SOLUTION MANUAL PART 1: ODD-NUMBERED EXERCISES
Multivariable feedback control - Analysis and design

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CLASSICAL FEEDBACK CONTROL

Exercise 2.1 *Use (2.31) to compute K_u and P_u for the process in (2.27).*

Note:

$$\boxed{G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)}}. \tag{2.27}$$

$$K_u = 1/|G(j\omega_u)|, \quad P_u = 2\pi/\omega_u \tag{2.31}$$

where ω_u is defined by $\angle G(j\omega_u) = -180°$.

Solution. According to (2.27), the following equation can be obtained:

$$\angle G(j\omega_u) = \tan^{-1}(-2\omega_u) - \tan^{-1}(5\omega_u) - \tan^{-1}(10\omega_u) = -180°$$

Hence,

$$\tan(180° + \tan^{-1}(-2\omega_u)) = \tan(\tan^{-1}(5\omega_u) + \tan^{-1}(10\omega_u))$$

leads to

$$-2\omega_u = \frac{5\omega_u + 10\omega_u}{1 - 50\omega_u^2}$$

which gives the solution of

$$\omega_u = \frac{\sqrt{17}}{10}$$

and

$$\begin{aligned} K_u &= \frac{1}{|G(j\frac{\sqrt{17}}{10})|} = \sqrt{\frac{(1+25\times 17/100)(1+100\times 17/100)}{9(1+4\times 17/100)}} = 2.5 \\ P_u &= \frac{2\pi}{\omega_u} = \frac{20\pi}{\sqrt{17}} = 15.23 \end{aligned}$$

Exercise 2.3 *Derive the approximation for* $K_u = 1/|G(j\omega_u)|$ *given in (5.73) for a first-order delay system.*

Note: A first-order delay system can be represented by $G(s) = ke^{-\theta s}/(1+\tau s)$, and

$$|G(j\omega_u)| \approx \frac{2}{\pi} k \frac{\theta}{\tau} \tag{5.73}$$

Solution. Assume $\tau > \theta$. Then $\angle(1 + j\omega_u\tau)^{-1} \approx -\theta\omega_u \approx \pi/2$. Hence, $\omega_u \approx (\pi/2)/\theta$ and

$$K_u = \frac{1}{|G(j\omega_u)|} \approx \frac{\tau\omega_u}{k} \approx \frac{\pi\tau}{2k\theta}$$

3

INTRODUCTION TO MULTIVARIABLE CONTROL

Exercise 3.1 *Derive the cascade and feedback rules.*

Solution.

Casecade rule As shown in Fig. 3.1(a), let the signal from G_1 to G_2 be x, then $z = G_2 x$ and $x = G_1 u$, thus, $z = G_2 G_1 u = Gu$, i.e. $G = G_1 G_2$.

Feedback rule As shown in Fig.3.1(b), we have, $v = u + z = u + G_2 y = u + G_2 G_1 v$, or, $(I - G_2 G_1)v = u$. Let $L = G_2 G_1$, we can obtain, $v = (I - L)^{-1} u$.

Exercise 3.3 *Use the MIMO rule to show that (2.18) corresponds to the negative feedback system in Figure 2.4.*

Solution. In Fig. 2.4, from point y the loop transfer function is KG and exiting the loop gives the term $(I + KG)^{-1}$. Therefore, the transfer function from r to y is $KG(I + KG)^{-1} = (I + KG)^{-1} KG$, from d to y is $(I + KG)^{-1} G_d$ and from n to y is $-(I + KG)^{-1} KG$. So (2.18) represents the negative feedback system in Fig. 2.4.

Exercise 3.5 *Compute the spectral radius and the five matrix norms mentioned above for the matrices in (3.27) and (3.30).*

Solution.

