SOLUTION
Sample Exam (p. 536)

Problem 1. Controllability analysis

a) 2×2 plant

$$G(s) = \frac{1}{(s+2)(s-1.1)} \begin{pmatrix} s-1 & 1 \\ 90 & 10(s-1) \end{pmatrix}$$

– For a controllability analysis we first check for possible RHP-zeros & poles.

Poles. First order minors yield: $\frac{*}{(s+2)(s-1.1)}$

Second order minor: $\det G = \frac{10(s-1)^2 - 90}{(s+2)^2(s-1.1)^2} = \frac{10\cancel{(s+2)}(s-4)}{(s+2)^{\cancel{2}}(s-1.1)^2}$ (*)

Conclusion: Pole polynomial: $(s+2)(s-1.1)^2$, that is two RHP-poles at $s=1.1$, i.e. need bandwidth $\omega_B > 1.1$ in all directions (since there are two RHP-poles which clearly appear in both directions). Also: I think this means that we need to close two loops in order to stabilize the plant.

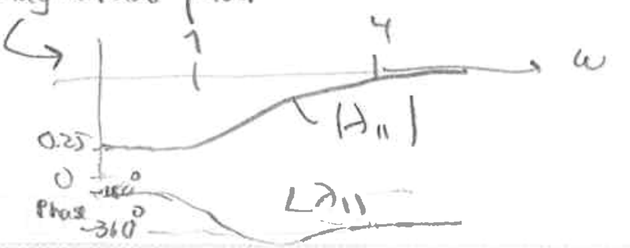
Zeros. First note that the two diagonal elements have a RHP zero at $s=1$. However, these do not pose a fundamental limitation. To compute the multivariable zeros we use the second order minor (*) and find the zero polynomial $z(s) = s-4$, that is there is a RHP-zero at $s=4$.

– Fortunately the RHP-zero at $z=4$ is to the right of the pole at $p=1$. Nevertheless, they are quite close so very tight control will be impossible for this plant

– Interactions: The 1,1-element of the RGA is

$$\lambda_{11} = \frac{1}{1 - \frac{g_{12}g_{21}}{g_{11}g_{22}}} = \frac{1}{1 - \frac{9}{(s-1)^2}} = \frac{(s-1)^2}{(s-1)^2 - 9} = \frac{(s-1)^2}{(s-4)(s+2)} = \begin{cases} -0.125 \text{ at } s=0 \\ 1 \text{ at } s=\infty \end{cases}$$

Magnitude plot



RGA is relatively small so expect no problem with diagonal input uncertainty.

- On the other hand the condition number is relatively large. At steady-state $G(0) = \frac{1}{-2.2}\begin{pmatrix} -1 & 1 \\ 90 & -10 \end{pmatrix}$

and one sees directly that for output 1 the gain is about 1 and for output 2 about 90, so the condition number is close to 100, and thus if a decoupler is used (such that $\gamma(C)=\gamma(G)$) it will be sensitive to "full" output uncertainty

- Constraints: There may be problems with input constraints for changes in setpoint for output 1 (Borderline case)

- Decentralized control: Because of the RHP-zero at s=1 for the individual elements, it will be essentially impossible to stabilize the individual elements, that is, one has to rely on the interactions to stabilize the plant — this means that the main advantages of decentralized control are lost, so pairing on the diagonal elements is not recommended.

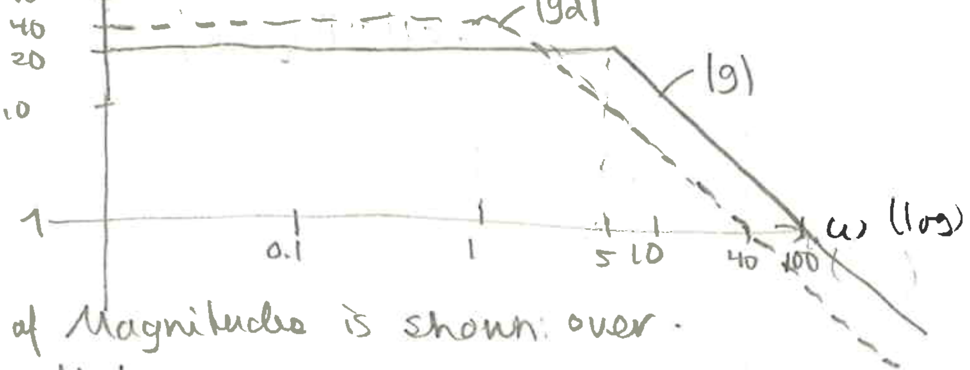
For the reverse pairing, we note that the RGA-elements approach 0 at $\omega \gtrsim 4$, which is within the desired bandwidth and thus unacceptable.

