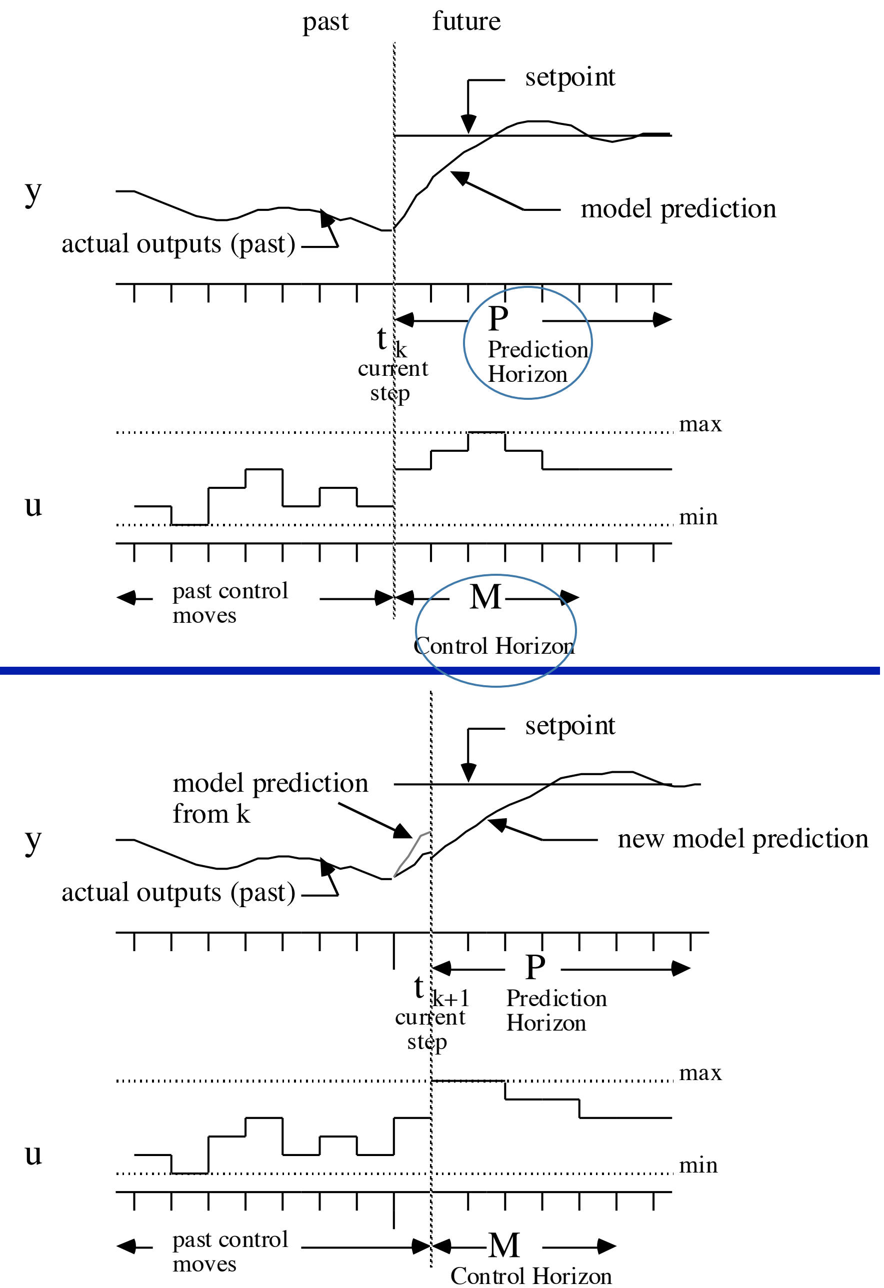
State Space-based MPC using a Kalman Filter

B. Wayne Bequette

- Quick Review
- State Space Models
  - Unconstrained and Constrained Solutions
- Kalman Filter for Disturbance Estimation
- Summary

Rensselaer
Chemical and Biological Engineering



Model Predictive Control (MPC)

- Constraints
- Multivariable
- Time-delays

- Quadratic objective function
- Quadratic program (QP)
  - constraints
- State space model
- Disturbances
  - additive output vs. state estimation

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Original DMC Approach to Plant-Model Mismatch

$$\hat{x}_{k|k-1} = \Phi \hat{x}_{k-1|k-1} + \Gamma u_{k-1}$$

$$\hat{y}_{k|k-1} = C\hat{x}_{k|k-1}$$

Prediction at step k, based on information at k-1

Measured output

$$\hat{p}_{k|k} = y_k - \hat{y}_{k|k-1}$$

Model output predicted from k-1

$$\hat{y}_{k|k} = \hat{y}_{k|k-1} + \hat{p}_{k|k}$$

"additive output" disturbance assumption

Forces the model "corrected output" equal to measured output

Notice that Model States are Not "Corrected"

$$\hat{x}_{k|k} = \hat{x}_{k|k-1}$$

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Model Prediction to k+1

$$\hat{x}_{k+1|k} = \Phi\hat{x}_{k|k} + \Gamma u_k$$

$$\hat{p}_{k+1|k} = \hat{p}_{k|k}$$

Assumes future corrections equal to current correction

$$\hat{y}_{k+1|k} = C\hat{x}_{k+1|k} + \hat{p}_{k+1|k} =$$

$$= C\Phi\hat{x}_{k|k} + C\Gamma u_k + \hat{p}_{k+1|k}$$

$$= C\Phi\hat{x}_{k|k} + C\Gamma u_{k-1} + C\Gamma\Delta u_k + \hat{p}_{k+1|k}$$

Continue Output Predictions

$$\underbrace{\begin{bmatrix} \hat{y}_{k+1|k} \\ \hat{y}_{k+2|k} \\ \vdots \\ \hat{y}_{k+P|k} \end{bmatrix}}_{\hat{Y}} = \underbrace{\begin{bmatrix} C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^P \end{bmatrix} \hat{x}_{k|k} + \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} \hat{p}_{k|k} + \begin{bmatrix} C\Gamma \\ C\Phi\Gamma + C\Gamma \\ \vdots \\ \sum_{i=1}^{P} C\Phi^{i-1}\Gamma \end{bmatrix} u_{k-1}}_{\text{"free" or "unforced response" (if no more control moves are made)}}$$

$\rightarrow f$

$$+ \underbrace{\begin{bmatrix} C\Gamma & 0 & \cdots & 0 \\ C\Phi\Gamma + C\Gamma & C\Gamma & 0 & 0 \\ \vdots & \vdots & & \\ \sum_{i=1}^{P} C\Phi^{i-1}\Gamma & \sum_{i=1}^{P-1} C\Phi^{i-1}\Gamma & \cdots & \end{bmatrix} \begin{bmatrix} \Delta u_k \\ \Delta u_{k+1} \\ \vdots \\ \Delta u_{k+M-1} \end{bmatrix}}_{\text{"forced" response}}$$

S_f (matrix) $\qquad \Delta u_f$ (vector)

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Output Predictions

$$\underbrace{\begin{bmatrix} \hat{y}_{k+1|k} \\ \hat{y}_{k+2|k} \\ \vdots \\ \hat{y}_{k+P|k} \end{bmatrix}}_{\hat{Y}} = \underbrace{\begin{bmatrix} C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^P \end{bmatrix} \hat{x}_{k|k} + \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} \hat{p}_{k|k} + \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_P \end{bmatrix} u_{k-1}}_{\text{"free" or "unforced response" (if no more control moves are made)}}$$

f

$$\underbrace{\begin{bmatrix} S_1 & 0 & \cdots & 0 \\ S_2 & S_1 & 0 & 0 \\ \vdots & \vdots & & \\ S_P & S_{P-1} & \cdots & S_{P-M+1} \end{bmatrix} \begin{bmatrix} \Delta u_k \\ \Delta u_{k+1} \\ \vdots \\ \Delta u_{k+M-1} \end{bmatrix}}_{\text{"forced" response}}$$

S_f

Δu_f

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Same Optimization Problem as Before

$$\min_{\Delta u_f} \quad J = \underbrace{\sum_{i=1}^{P} \left(r_{k+i|k} - \hat{y}_{k+i|k}\right)^T W^y \left(r_{k+i|k} - \hat{y}_{k+i|k}\right)}_{\hat{E}^T W^Y \hat{E}} + \underbrace{\sum_{i=0}^{M-1} \Delta u_{k+i}^T W^u \Delta u_{k+i}^T}_{\Delta u_f^T W^U \Delta u_f}$$

Where

$$W^Y = \begin{bmatrix} W^y & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & W^y \end{bmatrix} \qquad W^U = \begin{bmatrix} W^u & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & W^u \end{bmatrix}$$

future setpoints

and

$$\hat{E} = r - \hat{Y} = \underbrace{r - f}_{E} - S_f \Delta u_f$$

E ← **"unforced" error**

so

$$\hat{E} = E - S_f \Delta u_f$$

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Optimization Problem…

$$\hat{E}^T W^Y \hat{E} = \left(E - S_f \Delta u_f\right)^T W^Y \left(E - S_f \Delta u_f\right)$$

$$= E^T W^Y E - 2\Delta u_f^T S_f^T W^Y E + \Delta u_f^T S_f^T W^Y S_f \Delta u_f$$

so

$$\min_{\Delta u_f} \quad J = \hat{E}^T W^Y \hat{E} + \Delta u_f^T W^U \Delta u_f$$

can be written

$$\min_{\Delta u_f} \quad J = E^T W^Y E + \Delta u_f^T \left(S_f^T W^Y S_f + W^U\right) \Delta u_f - 2\Delta u_f^T S_f^T W^Y E$$

and the unconstrained solution is found from

$$\partial J / \partial \Delta u_f = 0$$

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Unconstrained Solution

Analytical Solution for Unconstrained System

$$\Delta u_f = \left(S_f^T W^Y S_f + W^U\right)^{-1} S_f^T W^Y E$$

"unforced" error

In practice, do not actually invert a matrix. Solve as set of simultaneous equations (or use \ in MATLAB)

$$\Delta u_f = \left(S_f^T W^Y S_f + W^U\right) \backslash S_f^T W^Y E$$

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Vector of Control Moves

$$\Delta u_f = \underbrace{\begin{bmatrix} \Delta u_k \\ \Delta u_{k+1} \\ \vdots \\ \Delta u_{k+M-1} \end{bmatrix}}_{\text{current and future moves}}$$

Although a set of control moves is computed, only the first move Δu_k is implemented. The next output at k+1 is obtained, then a new optimization problem is solved.