Matrix (3.27):

$$\begin{aligned}
G_1 &= \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \\
\rho(G_1) &= 7.2749 \\
\|G_1\|_F &= 7.3485 \\
\|G_1\|_{\text{sum}} &= 14 \\
\|G_1\|_{i1} &= 8 \\
\|G_1\|_{i\infty} &= 9 \\
\|G_1\|_{i2} &= 7.3434
\end{aligned}$$

Matrix (3.30):

$$
\begin{aligned}
G &= \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix} \\
\rho(G) &= 0 \\
\|G\|_F &= \|G\|_{\text{sum}} = \|G\|_{i1} = \|G\|_{i2} = \|G\|_{i\infty} = 100
\end{aligned}
$$

Exercise 3.7 *Design a SVD-controller $K = W_1 K_s W_2$ for the distillation process in (3.82), i.e. select $W_1 = V$ and $W_2 = U^T$ where U and V are given in (3.46). Select K_s in the form*

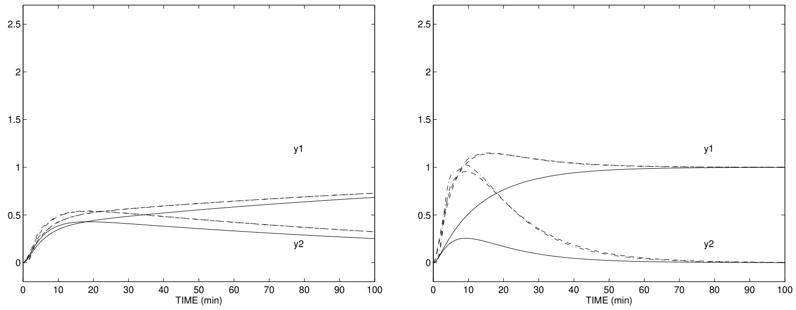
$$K_s = \begin{bmatrix} c_1 \frac{75s+1}{s} & 0 \\ 0 & c_2 \frac{75s+1}{s} \end{bmatrix}$$

and try the following values:

(a) $c_1 = c_2 = 0.005$;
(b) $c_1 = 0.005$, $c_2 = 0.05$;
(c) $c_1 = 0.7/197 = 0.0036$, $c_2 = 0.7/1.39 = 0.504$.

Simulate the closed-loop reference response with and without uncertainty. Designs (a) and (b) should be robust. Which has the best performance? Design (c) should give the response in Figure 3.12. In the simulations, include high-order plant dynamics by replacing $G(s)$ by $\frac{1}{(0.02s+1)^5}G(s)$. What is the condition number of the controller in the three cases? Discuss the results. (See also the conclusion on page 244).

Solution. The simulation results of designs (a) and (b) are shown in the following figures.



(a) Design (a) (b) Design (b)

Design (b) is the best if we want a fast settling time. By increasing c_2 we increase the loop gain in the weak direction.

The condition number of the controller: Design (a), $\gamma(K) = 1$; Design (b), $\gamma(K) = 10$ and Design (c), $\gamma(K) = 197/1.39 = 128.777$.

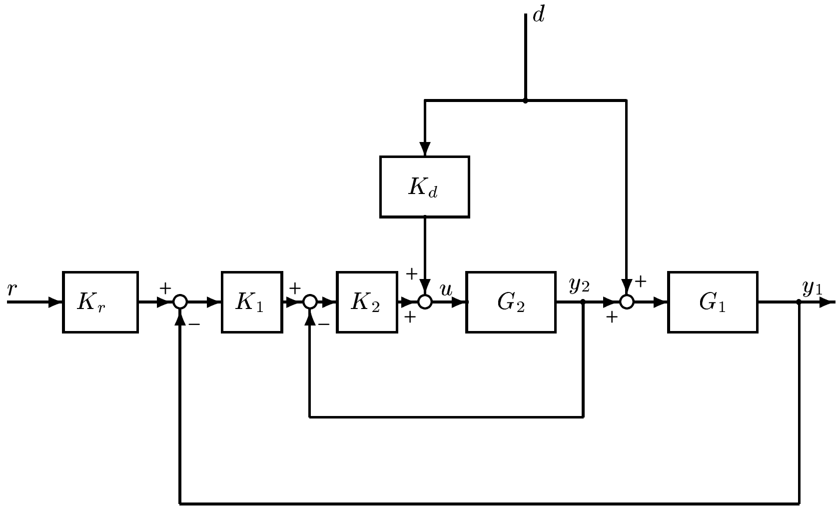
Exercise 3.9 *Consider again the distillation process $G(s)$ in (3.82). The response using the inverse-based controller $K_{\rm inv}$ in (3.84) was found to be sensitive to input gain errors. We want to see if the controller can be modified to yield a more robust system by using the Glover-McFarlane $\mathcal{H}_\infty$ loop-shaping procedure. To this effect, let the shaped plant be $G_s = GK_{\rm inv}$, i.e. $W_1 = K_{\rm inv}$, and design an $\mathcal{H}_\infty$ controller K_s for the shaped plant (see page 382 and Chapter 9), such that the overall controller becomes $K = K_{\rm inv}K_s$. (You will find that $\gamma_{min} = 1.414$ which indicates good robustness with respect to coprime factor uncertainty, but the loop shape is almost unchanged and the system remains sensitive to input uncertainty.)*