Conclusion: Forget decentralized control.

(Remark: In this case with two RHP-poles we would like to pair on positive RGA-elements, this is OK with the reverse pairing, but the interactions at high frequencies are unfavorable)

b) SISO plant with disturbance.

$$g(s) = \frac{20}{0.2s+1} \cdot \frac{-0.1s+1}{0.1s+1}, \quad g_d(s) = \frac{40}{s+1}$$



Plot of Magnitudes is shown over.

We find.

1. $|g_d| \gg 1$ for $\omega > \omega_d = 40$ so need $\omega_B > 40$
2. However, $g(s)$ has a RHP-zero at $z=10$ so in practice $\omega_B < 10$. Thus, disturbances at high frequencies

cannot be rejected satisfactorily.

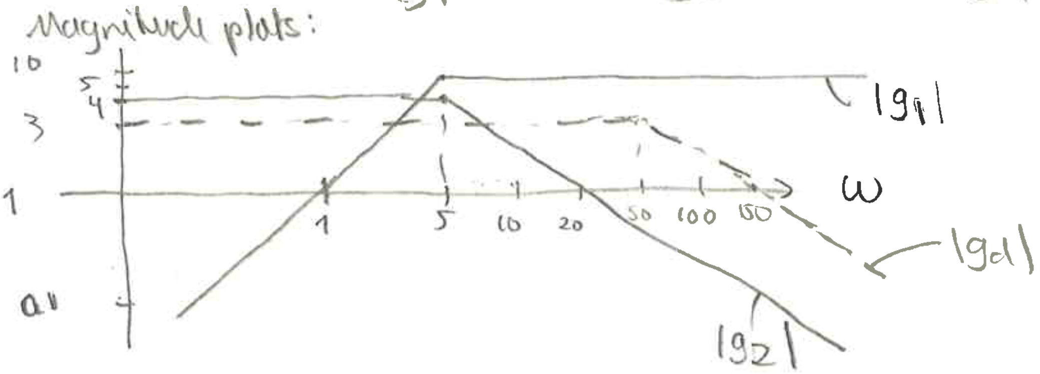
3. In addition, for $\omega < 1$ we have $|g| < |g_d|$ so there will be problem with input constraints, thus, only disturbances less than 0.5 in magnitude can be rejected at low frequencies.

In conclusion, reasonably acceptable control can only be achieved for disturbances in the frequency range 1 to 10, so the controllability of the plant is poor.

c) Plant with two inputs, one output, one disturbance:

$$y(s) = \underbrace{\frac{5}{0.2s+1}}_{g_1} u_1 + \underbrace{\frac{4}{0.2s+1}}_{g_2} \cdot u_2 + \underbrace{\frac{3}{0.02s+1}}_{g_d} \cdot d$$

Magnitude plots:



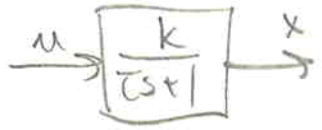
1. $|g_d| > 1$ for $\omega > \omega_d = 150$ so we need a bandwidth of greater than 150.