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Problems with "Classical MPC" (e.g. DMC)

- Finite Step/Impulse Models Limited
  - Limited to open-loop stable processes (there is no corrective feedback to model states)

- Additive Output Disturbance Assumption
  - Poor performance for input step disturbances
  - No explicit measurement noise trade-off

Ind. Eng. Chem. Res. **2002**, *41*, 3745–3750

Process Control: As Taught vs as Practiced

Francis G. Shinskey

LIMITATIONS OF DYNAMIC MATRIX CONTROL

P. Lundström,[1]† J. H. Lee[1,]‡, M. Morari[1,]§ and S. Skogestad[2]

Computers chem. Engng Vol. 19, No. 4, pp. 409–421, 1995

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MPC, Including Disturbance Models

$$\underset{(nx1)}{x_{k+1}} = \underset{(nxn)(nx1)}{\Phi x_k} + \underset{(nxn_u)(n_ux1)}{\Gamma u_k} + \underset{(nxn_d)}{\Gamma^d} \underset{(n_dx1)}{d_k}$$

$$\underset{(n_yx1)}{y_k} = \underset{(n_yxn)(nx1)}{C x_k}$$

For perturbations to manipulated inputs

$$\Gamma^d = \Gamma$$

with known current state, easy to propagate estimates

$$x_{k+1} = \Phi x_k + \Gamma u_k + \Gamma^d d_k$$

$$y_{k+1} = C x_{k+1} = C\Phi x_k + C\Gamma u_k + C\Gamma^d d_k$$

and, using control changes

$$u_k = u_{k-1} + \Delta u_k$$

$$y_{k+1} = C\Phi x_k + C\Gamma u_{k-1} + C\Gamma^d d_k + C\Gamma \Delta u_k$$

Next, use estimated disturbance

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Based on an Estimated Disturbance

Predicted output — State estimate — Previous input — Estimated disturbance — Current Input change

$$\hat{y}_{k+1|k} = C\Phi\hat{x}_{k|k} + C\Gamma u_{k-1} + C\Gamma^{d}\hat{d}_{k|k} + C\Gamma\Delta u_{k}$$

- $\hat{y}_{k+1|k}$: Estimate at k+1; Measurements through k
- $\hat{x}_{k|k}$: Estimate at k; Measurements through k
- $\hat{d}_{k|k}$: Estimate at k; Measurements through k

Now, propagate the prediction for P steps into the future

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Output Predictions

$$\underbrace{\begin{bmatrix} \hat{y}_{k+1|k} \\ \hat{y}_{k+2|k} \\ \vdots \\ \hat{y}_{k+P|k} \end{bmatrix}}_{\hat{Y}} = \underbrace{\begin{bmatrix} C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^P \end{bmatrix} \hat{x}_{k|k} + \begin{bmatrix} C\Gamma \\ C\Phi\Gamma + C\Gamma \\ \vdots \\ \sum_{i=1}^{P} C\Phi^{i-1}\Gamma \end{bmatrix} u_{k-1} + \begin{bmatrix} C\Gamma^d \\ C\Phi\Gamma^d + C\Gamma^d \\ \vdots \\ \sum_{i=1}^{P} C\Phi^{i-1}\Gamma^d \end{bmatrix} \hat{d}_{k|k}}_{\text{"free" or "unforced response" (if no more control moves are made)}}$$

$$f$$

Estimated disturbance (constant in future)

$$\underbrace{\begin{bmatrix} C\Gamma & 0 & \cdots & 0 \\ C\Phi\Gamma + C\Gamma & C\Gamma & 0 & 0 \\ \vdots & \vdots & & \\ \sum_{i=1}^{P} C\Phi^{i-1}\Gamma & \sum_{i=1}^{P-1} C\Phi^{i-1}\Gamma & \cdots & \end{bmatrix} \begin{bmatrix} \Delta u_k \\ \Delta u_{k+1} \\ \vdots \\ \Delta u_{k+M-1} \end{bmatrix}}_{\text{"forced" response}}$$

$$S_f \qquad \Delta u_f$$

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Output Predictions (using S notation)

$$\underbrace{\begin{bmatrix} \hat{y}_{k+1|k} \\ \hat{y}_{k+2|k} \\ \vdots \\ \hat{y}_{k+P|k} \end{bmatrix}}_{\hat{Y}} = \underbrace{\begin{bmatrix} C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^P \end{bmatrix} \hat{x}_{k|k} + \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_P \end{bmatrix} u_{k-1} + \begin{bmatrix} S_1^d \\ S_2^d \\ \vdots \\ S_P^d \end{bmatrix} \hat{d}_{k|k}}_{\text{"free" or "unforced response" (if no more control moves are made)}}$$

"free" or "unforced response" (if no more control moves are made) → f

$\hat{d}_{k|k}$: **Estimated disturbance (constant in future)**

$$\underbrace{\begin{bmatrix} S_1 & 0 & \cdots & 0 \\ S_2 & S_2 & 0 & 0 \\ \vdots & \vdots & & \\ S_P & S_{P-1} & \cdots & S_{P-M+1} \end{bmatrix} \begin{bmatrix} \Delta u_k \\ \Delta u_{k+1} \\ \vdots \\ \Delta u_{k+M-1} \end{bmatrix}}_{\text{"forced" response}}$$

S_f (matrix), Δu_f (vector)

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Same Optimization Problem as Before

$$\min_{\Delta u_f} \quad J = \underbrace{\sum_{i=1}^{P} \left(r_{k+i|k} - \hat{y}_{k+i|k}\right)^T W^y \left(r_{k+i|k} - \hat{y}_{k+i|k}\right)}_{\hat{E}^T W^Y \hat{E}} + \underbrace{\sum_{i=0}^{M-1} \Delta u_{k+i}^T W^u \Delta u_{k+i}^T}_{\Delta u_f^T W^U \Delta u_f}$$

Where

$$W^Y = \begin{bmatrix} W^y & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & W^y \end{bmatrix} \qquad W^U = \begin{bmatrix} W^u & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & W^u \end{bmatrix}$$

future setpoints (→ r)

and

$$\hat{E} = r - \hat{Y} = \underbrace{r - f}_{E} - S_f \Delta u_f$$

E ← **"unforced" error**

so

$$\hat{E} = E - S_f \Delta u_f$$

Optimization Problem…

$$\hat{E}^T W^Y \hat{E} = \left(E - S_f \Delta u_f\right)^T W^Y \left(E - S_f \Delta u_f\right)$$

$$= E^T W^Y E - 2\Delta u_f^T S_f^T W^Y E + \Delta u_f^T S_f^T W^Y S_f \Delta u_f$$

so

$$\min_{\Delta u_f} \quad J = \hat{E}^T W^Y \hat{E} + \Delta u_f^T W^U \Delta u_f$$

can be written

$$\min_{\Delta u_f} \quad J = E^T W^Y E + \Delta u_f^T \left(S_f^T W^Y S_f + W^U\right) \Delta u_f - 2\Delta u_f^T S_f^T W^Y E$$

and the unconstrained solution is found from

$$\partial J / \partial \Delta u_f = 0$$

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Unconstrained Solution

Analytical Solution for Unconstrained System

$$\Delta u_f = \left(S_f^T W^Y S_f + W^U\right)^{-1} S_f^T W^Y E$$

"unforced" error

In practice, do not actually invert a matrix. Solve as set of simultaneous equations (or use \ in MATLAB)

$$\Delta u_f = \left(S_f^T W^Y S_f + W^U\right) \backslash S_f^T W^Y E$$

How do we estimate the disturbance?

Concise Review of Optimal Estimation (Kalman Filtering)

Rudy Kalman

- Kalman & Bucy, 1960

  Kalman, R. E. (1960b). A new approach to linear filtering and prediction problems. *Transactions of ASME, Journal of Basic Engineering*, *87*, 35–45.