Solution. *The following MATLAB script could be used to do this exercise.*

```
G0 = [87.8 -86.4; 108.2 -109.6];
dyn = nd2sys(1,[75 1]); Dyn = daug(dyn,dyn);
G = mmult(Dyn,G0);

% Inverse-based controller
dynk = nd2sys([75 1],[1 1.e-6],0.7);
Dynk = daug(dynk,dynk);
Kinv = mmult(Dynk,minv(G0));

% Try to robustify with respect to coprime uncerainty
Gs = mmult(G,Kinv);
[a,b,c,d]=unpck(Gs);
gamrel=1.1;
% gammin=1.4142:
[Ac,Bc,Cc,Dc,gammin]=coprim(a,b,c,d,gamrel);
% Change from positive to negative feedback:
Ks=pck(Ac,-Bc,Cc,-Dc);
K = mmult(Kinv,Ks);

% TIME  simulation
% Nominal
GK = mmult(G,K); I2=eye(2); S = minv(madd(I2,GK));
T = msub(I2,S);
kr=nd2sys(1,[5 1]); Kr=daug(kr,kr); Tr = mmult(T,Kr);
y = trsp(Tr,[1;0],100,.1);
u = trsp(mmult(K,S,Kr),[1;0],100,.1);
% With 20% uncertainty
Unc = [1.2 0; 0 0.8]; GKu = mmult(G,Unc,K);
Su = minv(madd(I2,GKu)); Tu = msub(I2,Su);
Tru = mmult(Tu,Kr);
yu = trsp(Tru,[1;0],100,.1);
uu = trsp(mmult(K,Su,Kr),[1;0],100,.1);
subplot(211);vplot(y,yu,'--');title('OUTPUTS')
subplot(212);vplot(u,uu,'--');title('INPUTS');
xlabel('TIME (min)');
```

Exercise 3.11 Cascade implementation. Consider further Example 3.16. The local feedback based on y_2 is often implemented in a cascade manner; see also Figure 10.4. In this case the output from K_1 enters into K_2 and it may be viewed as a reference signal for y_2. Derive the generalized controller K and the generalized plant P in this case.

Solution. *The casecade structure is shown as follows.*



Since $u = K_2K_1K_rr - K_2K_1y_1 - K_2y_2 + K_dd$, *we get* $K = [\, K_2K_1K_r \quad -K_2K_1 \quad -K2 \quad K_d \,]$. *Also, we can get*

$$\begin{bmatrix} y_1 - r \\ r \\ y_1 \\ y_2 \\ d \end{bmatrix} = \begin{bmatrix} G_1 & -I & G_1G_2 \\ 0 & I & 0 \\ G_1 & 0 & G_1G_2 \\ 0 & 0 & G_2 \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} d \\ r \\ u \end{bmatrix}.$$

Thus,

$$P = \begin{bmatrix} G_1 & -I & G_1G_2 \\ 0 & I & 0 \\ G_1 & 0 & G_1G_2 \\ 0 & 0 & G_2 \\ I & 0 & 0 \end{bmatrix}.$$

Exercise 3.13 Mixed sensitivity. Use the above procedure (P. 107) to derive the generalized plant P for the stacked N in (3.94).