2. There are no RHP-zeros or delays to limit the achievable bandwidth, so this should be OK

3. We need to use both inputs to avoid constraints, i.e. use u_2 for "slow" control ($|g_2| > |g_d|$ for $\omega \lesssim 5$) and use u_1 for "fast" control ($|g_1| > |g_d|$ for $\omega \gtrsim 5$)

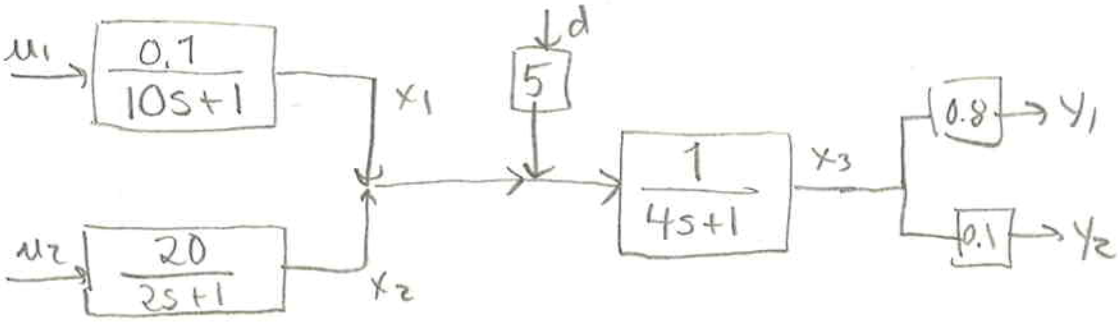
4. Thus the plant is controllable, but one must be quite careful with the design to avoid problems with input constraints.

d) i) Note that



corresponds to $\dot{x} = -\frac{1}{\tau}x + \frac{k}{\tau}u$

Thus for our case the equations correspond to the following block diagram.



Can also write as follows:

$$\underline{y} = \frac{1}{4s+1}\begin{pmatrix} \frac{0.08}{10s+1} & \frac{16}{2s+1} \\ \frac{0.01}{10s+1} & \frac{2}{2s+1} \end{pmatrix}\underline{u} + \frac{1}{4s+1}\begin{pmatrix} 4 \\ 0.5 \end{pmatrix} d$$

- Clearly the plant is not functionally controllable since everything has to go through a single state, and y_1 and y_2 can not be controlled independently.

- On the other hand, note that the gain from d to y_2 is 0.5 or less, so this output is actually "self-controlled", so we really only need to control y_1 (assuming we need not follow changes in the reference for y_2, but there is no mention of that so we assume $r_2 = 0$ and that we want to keep $|y_2|<1$.

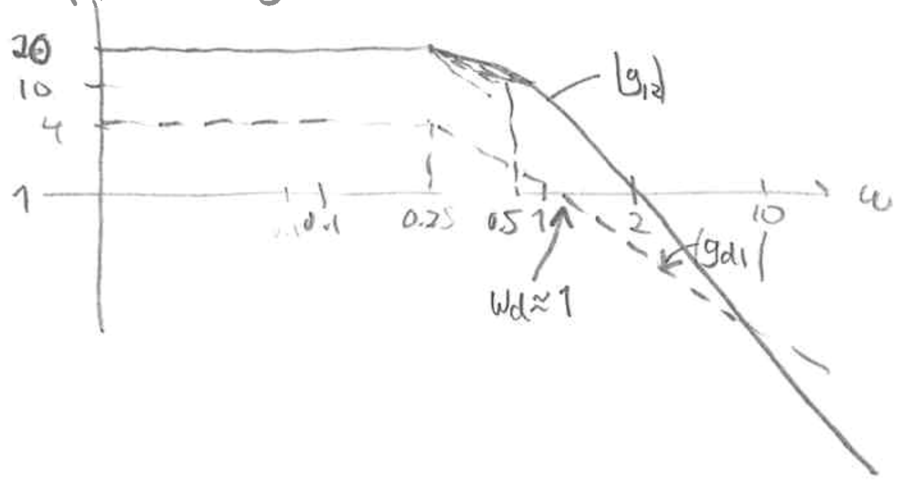
- Also: Even if the gain from d to y_2 was larger than 1 there would be no problem, since by controlling y_1 we also control y_2 (since $y_1 = 8y_2$).