- Developed for well-modeled systems with state input and measurement noise
- Success in Aerospace and other applications

- Problems with initial chemical process applications
  - Limited understanding of limitations
- Bias due to model uncertainty
- Need an "appended state" formulation

Stochastic Models

Input noise

$$x_{k+1} = \Phi x_k + \Gamma u_k + \Gamma^w w_k$$

$$\text{cov}(w) = Q$$

$$y_k = C x_k + v_k$$

Measurement noise

$$\text{cov}(v) = R$$

$$\text{cov}(x_0) = P_0$$

Initial State Uncertainty

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Optimal State Estimate

$$J(\hat{y}) = \sum_{i=1}^{k} (y_i - C\hat{x})^T R^{-1} (y_i - C\hat{x})$$

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State Estimation Procedure

$$\hat{x}_{k|k-1} = \Phi \hat{x}_{k-1|k-1} + \Gamma u_{k-1}$$

Prediction

Estimate at k, based on measurements through k-1

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k \left(y_k - C\hat{x}_{k|k-1} \right)$$

Correction

Estimate at k, based on measurements through k

$$P_k = \Phi P_{k-1} \Phi^T + \Gamma^w Q \Gamma^{wT} - \Phi P_{k-1} C^T \left(C P_{k-1} C^T + R \right)^{-1} C P_{k-1} \Phi^T$$

State error covariance estimate

Kalman gain

$$L_k = P_k C^T \left(C P_k C^T + R \right)^{-1}$$

Discuss steady-state Kalman gain

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State Covariance Comparison

"Open-loop"

$$P_k = \Phi P_{k-1} \Phi^T + \Gamma^w Q \Gamma^{wT}$$

With measurement (Kalman Filter formulation)

$$P_k = \Phi P_{k-1} \Phi^T + \Gamma^w Q \Gamma^{wT} - \underbrace{\Phi P_{k-1} C^T \left(C P_{k-1} C^T + R \right)^{-1} C P_{k-1} \Phi^T}$$

Measurement update reduces state estimate covariance

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Kalman Filter w/Augmented States

$$\underbrace{\begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix}}_{x_{k+1}^a} = \underbrace{\begin{bmatrix} \Phi & \Gamma^d \\ 0 & I \end{bmatrix}}_{\Phi^a} \underbrace{\begin{bmatrix} x_k \\ d_k \end{bmatrix}}_{x_k^a} + \underbrace{\begin{bmatrix} \Gamma \\ 0 \end{bmatrix}}_{\Gamma^a} u_k + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{\Gamma^{a,w}} w_k$$

process noise

$$\text{cov}(w) = Q$$

Augmented state (includes disturbance)

$$y_k = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{C^a} \underbrace{\begin{bmatrix} x_k \\ d_k \end{bmatrix}}_{x_k^a} + v_k$$

Measurement noise

$$\text{cov}(v) = R$$

Predictor-corrector equations:

$$\hat{x}_{k|k-1}^a = \Phi^a \hat{x}_{k-1|k-1}^a + \Gamma^a u_{k-1}$$

Manipulated input

$$\hat{x}_{k|k}^a = \hat{x}_{k|k-1}^a + L_k \left(y_k - C^a \hat{x}_{k|k-1}^a \right)$$

Aug. state estimate

Kalman gain

Measured output

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Disturbance Portion of Augmented State

$$\hat{x}^{a}_{k|k} = \begin{bmatrix} \hat{x}_{k|k} \\ \hat{d}_{k|k} \end{bmatrix}$$

Constant Disturbance Into the Future

$$\hat{d}_{k+j|k} = \hat{d}_{k|k}$$

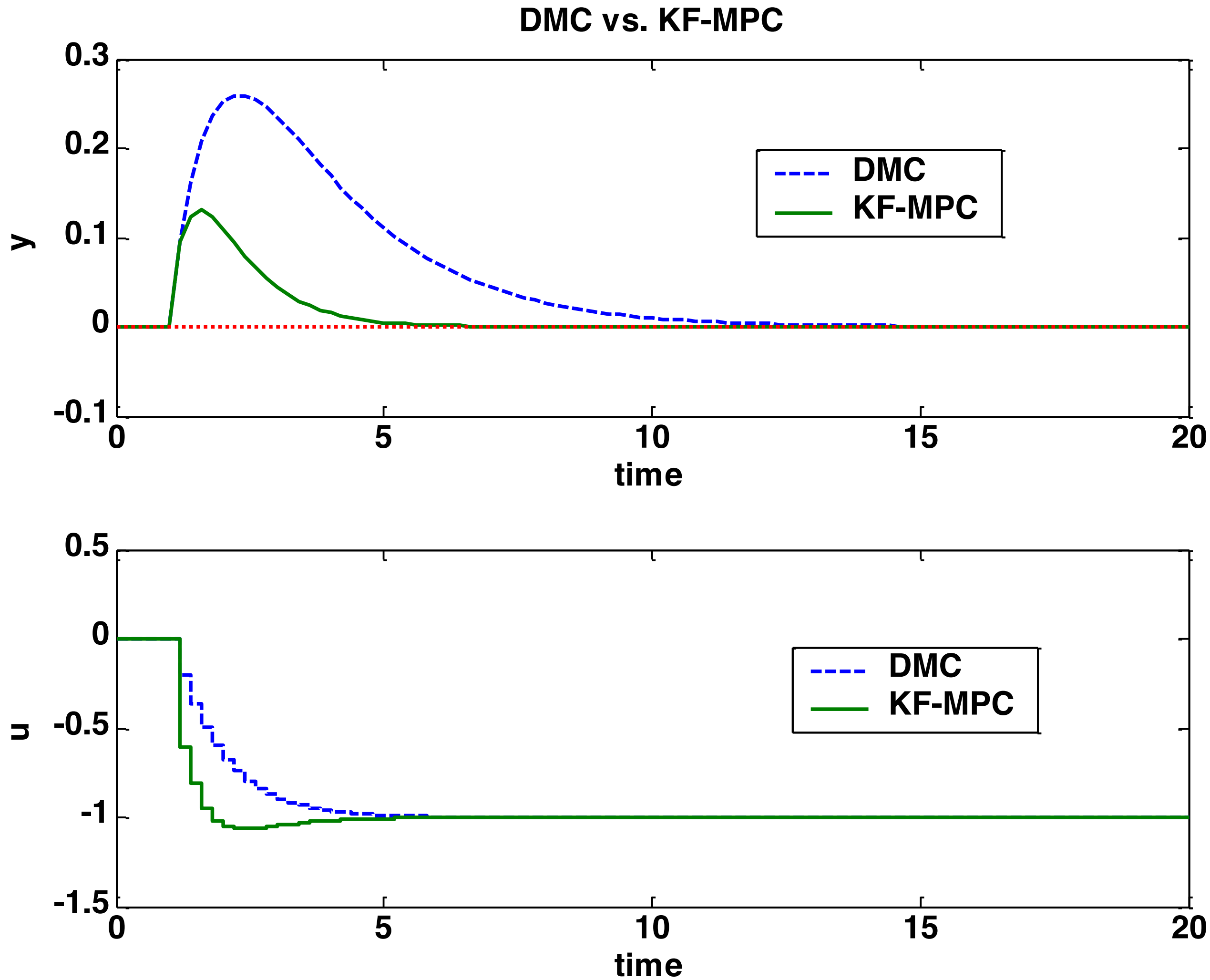
Can use physical knowledge to make better assumptions (periodic effects, etc.)

Example

$$g_p(s) = \frac{1}{10s+1}$$

Comparison of DMC and KF-based MPC for a step input disturbance

Step Input Disturbance



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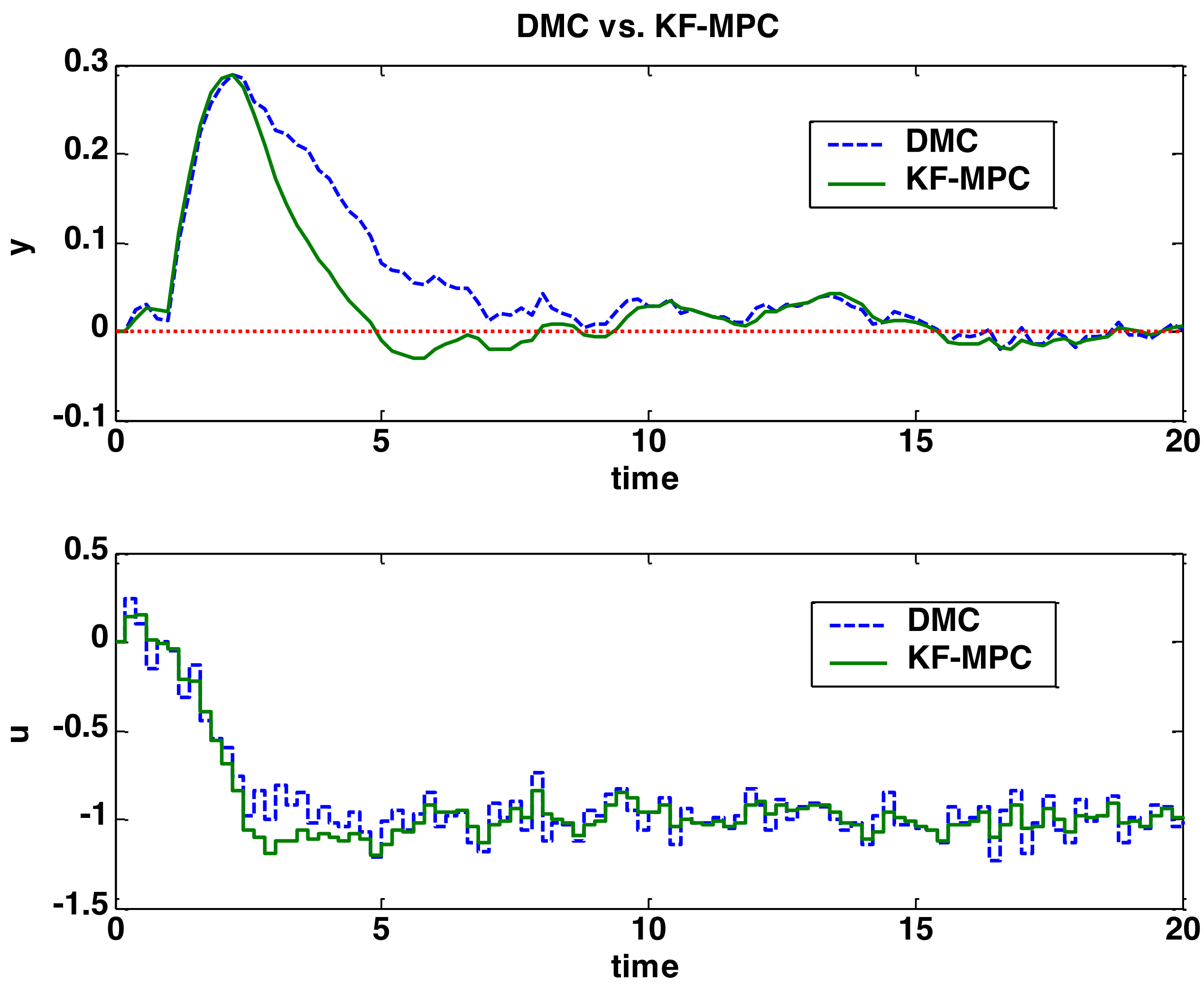
Measurement Noise