Solution. *In (3.94),*

$$N = \begin{bmatrix} W_uKS \\ W_TT \\ W_PS \end{bmatrix}$$

.

1. Let $K = 0$ in N, we get,

$$P_{11} = N(K=0) = \begin{bmatrix} 0 \\ 0 \\ W_P I \end{bmatrix}.$$

2. From

$$N - P_{11}$$

, we get,

$$Q = N - P_{11} = \begin{bmatrix} W_u KS \\ W_T T \\ W_P(S-I) \end{bmatrix} = \begin{bmatrix} W_u K(I+GK)^{-1} \\ W_T GK(I+GK)^{-1} \\ -W_P GK(I+GH)^{-1} \end{bmatrix}.$$

The common factor is $R = K(I+GK)^{-1}$. Thus, $G_{22} = -G$.

3. Since $Q = P_{12}RP_{21}$, we have,

$$P_{12} = \begin{bmatrix} W_u \\ W_T G \\ -W_P G \end{bmatrix},$$

and $P_{21} = I$.

As conclusion, we get,

$$P = \begin{bmatrix} 0 & W_u \\ 0 & W_T G \\ W_P I & -W_P G \\ I & -G \end{bmatrix}$$

Exercise 3.15 Consider the performance specification $\|w_P S\|_\infty < 1$. Suggest a rational transfer function weight $w_P(s)$ and sketch it as a function of frequency for the following two cases:

1. We desire no steady-state offset, a bandwidth better than 1 rad/s and a resonance peak (worst amplification caused by feedback) lower than 1.5.
2. We desire less than 1% steady-state offset, less than 10% error up to frequency 3 rad/s, a bandwidth better than 10 rad/s, and a resonance peak lower than 2. Hint: See (2.72) and (2.73).

Solution. $w_{P1} = \frac{s+1.5}{1.5s}$*, and* $w_{P2} = \frac{1}{2} \cdot \frac{(s+14.1)^2}{(s+1)^2}$*. Comment: In the latter case we require that the magnitude should increase by a factor 10 when the frequency increases by approximately a factor* $10^{1/2}$ *(from 3 rad/s to 10 rad/s).*

Exercise 3.17 What is the relationship between the RGA-matrix and uncertainty in the individual elements? Illustrate this for perturbations in the 1, 1-element of the matrix

$$A = \begin{bmatrix} 10 & 9 \\ 9 & 8 \end{bmatrix} \qquad (3.114)$$

Solution. $\mathrm{RGA}(A) = \begin{bmatrix} -80 & 81 \\ 81 & -80 \end{bmatrix}$*. RGA-matrix has 1,1-element of -80, so G becomes singular if 1,1-element is perturbed from 10 to 10(1+1/80)=10.125.*

Exercise 3.19 Compute $\|A\|_{i1}, \bar{\sigma}(A) = \|A\|_{i2}, \|A\|_{i\infty}, \|A\|_F, \|A\|_{\max}$ and $\|A\|_{\text{sum}}$ for the following matrices and tabulate your results:

$$A_1 = I; \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; A_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_5 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Show using the above matrices that the following bounds are tight (i.e. we may have equality) for 2×2 matrices ($m = 2$):

$$\bar{\sigma}(A) \leq \|A\|_F \leq \sqrt{m}\,\bar{\sigma}(A)$$

$$\|A\|_{\max} \leq \bar{\sigma}(A) \leq m\|A\|_{\max}$$
$$\|A\|_{i1}/\sqrt{m} \leq \bar{\sigma}(A) \leq \sqrt{m}\|A\|_{i1}$$
$$\|A\|_{i\infty}/\sqrt{m} \leq \bar{\sigma}(A) \leq \sqrt{m}\|A\|_{i\infty}$$
$$\|A\|_F \leq \|A\|_{\text{sum}}$$

Solution. *The result table is as follows:*

A	$\|A\|_{i1}$	$\|A\|_{i2}$	$\|A\|_{i\infty}$	$\|A\|_F$	$\|A\|_{\max}$	$\|A\|_{\text{sum}}$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	1	1	1.4142	1	2
$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	1	1	1	1	1	1
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	2	2	2	2	1	4
$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$	1	1.4142	2	1.4142	1	2
$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$	2	1.4142	1	1.4142	1	2

From the table, the above bounds can be checked (omitted).