- We also note that to control y_1, we really only need to use input 2 (Input 1 has a much smaller effect at all frequencies; 1/20 at low frequencies and even less at higher frequencies)

- So we conclude that we only need to consider using u_2 to control y_1, i.e. we are left with the problem

$$y_1 = \underbrace{\frac{16}{(2s+1)(4s+1)}}_{g_{12}} u_2 + \underbrace{\frac{4}{4s+1}}_{g_{d1}} d$$

We note that the transfer function for the input is of higher order, so there is potentially a problem with input constraints at high frequencies. A more careful analysis is needed:



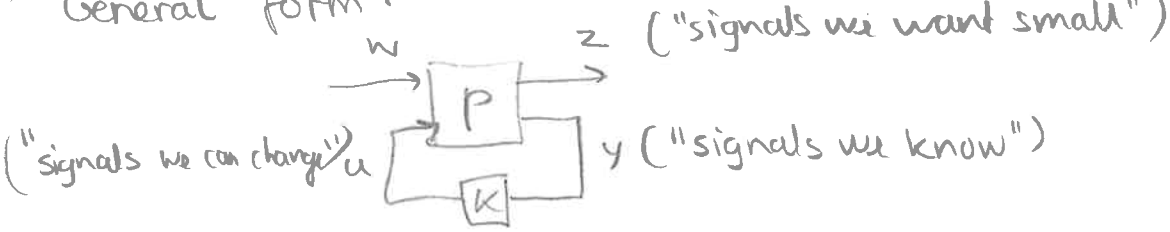
- From disturbance we require $\omega_B > \omega_d = 1$
- Fortunately $|g| > |g_d|$ for frequencies up to $\omega = 2$ so we expect no problems with input constraints (provided we do not tune the controller too fast).

(I used 70 min on problem 1)

2150

Problem 2. General Control Problem Formulation.

a) General form:



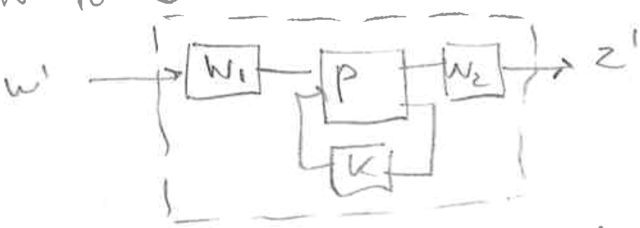
$$w = \begin{pmatrix} d \\ y_{2s} \\ u_{2s} \end{pmatrix}, \quad z = \begin{pmatrix} y_2 - y_{2s} \\ u_2 - u_{2s} \end{pmatrix}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_{2s} \\ u_{2s} \end{pmatrix}$$

$$P = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc|cc} \multicolumn{3}{c|}{\overbrace{\hphantom{g_2g_1 \quad -1 \quad 0}}^{w}} & \multicolumn{2}{c}{\overbrace{\hphantom{g_2g_1 \quad g_2}}^{u}} \\ d & y_{2s} & u_{2s} & u_1 & u_2 \\ \hline g_2g_1 & -1 & 0 & g_2g_1 & g_2 \\ 0 & 0 & -1 & 0 & 1 \\ \hline g_1 & 0 & 0 & g_1 & 0 \\ g_2g_1 & 0 & 0 & g_2g_1 & g_2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline \multicolumn{3}{c|}{P_{21}} & \multicolumn{2}{c}{P_{22}} \end{array} \begin{array}{l} \\ \\ \left.\begin{array}{l} y_2 - y_{2s} \\ u_2 - u_{2s} \end{array}\right\} z \\ \left.\begin{array}{l} y_1 \\ y_2 \\ y_{2s} \\ u_{2s} \end{array}\right\} y \\ \\ \end{array}$$

Note: Generalized controller K has 4 inputs and 2 outputs

b) H^∞-control problem. Consider transfer function from w' to z':