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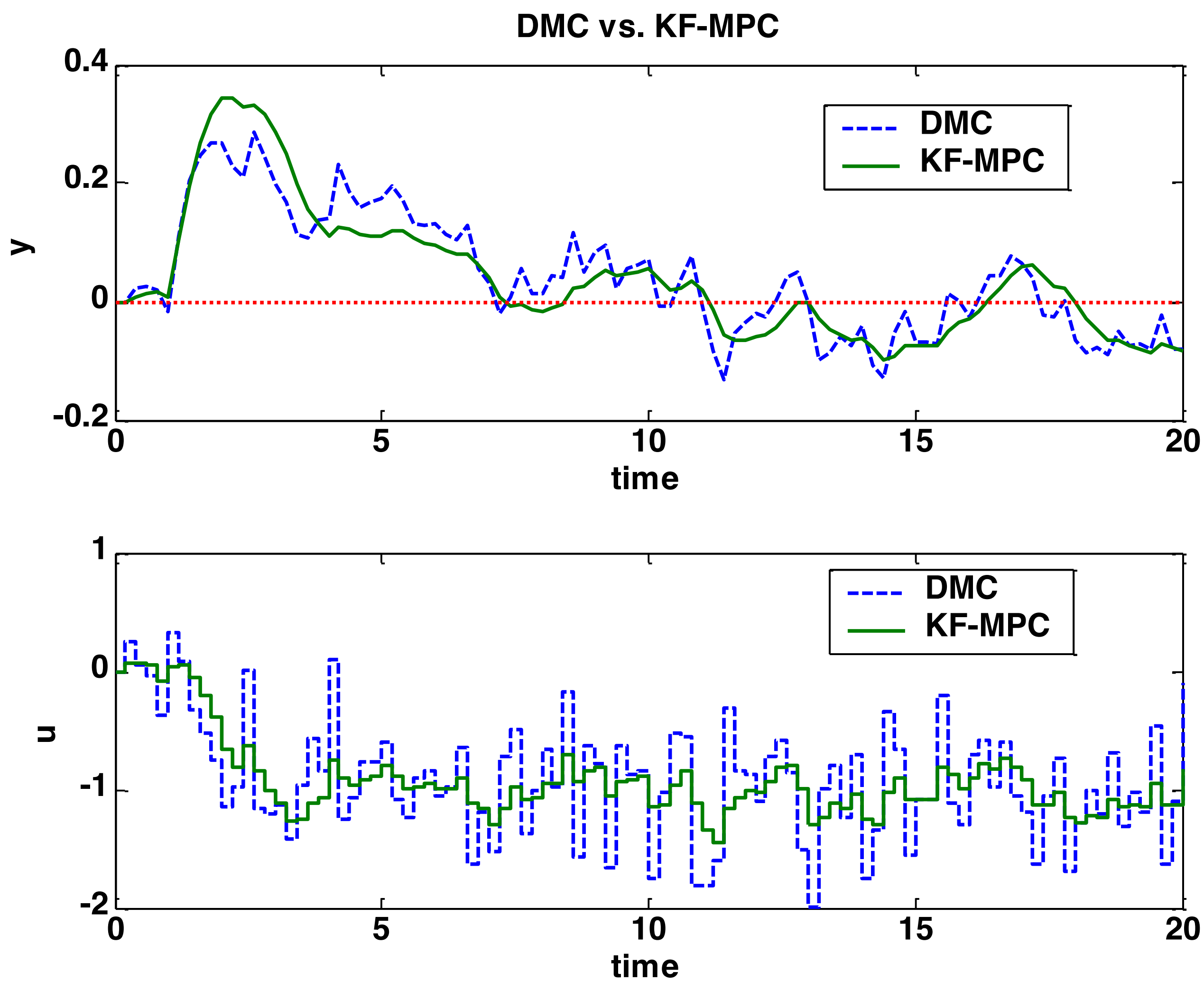
Substantial Measurement Noise



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Constrained MPC

Input constraints

$$u_{\min} \le u_{k+i} \le u_{\max}$$

$(n_u x1)$ $(n_u x1)$ $(n_u x1)$

where

$$u_k = u_{k-1} + \Delta u_k$$

$$u_{k+1} = u_{k-1} + \Delta u_k + \Delta u_{k+1}$$

etc.

Input "velocity" constraints

$$\Delta u_{\min} \le \Delta u_{k+i} \le \Delta u_{\max}$$

Output constraints

$$y_{\min} \le \hat{y}_{k+i|k} \le y_{\max}$$

MPC: Constraint Formulation

Input

$$\begin{bmatrix} u_{\min} \\ u_{\min} \\ \vdots \\ u_{\min} \end{bmatrix} \le \begin{bmatrix} u_{k-1} \\ u_{k-1} \\ \vdots \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ I & I & 0 & \cdots & 0 \\ \vdots & \vdots & & & \\ I & I & I & \cdots & I \end{bmatrix} \begin{bmatrix} \Delta u_k \\ \Delta u_{k+1} \\ \vdots \\ \Delta u_{k+M-1} \end{bmatrix} \le \begin{bmatrix} u_{\max} \\ u_{\max} \\ \vdots \\ u_{\max} \end{bmatrix}$$

Velocity

$$\begin{bmatrix} \Delta u_{\min} \\ \Delta u_{\min} \\ \vdots \\ \Delta u_{\min} \end{bmatrix} \le \begin{bmatrix} \Delta u_k \\ \Delta u_{k+1} \\ \vdots \\ \Delta u_{k+M-1} \end{bmatrix} \le \begin{bmatrix} \Delta u_{\max} \\ \Delta u_{\max} \\ \vdots \\ \Delta u_{\max} \end{bmatrix}$$

Output constraints

$$y_{\min} - f \le S_f \Delta u_f \le y_{\max} - f$$

"Free" or "unforced" response

$$f = \underbrace{\begin{bmatrix} C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^P \end{bmatrix} \hat{x}_{k|k} + \begin{bmatrix} C\Gamma \\ C\Phi\Gamma + C\Gamma \\ \vdots \\ \sum_{i=1}^{P} C\Phi^{i-1}\Gamma \end{bmatrix} u_{k-1} + \begin{bmatrix} C\Gamma^d \\ C\Phi\Gamma^d + C\Gamma^d \\ \vdots \\ \sum_{i=1}^{P} C\Phi^{i-1}\Gamma^d \end{bmatrix} \hat{d}_{k|k}}_{\text{"free" or "unforced response" (if no more control moves are made)}}$$

QP Codes: "one-sided" form

$$\underbrace{\begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ I & I & 0 & \cdots & 0 \\ \vdots & \vdots & & & \\ I & I & I & \cdots & I \end{bmatrix}}_{\mathbf{A_1}} \underbrace{\begin{bmatrix} \Delta u_k \\ \Delta u_{k+1} \\ \vdots \\ \Delta u_{k+M-1} \end{bmatrix}}_{\mathbf{\Delta u_f}} \geq \underbrace{\begin{bmatrix} u_{\min} - u_{k-1} \\ u_{\min} - u_{k-1} \\ \vdots \\ u_{\min} - u_{k-1} \end{bmatrix}}_{\mathbf{b_1}}$$

$$-\underbrace{\begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ I & I & 0 & \cdots & 0 \\ \vdots & \vdots & & & \\ I & I & I & \cdots & I \end{bmatrix}}_{\mathbf{A_2}} \underbrace{\begin{bmatrix} \Delta u_k \\ \Delta u_{k+1} \\ \vdots \\ \Delta u_{k+M-1} \end{bmatrix}}_{\mathbf{\Delta u_f}} \geq \underbrace{\begin{bmatrix} u_{k-1} - u_{\max} \\ u_{k-1} - u_{\max} \\ \vdots \\ u_{k-1} - u_{\max} \end{bmatrix}}_{\mathbf{b_2}}$$

“one-sided” form for input constraints

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \Delta u_f \geq \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

“one-sided” form for output constraints

$$S_f \Delta u_f \geq y_{\min} - f$$

$$\underbrace{-S_f \Delta u_f \geq -y_{\max} + f}$$

$$\begin{bmatrix} A_3 \\ A_4 \end{bmatrix} \Delta u_f \geq \begin{bmatrix} b_3 \\ b_4 \end{bmatrix}$$

Combining all constraints

$$\underbrace{\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \Delta u_f \geq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}$$

$$A \Delta u_f \geq b$$

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Quadratic Program (QP)

Objective Function $$\min_{\Delta u_f} \quad \Phi = \frac{1}{2} \cdot \Delta u_f^T H \Delta u_f + c^T \Delta u_f$$

Decision variables

s.t.