Exercise 3.21 Do the extreme singular values bound the magnitudes of the elements of a matrix? That is, is $\bar{\sigma}(A)$ greater than the largest element (in magnitude), and is $\underline{\sigma}(A)$ smaller than the smallest element? For a non-singular matrix, how is $\underline{\sigma}(A)$ related to the largest element in A^{-1}?

Solution. *The answer for the first question is "yes", because* $\|A\|_{\max} \leq \bar{\sigma}(A)$. *But for the second question, the answer is "no". For a non-singular matrix,* $\underline{\sigma}(A)$ *is smaller than inverse of the largest element in* A^{-1}, *because* $\underline{\sigma}(A) = 1/\bar{\sigma}(A^{-1}) \leq 1/\|A^{-1}\|_{\max}$.

Exercise 3.23 Find two matrices A and B such that $\rho(A + B) > \rho(A) + \rho(B)$ which proves that the spectral radius does not satisfy the triangle inequality and is thus not a norm.

Solution. *Let* $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, *and* $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. $\rho(A) = \rho(B) = 1$, *but* $\rho(A + B) = 3 > \rho(A) + \rho(B) = 2$.

Exercise 3.25 Write K as an LFT of $T = GK(I+GK)^{-1}$, i.e. find J such that $K = F_l(J,T)$.

Solution. $T = GK(I+GK)^{-1}$ *gives* T *as an LFT of* K*, but we want the inverse. We get* $K = G^{-1}T(I-T)^{-1}$*, so*

$$J = \begin{bmatrix} 0 & G^{-1} \\ I & I \end{bmatrix}.$$

Exercise 3.27 Show that the set of all stabilizing controllers in (4.91) can be written as $K = F_l(J,Q)$ and find J.

Solution. *In (4.91),* $K = (V_r - QN_l)^{-1}(U_r + QM_l)$.

$$\begin{aligned} K &= (V_r - QN_l)^{-1}(U_r + QM_l) \\ &= V_r^{-1}U_r + (V_r - QN_l)^{-1}Q(M_l + N_lV_r^{-1}U_r) \\ &= V_r^{-1}U_r + V_r^{-1}(I - QN_lV_r^{-1})^{-1}Q(M_l + N_lV_r^{-1}U_r) \\ &= V_r^{-1}U_r + V_r^{-1}Q(I - N_lV_r^{-1}Q)^{-1}(M_l + N_lV_r^{-1}U_r) \end{aligned}$$

We get

$$J = \begin{bmatrix} V_r^{-1}U_r & V_r^{-1} \\ M_l + N_lV_r^{-1}U_r & N_lV_r^{-1} \end{bmatrix}$$

4

ELEMENTS OF LINEAR SYSTEM THEORY

Exercise 4.1 We want to find the normalized coprime factorization for the scalar system in (4.22). Let N and M be as given in (4.23), and substitute them into (4.24). Show that after some algebra and comparing of terms one obtains: $k = \pm 0.71$, $k_1 = 5.67$ and $k_2 = 8.6$.

Solution. *Since $M^*(s) = M(-s)$, (4.24) can be written as $M(-s)M(s) + N(-s)N(s) = 1$. Thus:*

$$\frac{k^2(2s^4 - 30s^2 + 148)}{s^4 - (k_1^2 - 2k_2)s^2 + k_2^2} = 1.$$

By comparing the terms of the denominator and numerator, it can be obtained that:

$$\begin{aligned} 2k^2 = 1 \quad &\Longrightarrow \quad k = \pm 0.71, \\ 148k^2 = k_2^2 \quad &\Longrightarrow \quad k_2 = 8.60, \\ 30k^2 = k_1^2 - 2k_2 \quad &\Longrightarrow \quad k_1 = 5.67. \end{aligned}$$

Exercise 4.3 Consider a SISO system $G(s) = C(sI - A)^{-1}B + D$ with just one state, i.e. A is a scalar. Find the zeros. Does $G(s)$ have any zeros for $D = 0$?