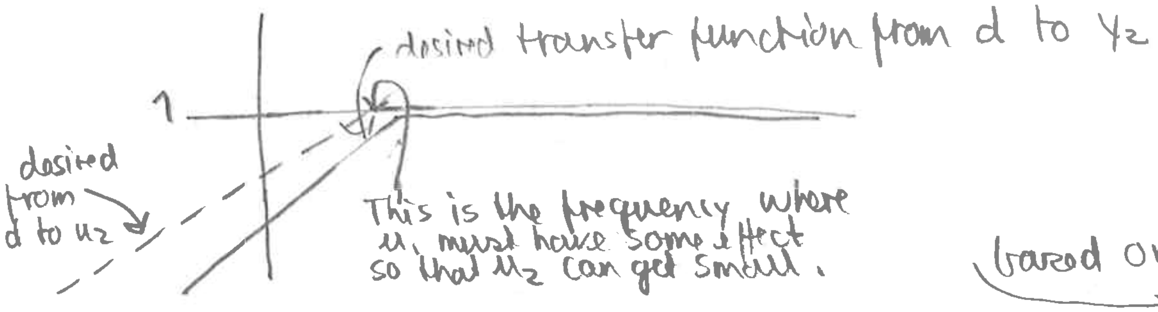
$$W_2 F_l(P, K) W_1 = W_2 \left[P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21} \right] W_1$$

The H_∞ - problem is then

$$\min_K \| W_2 T_l(P,K) W_1 \|_\infty$$

Unweighted transfer function from d to $z = \begin{pmatrix} y_2 - y_{2s} \\ u_2 - u_{2s} \end{pmatrix}$:

would like d to have small effect (less than 1) on $y_2 - y_{2s}$ over all frequencies (and zero at steady-state), and d to have a small effect on $u_2 - u_{2s}$ at low frequencies. Desired transfer functions

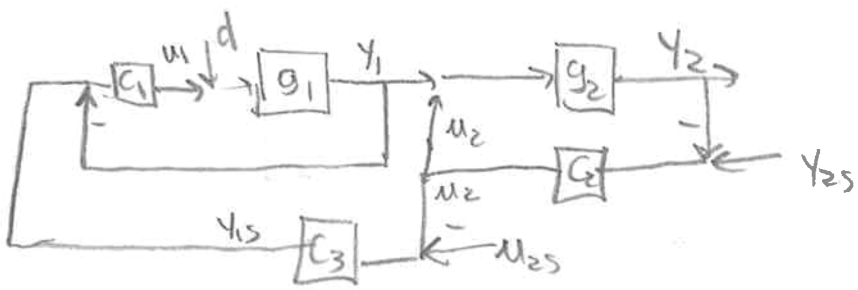


(based on the information in the text)

Note: It is not possible to put down an exact bound. The above assumes that the variables have already been scaled -1 to 1 (as usual) for constraints etc.

The weights will approximately the inverse of the desired unweighted transfer functions.

c) Proposed control structure based on single loops.



1. Note that the external inputs are $w = \begin{pmatrix} d \\ y_{2s} \\ u_{2s} \end{pmatrix}$ as in a)

2. There is an individual local feedback loop around each tank (use u_1 to control y_1; use u_2 to control y_2).

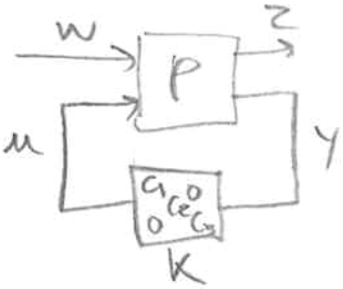
3. In addition, there is a controller C_3 which changes the setpoint, y_{1s}, for loop 1 such that eventually $u_2 = u_{2s}$ (assuming C_3 has integral action). Note that y_{1s} may be viewed as

an internal variable within the controller part of the system. In summary, the idea is to to use the loop u_1-y_1 to reject d locally, use u_2 to reject fast disturbances only, and finally reset $u_2 = u_{2s}$ by use of u_1.

The generalized plant becomes in this case:

$$w = \begin{pmatrix} d \\ y_{2s} \\ u_{2s} \end{pmatrix} \qquad z = \begin{pmatrix} y_2 - y_{2s} \\ u_2 - u_{2s} \end{pmatrix}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ y_{1s} \end{pmatrix} \qquad y = \begin{pmatrix} y_{1s} - y_1 \\ y_{2s} - y_2 \\ u_{2s} - u_2 \end{pmatrix}$$



$$\begin{array}{ccc|ccc|l}
d & y_{2s} & u_{2s} & u_1 & u_2 & y_{1s} & \\
\hline
g_{21}g_1 & -1 & 0 & g_{21}g_1 & g_2 & 0 & y_2 - y_{2s} \\
0 & 0 & -1 & 0 & 1 & 0 & u_2 - u_{2s} \\
\hline
-g_1 & 0 & 0 & -g_1 & 0 & 1 & y_{1s} - y_1 \\
-g_{21}g_1 & 1 & 0 & -g_{21}g_1 & -g_2 & 0 & y_{2s} - y_2 \\
0 & 0 & 1 & 0 & -1 & 0 & u_{2s} - u_2
\end{array}$$