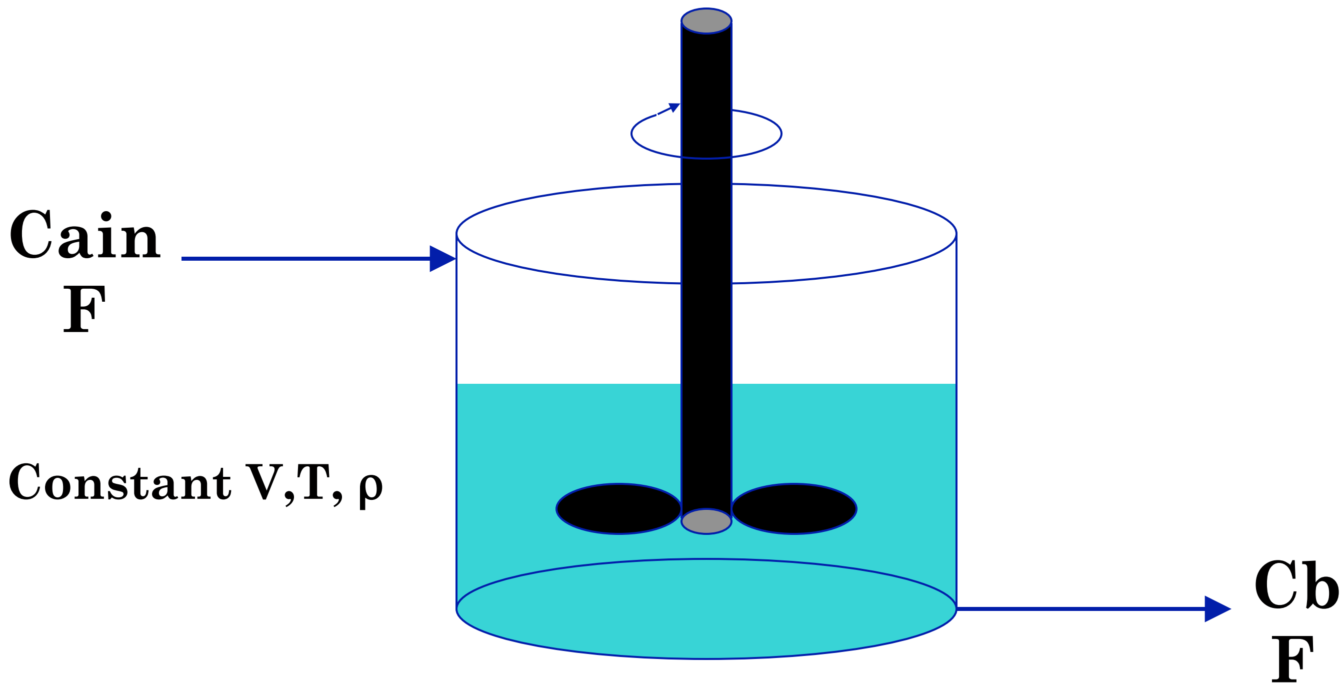
$$A \Delta u_f \geq b$$

$$\Delta u_{\min} \leq \Delta u_f \leq \Delta u_{\max}$$

Where $$H = S_f^T W^Y S_f + W^U$$

$$c = -S_f^T W^Y E$$

Inverse Response Process: Van de Vusse Reaction



$$A \xrightarrow{k_1} B \xrightarrow{k_2} C \qquad A + A \xrightarrow{k_3} D$$

$$\frac{dCa}{dt} = -k_1 Ca - k_3 Ca^2 + (Cain - Ca)u$$

$$\frac{dCb}{dt} = k_1 Ca - k_2 Cb - Cbu \qquad \text{where } u = F/V$$

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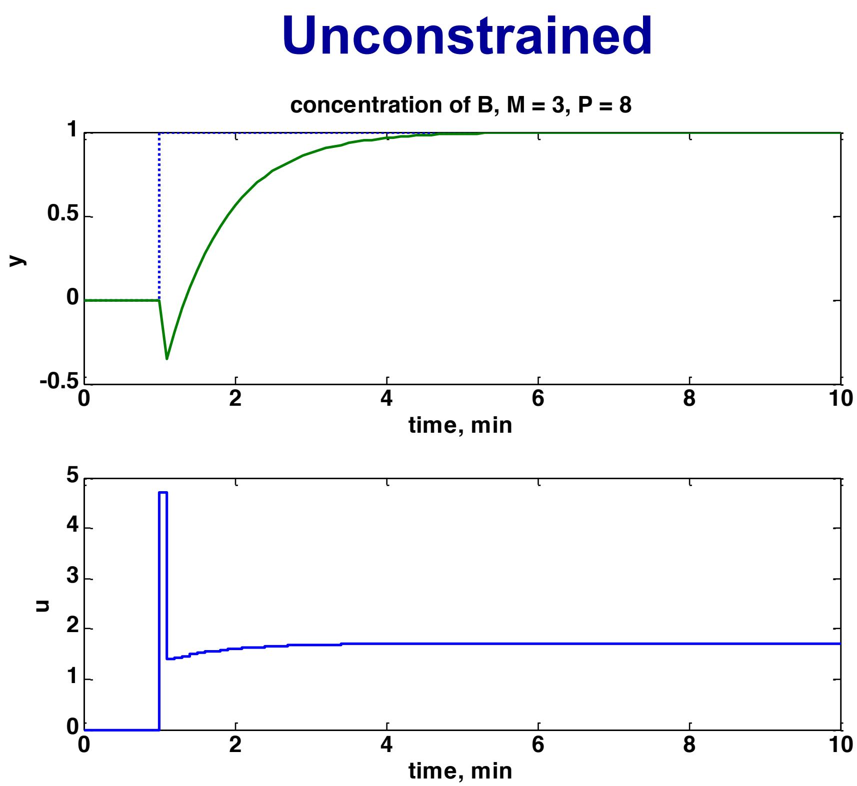
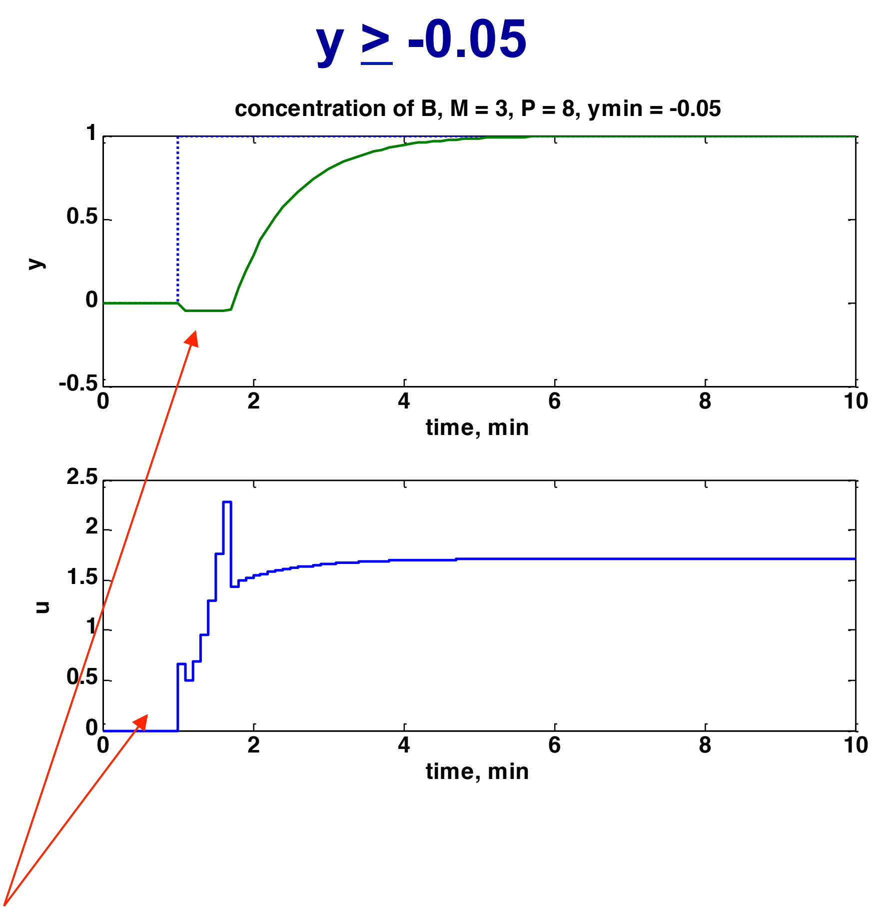
Output Constraints: Inverse Response Example