Solution. *Zero at $z = A - (CB)/D$. When $D \to 0$ the zero moves to infinity, i.e. no zero.*

Exercise 4.5 Given $y(s) = G(s)u(s)$, with $G(s) = \frac{1-s}{1+s}$, determine a state-space realization of $G(s)$ and then find the zeros of $G(s)$ using the generalized eigenvalue problem. What is the transfer function from $u(s)$ to $x(s)$, the single state of $G(s)$, and what are the zeros of this transfer function?

Solution. *A state-space realization of $G(s)$ is*

$$\begin{aligned} \dot{x} &= -x + 2u \\ ,y &= x - u, \end{aligned}$$

and

$$P(s) = \begin{bmatrix} s+1 & -2 \\ 1 & -1 \end{bmatrix}.$$

Thus, zero at $s = 1$. From $u(s)$ to $x(s)$, the polynomial system matrix is:

$$P(s) = \begin{bmatrix} s+1 & -2 \\ 1 & 0 \end{bmatrix},$$

and the corresponding transfer function is $G(s) = \frac{s}{1+s}$. Thus, there are no zeros in this transfer function.

Exercise 4.7 For what values of c_1 does the following plant have RHP-zeros?

$$A = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = I, \quad C = \begin{bmatrix} 10 & c_1 \\ 10 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.1}$$

Solution.

$$P(s) = \begin{bmatrix} s-10 & 0 & -1 & 0 \\ 0 & s+1 & 0 & -1 \\ 10 & c_1 & 0 & 0 \\ 10 & 0 & 0 & 1 \end{bmatrix}$$

If $s = c_1 - 1$, the sum of the 2nd and 4th rows of $P(s)$ is equal to the 3rd row. Thus $z = c_1 - 1$. So the plant has RHP-zero if $c_1 > 1$.

Exercise 4.9 Use (A.7) to show that $M(s)$ in (4.82) is identical to the block transfer function matrix implied by (4.80) and (4.81).

Solution. *Using (A.7) M can be derived from (4.82) as:*

$$\begin{aligned} M &= \begin{bmatrix} I & K \\ -G & I \end{bmatrix} \\ &= \begin{bmatrix} I + KX^{-1}(-G) & -KX^{-1} \\ -X^{-1}(-G) & X^{-1} \end{bmatrix}. \end{aligned}$$

Since $X = I + GK$ and noting $(I+GK)^{-1}G = G(I+KG)^{-1}$ and $I - KG(I+KG)^{-1} = (I+KG)^{-1}$,

$$\begin{aligned} M &= \begin{bmatrix} I - K(I+GK)^{-1}G & -K(I+GK)^{-1} \\ (I+GK)^{-1}G & (I+GK)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (I+KG)^{-1} & -K(I+GK)^{-1} \\ G(I+KG)^{-1} & (I+GK)^{-1} \end{bmatrix}. \end{aligned}$$

This is the block transfer function matrix implied by (4.80)and (4.81).

Exercise 4.11 Given the complementary sensitivity functions

$$T_1(s) = \frac{2s+1}{s^2+0.8s+1} \quad T_2(s) = \frac{-2s+1}{s^2+0.8s+1}$$

what can you say about possible RHP-poles or RHP-zeros in the corresponding loop transfer functions, $L_1(s)$ and $L_2(s)$?

Solution. *Assume internal stability. Then L_1 has a RHP-pole and L_2 has a RHP-zero. Compute $S = 1 - T$. L_1 is unstable with pole at $s = 1.2$ (close to bandwidth) and L_2 has RHP-zero at $s = 0.5$ (also close to bandwidth).*

Exercise 4.13 Show that the IMC-structure in Figure 4.5 is internally unstable if either Q or G is unstable.