$\underline{P}$

<u>Note</u>: In this case it is not possible to synthesize an optimal H_∞-controller using conventional software since K has a special structure. The reason is of course that the proposed control structure, although reasonable, is suboptimal.

(I used 35 min on Problem 2.)

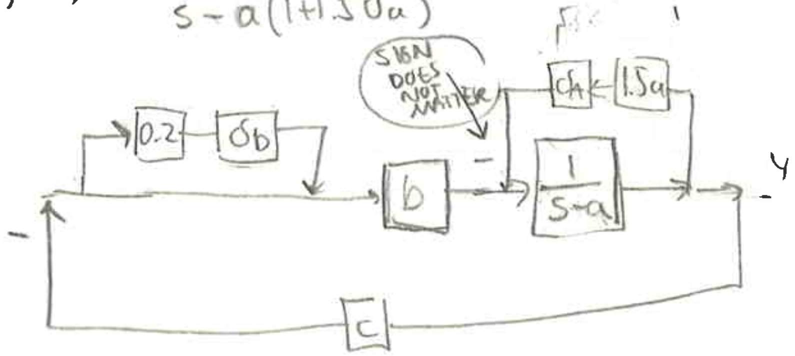
23^{05}

Problem 3.

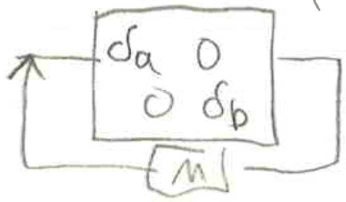
(This is a good example to include in the book)

a) $\dot{y}(t) = a(1+1.5\,\delta_a)\,y + b(1+0.2\,\delta_b)\,u$, $|\delta_a| \le 1$, $|\delta_b| \le 1$

$$\Rightarrow \quad y(s) = \frac{b(1+0.2\,\delta_b)}{s-a(1+1.5\,\delta_a)}\,u(s)$$



Interconnection structure for RS :



where $M =$

$$\begin{array}{c|cc} & \delta_a & \delta_b \\ \hline \delta_a & -\frac{1.5a}{s-a}\,S & \frac{1.5a}{s-a}\,b\cdot S \\ \delta_b & -\frac{0.2}{s-a}\cdot c\,S & -0.2\cdot T \end{array}$$

$1.5a\cdot gS$

$0.2\cdot\frac{1}{b}\cdot T$

Here: $g(s) = \frac{b}{s-a}$, $S = \frac{1}{1+gc}$, $T = \frac{gc}{1+gc}$

Note: 1) May "remove" negative signs and "move" 1.5 or 0.2 from rows to columns without changing RS-condition. Reason: $\mu(M) = \mu(MW) = \mu(WM)$ when $W = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}$

2) So we could get (!!): $M = \begin{pmatrix} -w_1 S & w_1 g S \\ w_2 cS & w_2 T \end{pmatrix}$, $w_1 = \frac{1.5a}{s-a}$, $w_2 = 0.2$

and can get $M = \begin{pmatrix} w_1 S & w_1 S \\ w_2 T & w_2 T \end{pmatrix}$

4) So we can get $M = \begin{pmatrix} w_1 S & w_1 S \\ w_2 T & w_2 T \end{pmatrix}$

Note that this is singular so we always have $\rho(M\Delta) = |w_1 S\delta_1 + w_2 T\delta_2|$ $(\delta_1 \delta_2)$ etc.