Less control action in order to meet output constraint

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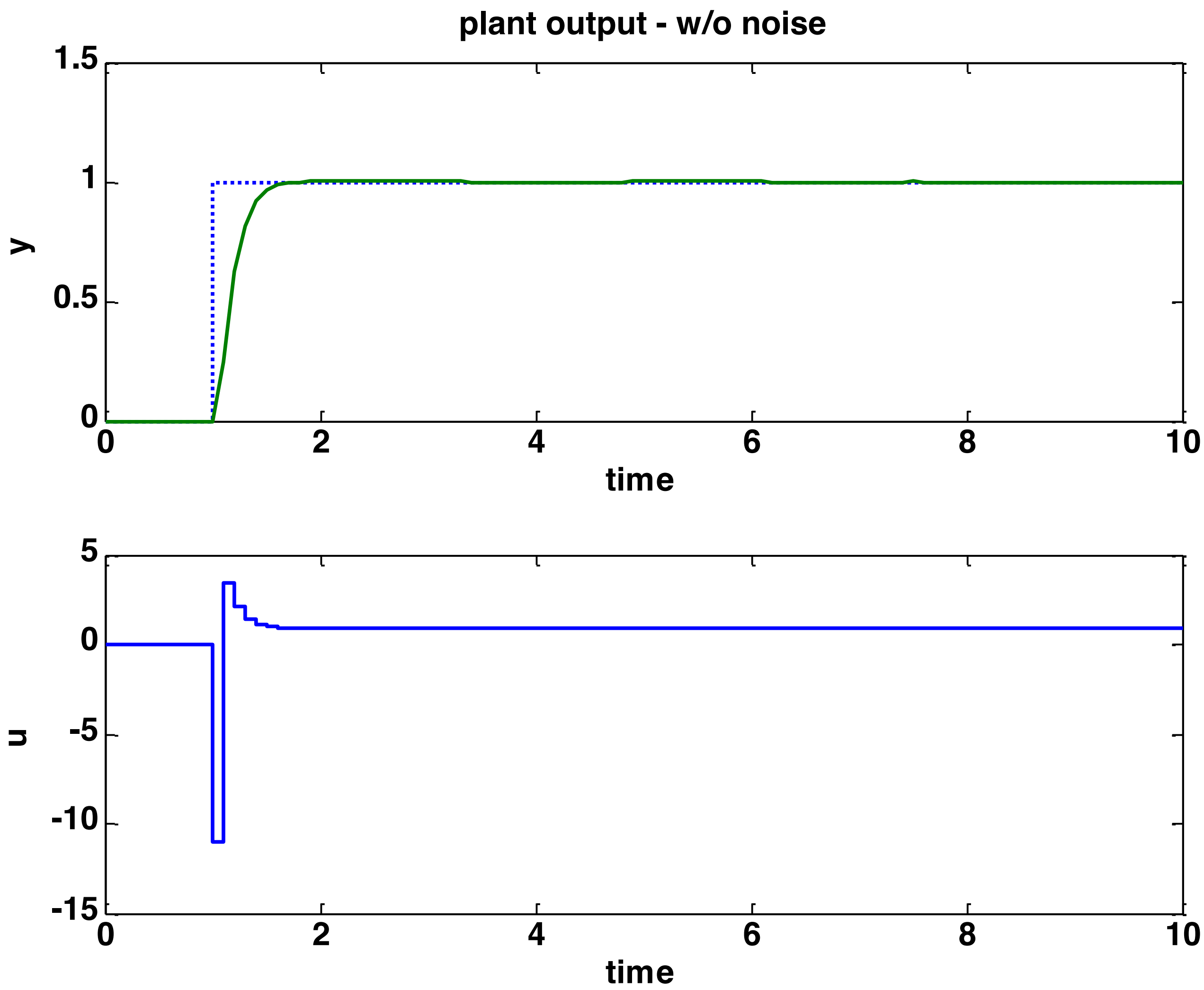
Output Constraints

- May not exist a feasible solution
- Relax constraints until feasible
- "Soft constraints" – Penalize in Objective Function

Unstable CSTR Example



DMC

Artifact of perfect model

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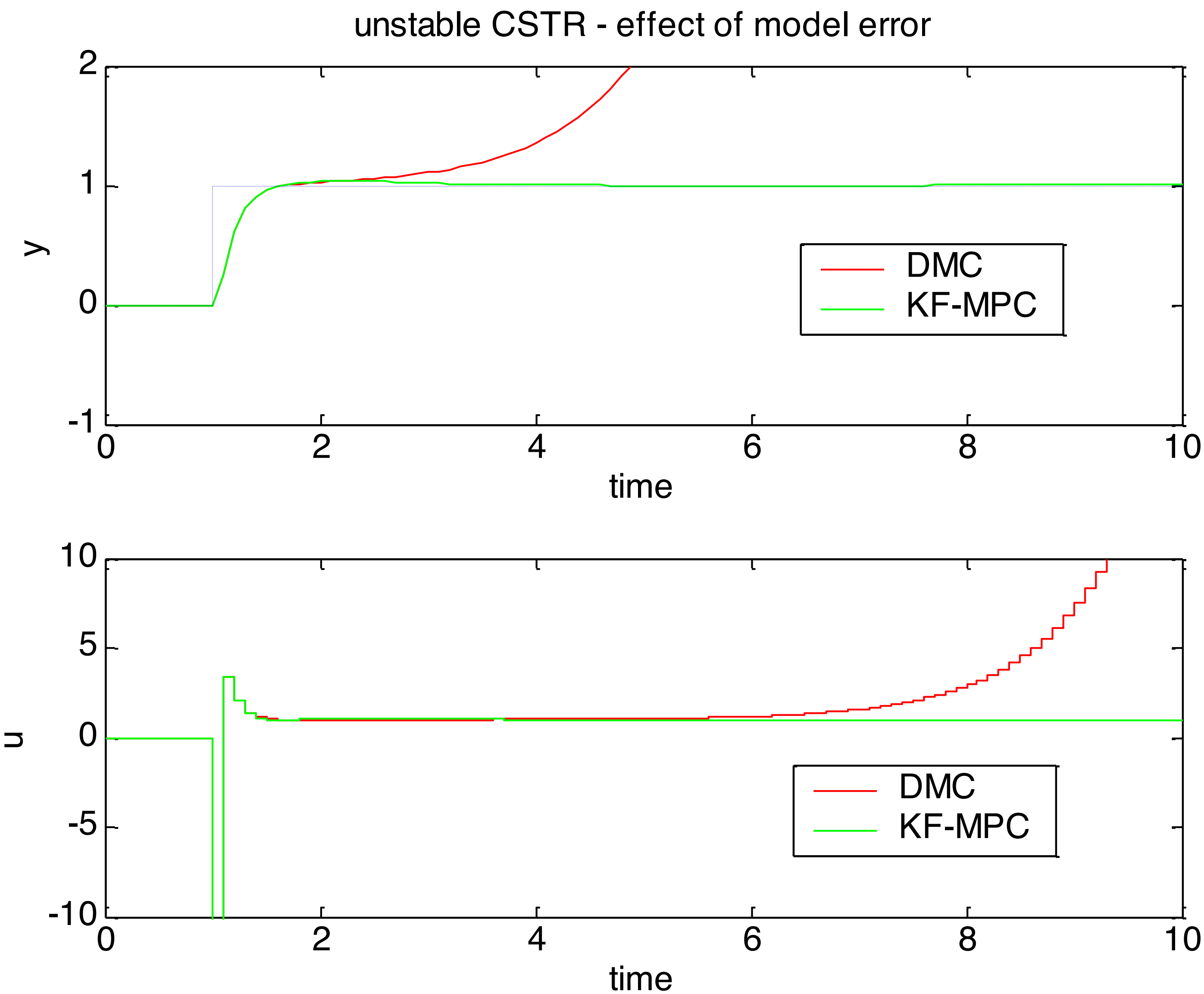
Effect of Model Error



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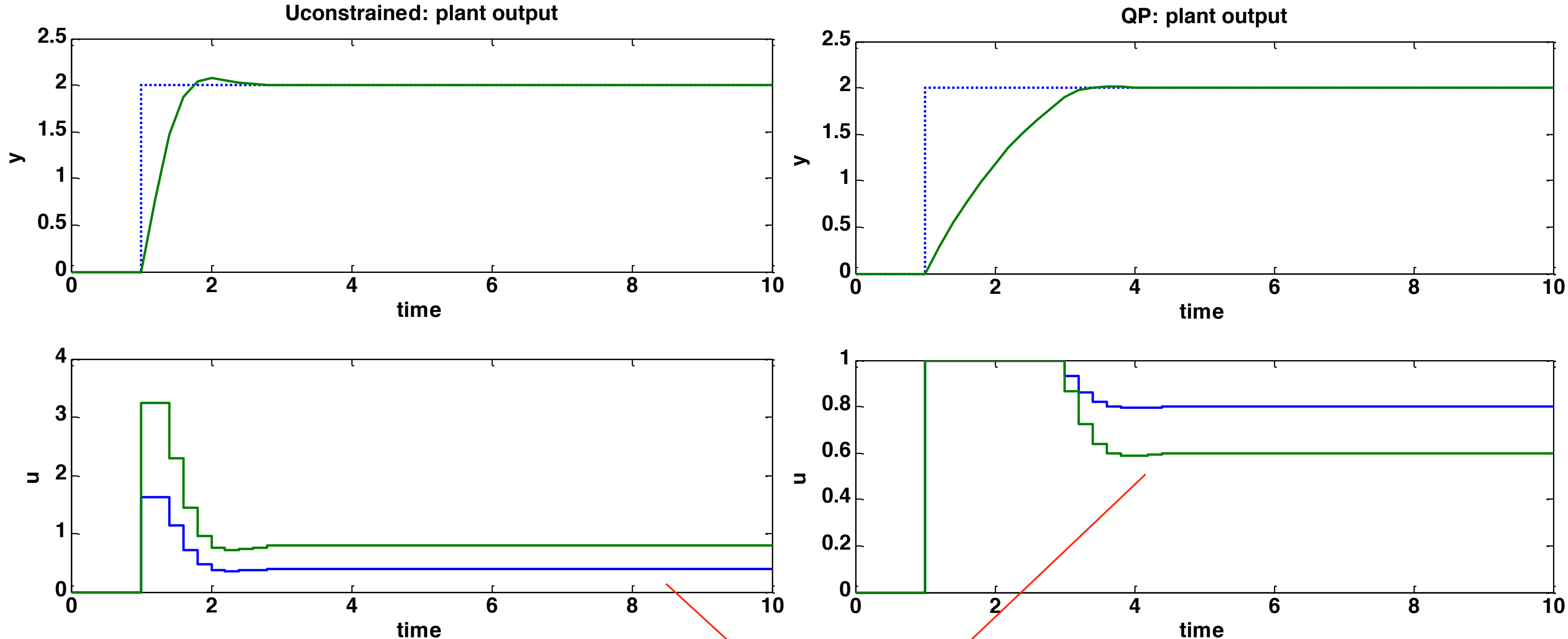
Multivariable Examples