Solution. *In Figure 4.5, it can be derived that $u = K(r - y)$ where $K = (I - QG)^{-1}Q$, i.e. equation (4.89). Thus, $Q = K(I + GK)^{-1}$. According to Lemma 4.6 if either Q or G is unstable then the IMC system is internally unstable.*

Exercise 4.15 Given a stable controller K. What set of plants can be stabilized by this controller? (Hint: interchange the roles of plant and controller.)

Solution. *The set of the plants can be parameterized as:*

$$G = (I - QK)^{-1}Q = Q(I - KQ)^{-1},$$

where "parameter" Q is any stable transfer function matrix.

5

LIMITATIONS ON PERFORMANCE IN SISO SYSTEMS

Exercise 5.1 Kalman inequality The Kalman inequality for optimal state feedback, which also applies to unstable plants, says that $|S| \leq 1\ \forall\omega$, see Example 9.1. Explain why this does not conflict with the above sensitivity integrals.

Solution. *1. Optimal control with state feedback yields a loop transfer function with a pole-zero excess of one so (5.6) does not apply.*
2. There are no RHP-zeros when all states are measured so (5.9) does not apply.

Exercise 5.3 Consider the weight $w_P(s) = \frac{1}{M} + (\frac{\omega_B^*}{s})^n$ which requires $|S|$ to have a slope of n at low frequencies and requires its low-frequency asymptote to cross 1 at a frequency ω_B^*. Note that $n = 1$ yields the weight (5.32) with $A = 0$. Derive an upper bound on ω_B^* when the plant has a RHP-zero at z. Show that the bound becomes $\omega_B^* \leq |z|$ as $n \to \infty$.

Solution. *When $z > 0$ is real, from $|w_P(z)| < 1$ we have:*

$$w_B^* < z(1 - 1/M)^{1/n}.$$

When z is imaginary, we get:

$$\omega_B^* < \begin{cases} |z|\left(1 - \frac{1}{M^2}\right)^{1/n} & n = 2k-1,\ k = 1, 2, \ldots \\ |z|\left(1 - \frac{(-1)^k}{M}\right)^{1/n} & n = 2k,\ k = 1, 2, \ldots \end{cases}$$

In all these cases, $\omega_B^* < |z|$ as $n \to \infty$.

Exercise 5.5 Consider the case of a plant with a RHP-zero where we want to limit the sensitivity function over some frequency range. To this effect let

$$w_P(s) = \frac{(1000s/\omega_B^* + \frac{1}{M})(s/(M\omega_B^*) + 1)}{(10s/\omega_B^* + 1)(100s/\omega_B^* + 1)} \tag{5.1}$$

This weight is equal to $1/M$ at low and high frequencies, has a maximum value of about $10/M$ at intermediate frequencies, and the asymptote crosses 1 at frequencies $\omega_B^*/1000$ and ω_B^*.

Thus we require "tight" control, $|S| < 1$, in the frequency range between $\omega_{BL}^* = \omega_B^*/1000$ and $\omega_{BH}^* = \omega_B^*$.

a) Make a sketch of $1/|w_P|$ (which provides an upper bound on $|S|$).

b) Show that the RHP-zero cannot be in the frequency range where we require tight control, and that we can achieve tight control either at frequencies below about $z/2$ (the usual case) or above about $2z$. To see this select $M = 2$ and evaluate $w_P(z)$ for various values of $\omega_B^* = kz$, e.g. $k = .1, .5, 1, 10, 100, 1000, 2000, 10000$. (You will find that $w_P(z) = 0.95$ (≈ 1) for $k = 0.5$ (corresponding to the requirement $\omega_{BH}^* < z/2$) and for $k = 2000$ (corresponding to the requirement $\omega_{BL}^* > 2z$))

Solution. *a)The asymptotic magnitude Bode-plot is shown as follows: b) In the above plot*