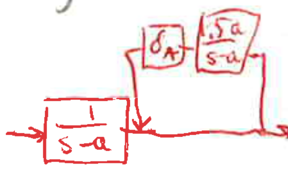
b) RS $\Leftarrow \min_D \bar{\sigma}(DMD^{-1}) < 1$. In this case $\Delta = \begin{pmatrix} \delta_a & 0 \\ 0 & \delta_b \end{pmatrix}$ so

$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ (Need $D\Delta D^{-1} = \Delta$)

But can set $d_2 = 1$ so we have only one parameter left for optimization.

This RS-condition would have been exact if δ_a and δ_b were complex (only two blocks), but this is unfortunately not the case so the RS-condition is not tight in this case.

3) Comment: we could have "moved" the inverse uncertainty to and we would get the more familiar M-matrix (see Problem 3d)



$M = \begin{pmatrix} w_2 S & w_2 gS \\ w_1 cS & w_1 T \end{pmatrix}$ where $w_1 = 0.2$, $w_2 = \frac{1.5a}{s-a}$

c) i) Condition RS $\Leftrightarrow \rho(M\Delta) < 1, \forall \omega$.

The condition is tight (necessary and sufficient) when we allow for a class of complex Δ, such that, if Δ' is an allowed perturbation, then so is $\Delta'' = c\Delta'$ where c is any complex number such that $|c| \le 1$. (In particular it is tight when Δ is any normalized complex matrix with $\|\Delta\| < r, \forall \omega$). ← ANY NORM. Δ may have any prespecified structure.

ii) To derive the μ-condition we use (this is essentially the defin. of μ)

$$\sup_{\Delta} \rho(M\Delta) = \mu_\Delta(M) \cdot \underbrace{\sup_{\Delta} \bar{\sigma}(\Delta)}_{1}$$

The definition of μ relies on

1) Δ has a block diagonal structure (or full matrix)
2) All complex Δ's such that $\|\Delta\|_\infty \le 1$ (H_∞-norm less than 1) are allowed.

$\Downarrow$ So the RS-condition $\mu_\Delta(M) < 1$ will be nec. and suff. when these two conditions are satisfied.

3 d, e, f, g): no soln. provided

$$RP \Leftrightarrow \mu_{(\Delta_I \Delta_P)}(N) < 1, \forall \omega$$

Assume Δ_I is full then

$$RP \Leftrightarrow \min_D \bar{\sigma}(DND^{-1}) < 1 \quad \text{where } D = \begin{pmatrix} dI & 0 \\ 0 & I \end{pmatrix}$$

$$\Leftrightarrow \min_d \bar{\sigma}\begin{pmatrix} \underbrace{w_I\, CG(I+CG)^{-1}}_{T_I} & d\, w_I\, \overbrace{C(I+GC)^{-1}}^{T_I G^{-1}} \\ \frac{1}{d} w_P \underbrace{G(I+CG)^{-1}}_{GS} & w_P \underbrace{(I+GC)^{-1}}_{S} \end{pmatrix}$$

← "select d to balance the off-diag elements"

Here $\bar{\sigma}\begin{pmatrix} A & B \\ C & D \end{pmatrix} \le \bar{\sigma}(A) + \bar{\sigma}(B) + \bar{\sigma}(C) + \bar{\sigma}(D)$ so:

$$\bar{\sigma}(DND^{-1}) \le \bar{\sigma}(w_I T_I) + \bar{\sigma}(w_P S) + d\,\bar{\sigma}(G^{-1})\,\bar{\sigma}(w_I T_I) + \frac{1}{d}\bar{\sigma}(G)\,\bar{\sigma}(w_P S)$$

Select $d = \bar{\sigma}(G)^{1/2} \cdot \underline{\sigma}(G)^{1/2}$ and note that $\bar{\sigma}(G^{-1}) = 1/\underline{\sigma}(G)$. We then get $d\,\bar{\sigma}(G^{-1}) = \frac{\bar{\sigma}(G)}{d} = \gamma(G)^{1/2}$

Then:

$$RP \Leftarrow \Big[\underbrace{\bar{\sigma}(W_I T_I)}_{RS-\text{cond.}} + \underbrace{\bar{\sigma}(W_P S)}_{NP-\text{cond.}}\Big]\Big[1+\sqrt{\gamma(G)}\Big] < 1 \quad, \forall \omega$$

(Can derive easily similar condition with $\gamma(G)$ replaced by $\gamma(C)$)

- This condition is most useful when Δ_I is a full matrix (also in this case it is of course conservative, for example, we get a factor 2 instead of 1 for SISO plants).
- If the input uncertainty, Δ_I, is diagonal, then the RGA (which is closely related to γ^* is a more useful measure)

e) – $\bar{\sigma}(AB) \neq \bar{\sigma}(BA)$ (In general!)

Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We have $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ with $\bar{\sigma}(AB) = 0$ whereas $BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ with $\bar{\sigma}(BA) = 1$.

– $\mu(AB) = \mu(BA)$?

Obviously, not true in general – se above example.

Let is here assume A and B are square. 2) Recall the fact that $\mu(DMD^{-1}) = \mu(M)$ provided $D\Delta = \Delta D$ ("D commutes with Δ"). Now introduce $A = MD^{-1}$ and we get

$$\mu(DA) = \mu(AD)$$

provided $D\Delta = \Delta D$. So we have proved the result that $\mu(AB) = \mu(BA)$ provided A or B commute with Δ. (Simple proof: $\det(I - AB\Delta) \overset{\text{ALWAYS}}{=} \det(I - B\Delta A) = \det(I - BA\Delta)$ provided $B\Delta = \Delta B$ or $\Delta A = A\Delta$). For example, if Δ is full then, B must be scalar times identity so we simply have

$$\bar{\sigma}(A \cdot b) = \bar{\sigma}(b \cdot A) \text{ where } b \text{ is scalar}$$

If $\Delta = \delta \cdot I$, then B may be a full matrix, and we have $\rho(AB) = \bar{\sigma}(BA)$ (for any B) which is well known.

1) Clearly holds when $AB = BA$ (A and B commute) but this is not very interesting

3) A or B are unitary with same structure as Δ (not very likely)

f) $\text{PRGA} = \tilde{G} G^{-1} = (G\tilde{G})^{-1} = \Gamma$, $\tilde{G} = \text{diag}\{G\}$

(Essentially, the PRGA is equal to the scaled inverse - this follows since $G\tilde{G}$ is the matrix with input scaling applied such that all the diagonal elements are 1).

Relationship to RGA: The diagonal elements are identical.

"Performance conditions" for decentralized control.

A bit imprecise. I was thinking about $\|w_P S\|_\infty < 1$ as a typical performance condition

We have

$$S = \tilde{S}(I + E\tilde{T})^{-1} \quad \text{where } E = (G - \tilde{G})\tilde{G}^{-1} = \Gamma^{-1} - I$$

Consider low frequencies where $\tilde{T} \approx I$. We get

$$S \approx \tilde{S}\Gamma$$

so the PRGA $\approx (I + E\tilde{T})^{-1}$ at frequencies where feedback is effective.

g)

	$\|A\|_{i1}$ "max column sum"	$\|A\|_{i2} = \bar{\sigma}(A) = \sqrt{\rho(A^H A)}$	$\|A\|_{i\infty}$ "max row sum"	$\|A\|_F = \sqrt{\sum \lvert a_{ij}\rvert^2}$	$\|A\|_{max}$
$A_1 = I$	1	1	1	$\sqrt{2}$	1
$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	1	1	1	1	1
$A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	2	2	2	2	1
$A_4 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$	1	$\sqrt{2}$	2	$\sqrt{2}$	1

Bounds: (have marked for which matrices we have equality)

$$\bar{\sigma}(A) \overset{A_2}{\leq} \|A\|_F \overset{A_1}{\leq} \sqrt{n}\,\bar{\sigma}(A)$$

$$\|A\|_{max} \overset{A_1}{\leq} \bar{\sigma}(A) \overset{A_3}{\leq} n \cdot \|A\|_{max}$$

$$\frac{1}{\sqrt{n}}\|A\|_{i1} \leq \bar{\sigma}(A) \underset{A_4}{\leq} \sqrt{n}\,\|A\|_{i1}$$

consider $A_5 = A_4^T$ for which $\bar{\sigma}(A_5) = \sqrt{2}$ and $\|A_5\|_{i1} = 2$

(Problem 3: I used about 1 hour)