- Have shown clear advantages of KF-based MPC over DMC for SISO
  - Disturbance rejection
  - Unstable processes
- In general, MPC is especially powerful for handling constrained, multivariable systems (including "nonsquare")
- Two "simple" example systems follow
  - Two-input, one-output (parallel valves)
  - Quadruple tank (two-input, two-output)

2 input – 1 output

$$y(s)=\frac{1}{2s+1}u_1(s)+\frac{2}{2s+1}u_2(s)+\frac{1}{2s+1}d(s)$$

$$W^u=diag(0.1,0.1), W^y=1, P=10, M=5$$



Unconstrained **Constrained**

Inputs converge to different values

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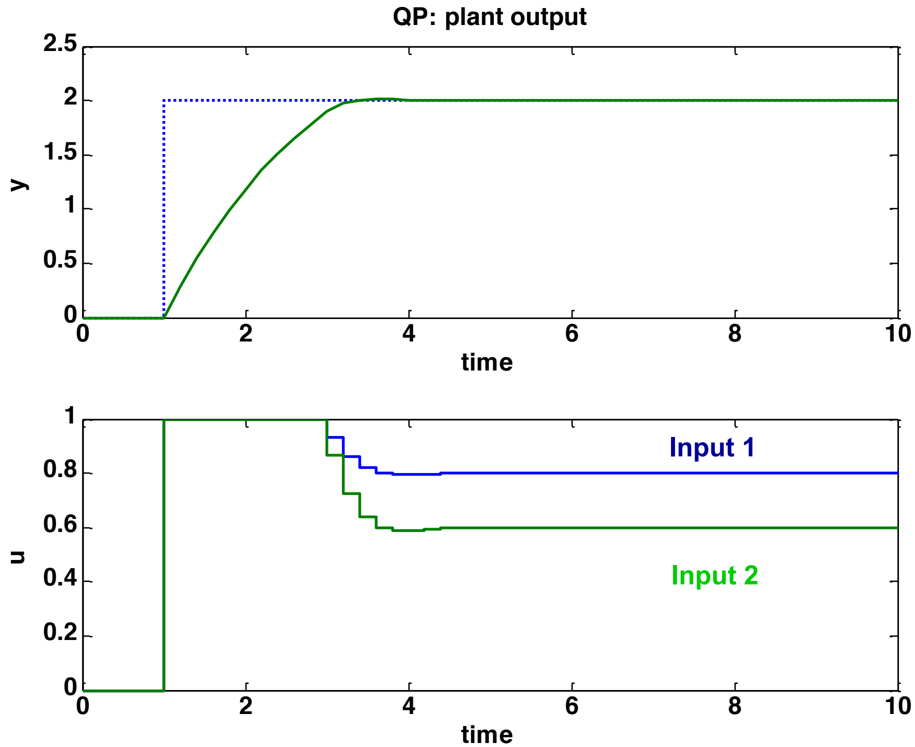
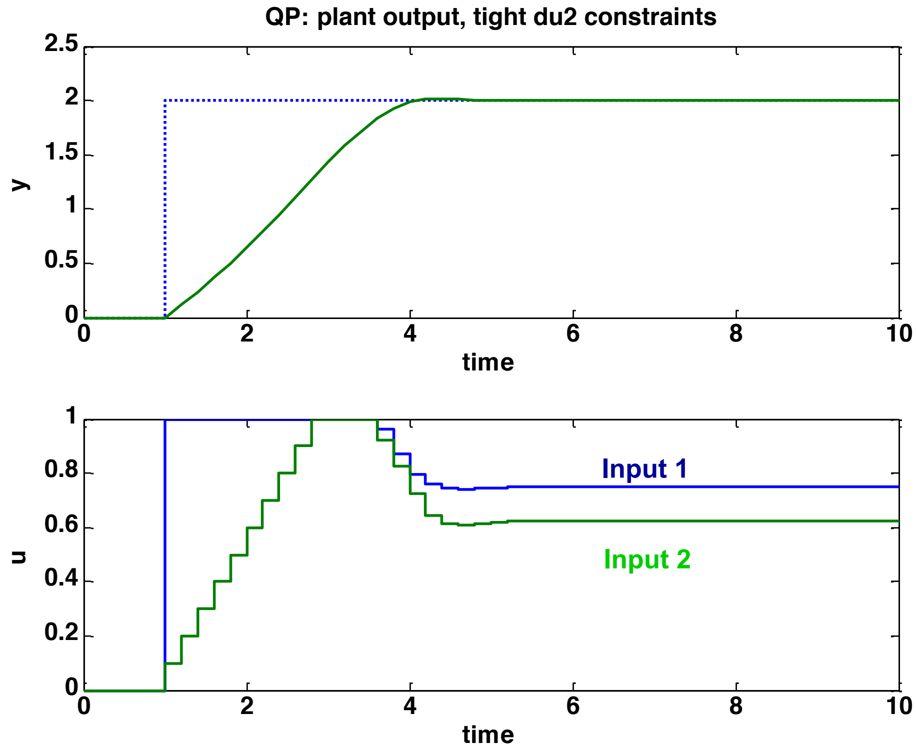
2 input – 1 output

Setpoint Change, with constraints





Constrained

Constrained, Δu_2 <= 0.1

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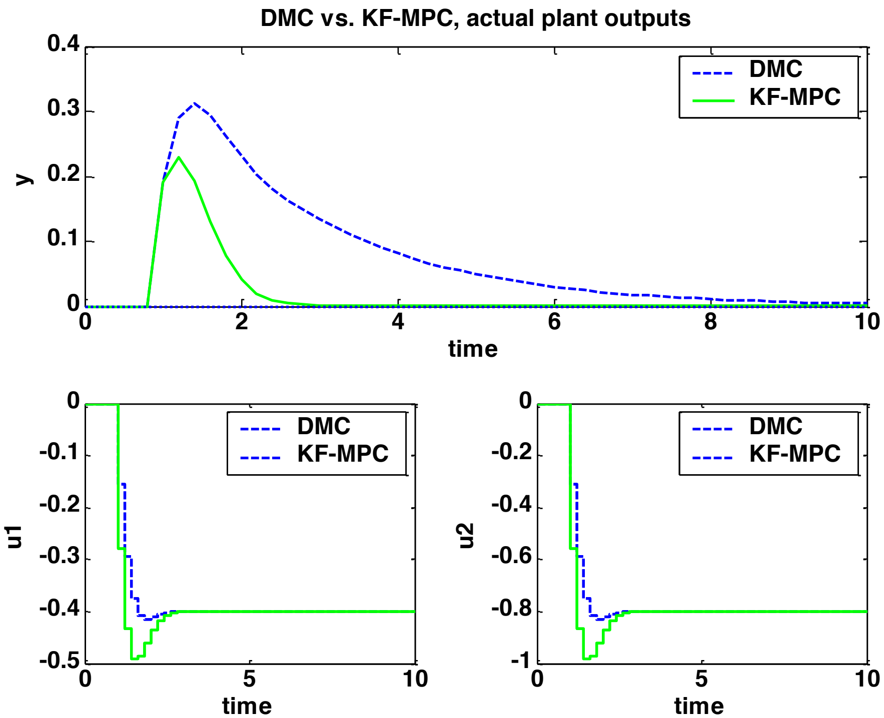
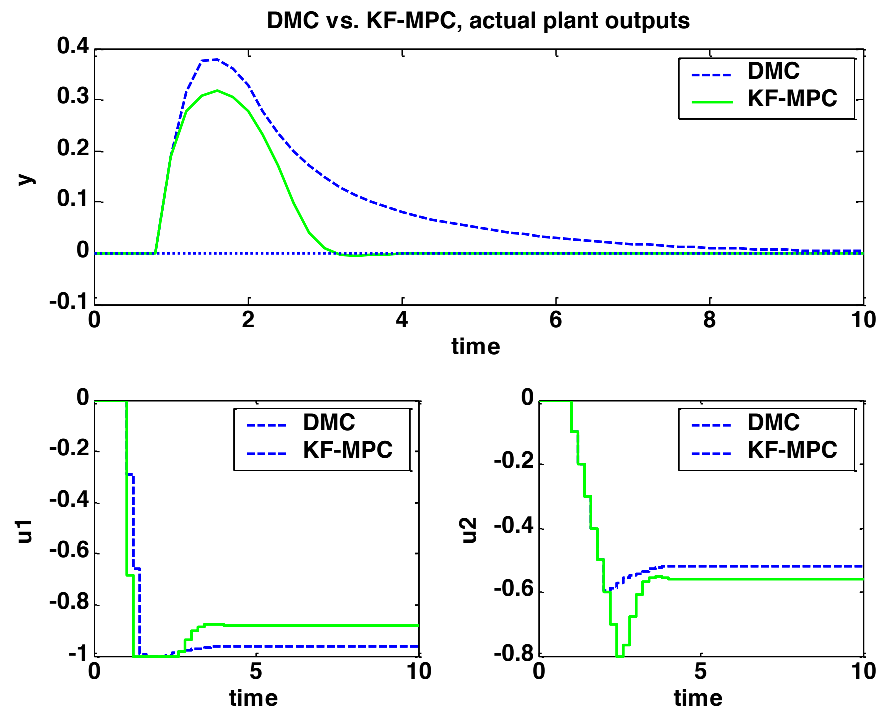
2 input – 1 output

Disturbance Rejection: unknown step in input





Tight velocity bounds (0.1) on u_1

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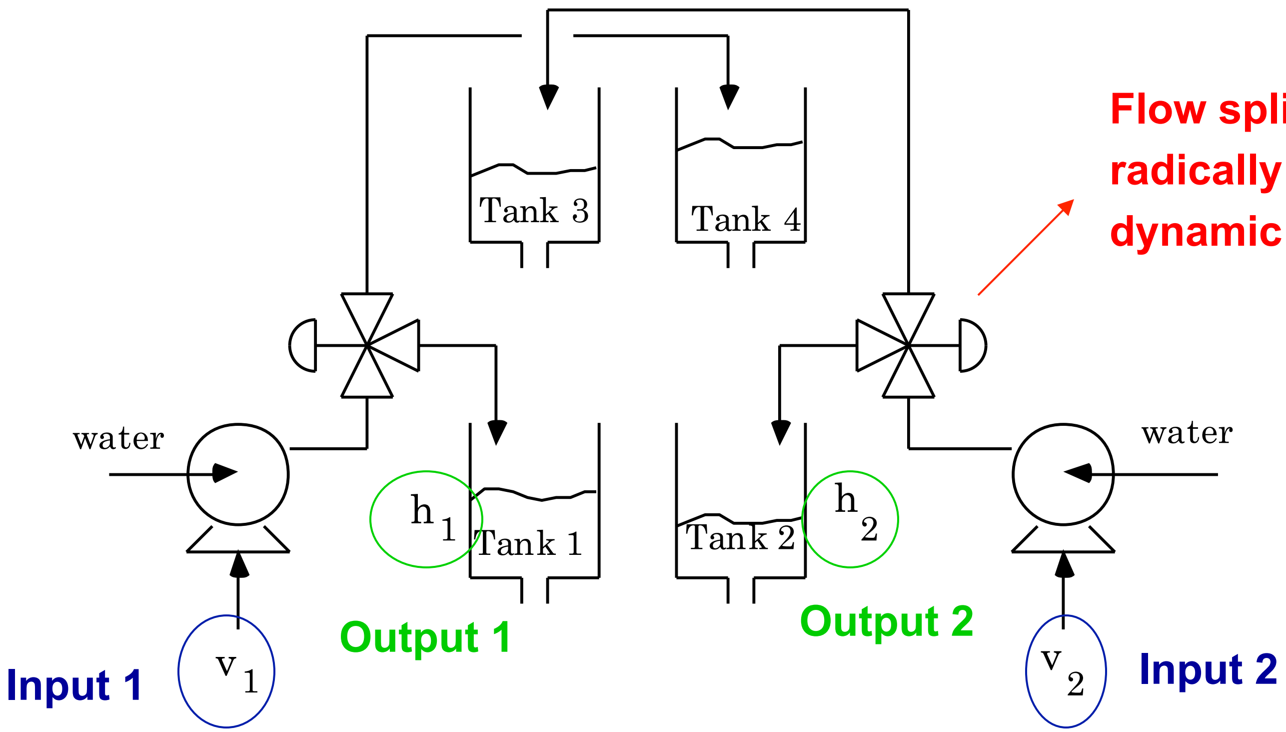
2 inputs – 1 output

- Many input combinations can achieve same output in steady-state
  - Tuning/constraints determine input values
- "Habituating Control" idea can be used
  - Fast, but "expensive" input for immediate disturbance rejection
  - Slow, but "cheap" input for long-term compensation

Quadruple Tank Problem



Flow splits can radically change dynamic behavior

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Quadruple Tank Problem

Operating Point 1

$$G_1(s) = \begin{bmatrix} \dfrac{2.6}{62s+1} & \dfrac{1.5}{(23s+1)(62s+1)} \\ \dfrac{1.4}{(30s+1)(90s+1)} & \dfrac{2.8}{(90s+1)} \end{bmatrix}$$

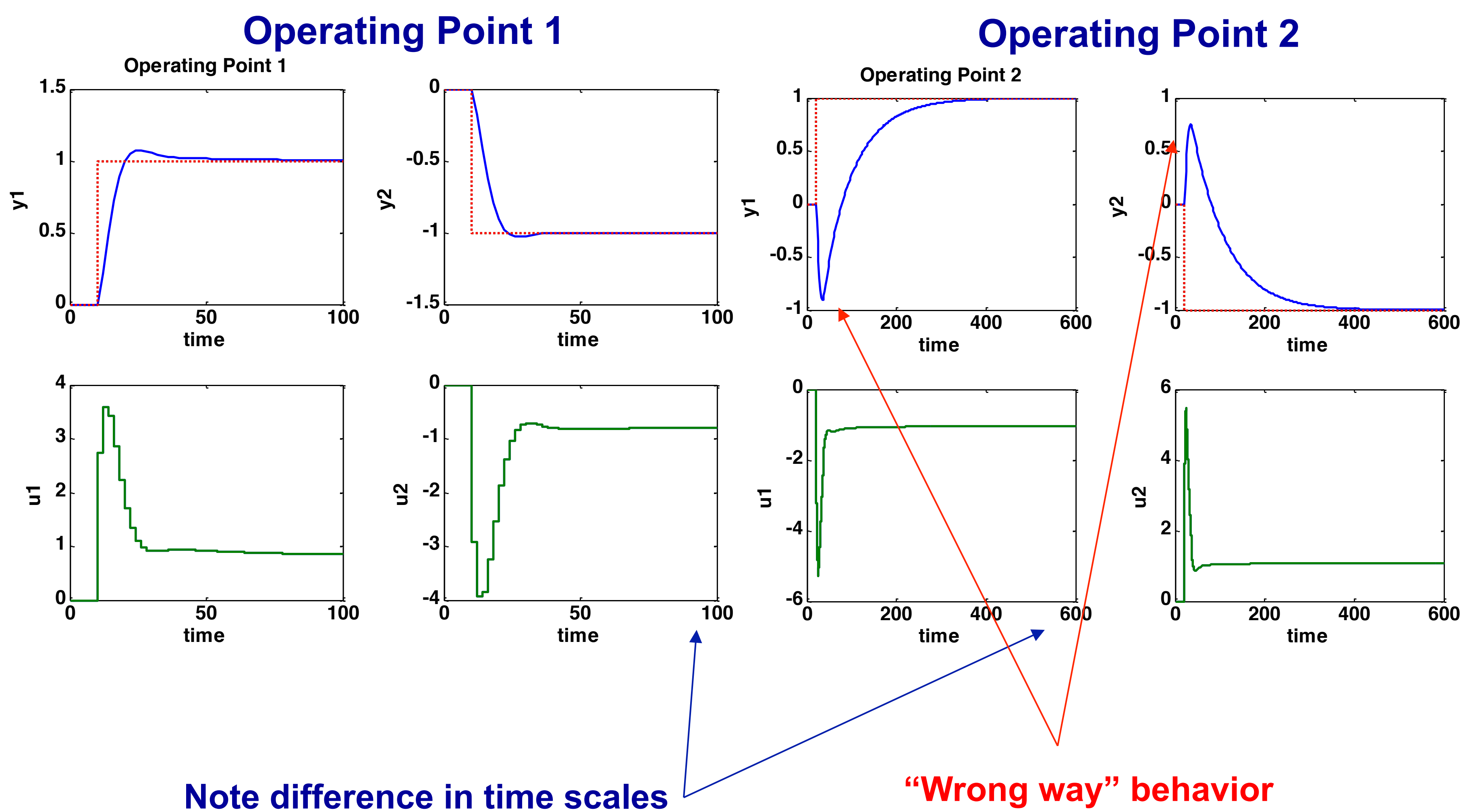
Operating Point 2

$$G_2(s) = \begin{bmatrix} \dfrac{1.5}{63s+1} & \dfrac{2.5}{(39s+1)(63s+1)} \\ \dfrac{2.5}{(56s+1)(91s+1)} & \dfrac{1.6}{(91s+1)} \end{bmatrix}$$

No time-delays or right-half-plane zeros.

Should we expect similar closed-loop performance?

Setpoint Responses



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Quadruple Tank Problem

Operating Point 1 – Minimum Phase

$$G_1(s) = \begin{bmatrix} \dfrac{2.6}{62s+1} & \dfrac{1.5}{(23s+1)(62s+1)} \\ \dfrac{1.4}{(30s+1)(90s+1)} & \dfrac{2.8}{(90s+1)} \end{bmatrix} \qquad z = -0.060 \text{ and } -0.018 \text{ sec}^{-1}$$

Operating Point 2 – Nonminimum Phase (RHPT zero)

$$G_2(s) = \begin{bmatrix} \dfrac{1.5}{63s+1} & \dfrac{2.5}{(39s+1)(63s+1)} \\ \dfrac{2.5}{(56s+1)(91s+1)} & \dfrac{1.6}{(91s+1)} \end{bmatrix} \qquad z = -0.057 \text{ and } +0.013 \text{ sec}^{-1}$$

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Multivariable Systems

- Can have right-half-plane "transmission zeros" even when no individual transfer function has a RHP zero
- Can have individual RHP zeros yet not have a RHPT zero
  - Fine performance when constraints are not active
  - May fail when one constraint becomes active or a loop is "opened"
- Can exhibit "directional sensitivity" – with some setpoint directions much easier to achieve than others

- Some of these MV properties cause challenges *independent of control strategy selected*

State Space Form not Limiting

- Step Response or Discrete TF can be written in State Space form

- Virtually all recent theoretical results are based on a state space formulation

Summary

- State Space MPC
- Disturbance Models
  - State/input or/and output disturbance
  - State/input form handles unstable systems
- Observer Design
  - Kalman Filter

- Can Control Outputs that are not measured
- Manipulated Input Blocking
  - Reduce number of decision variables

Stability Proofs: Linear Models

- Unconstrained
  - Infinite horizon, State and Input Weights
    - LQ Control
    - Couple with Kalman Filter if some states are unmeasured
  - State penalty at end of horizon
- Constrained
  - Terminal state constraint
  - Stabilizing state feedback at end of prediction horizon