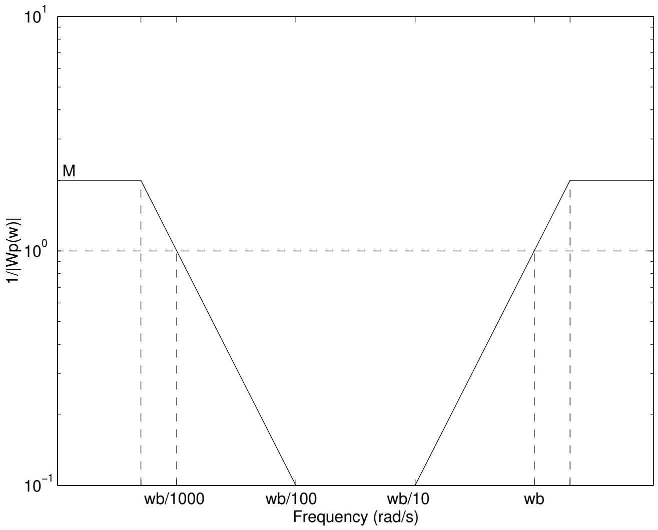


Figure for exercise 5.5

betweem $\omega_B^*/1000$ and ω_B^* $1/w_P(\omega) \leq 1$. Hence to satisfy the condition $|w_P(z)| < 1$, z cannot be in the frequency range from $\omega_B^*/1000$ to ω_B^* where we require tight control.

The values of $w_P(z)$ via $k = \omega_B^*/z$ for $M = 2$ are shown in the following table:

k	$w_P(z)$
0.1	*0.59350*
0.5	*0.94788*
1.0	*1.3508*
10.0	*4.7966*
100.0	*4.7966*
1000.0	*1.3508*
2000.0	*0.94788*
10000.0	*0.59350*

Exercise 5.7 Derive the bound in (5.47).

Solution. $|w_T(p)| = \sqrt{\frac{|p|^2}{\omega_{BT}^{*2}} + \frac{1}{M_T^2}} < 1 \Rightarrow \omega_{BT}^* > \frac{|p| M_T}{\sqrt{M_T^2 - 1}} = 1.1547|p|$.

Exercise 5.9 (a) The effect of a concentration disturbance must be reduced by a factor of 100 at the frequency 0.5 rad/min. The disturbances should be dampened by use of buffer tanks and the objective is to minimize the total volume. How many tanks in series should one have? What is the total residence time?

(b) The feed to a distillation column has large variations in concentration and the use of one buffer tank is suggested to dampen these. The effect of the feed concentration d on the product composition y is given by (scaled variables, time in minutes)

$$G_d(s) = e^{-s}/3s$$

That is, after a step in d the output y will, after an initial delay of 1 min, increase in a ramp-like fashion and reach its maximum allowed value (which is 1) after another 3 minutes. Feedback control should be used and there is an additional measurement delay of 5 minutes. What should be the residence time in the tank?

(c) Show that in terms of minimizing the total volume for buffer tanks in series, it is optimal to have buffer tanks of equal size.

(d) Is there any reason to have buffer tanks in parallel (they must not be of equal size because then one may simply combine them)?

(e) What about parallel pipes in series (pure delay). Is this a good idea?

Solution.

(a) From (5.93) we must require

$$|h_n(j\omega_0)| = \frac{1}{(\frac{\tau_n}{n}\omega_0)^2 + 1} = 0.01$$

This gives

$$\tau_n = \frac{n}{\omega_0}\sqrt{100^{2/n} - 1}$$

With $\omega_0 = 0.5$ rad/min we find

n	τ_n
1	200
2	39.80
3	27.20
4	24.00
5	23.04
6	22.90
7	23.12
10	24.59

and we find that to minimize τ_n we should have 6 tanks in series with a total residence time $\tau_n = 22.90$ min. (In practice, we would prefer fewer tanks because the number of tanks increases the cost, and probably use only 2 or at most 3 tanks).

(b) Assume the transfer function of the tank is

$$h(s) = \frac{1}{\tau_h s + 1}$$

. The total delay in the feedback loop is $\theta = 1+5 = 6$ [min]. To make the system controllable, we need

$$|G_d(j/\theta) \cdot h(j/\theta)| \leq 1.$$

This leads to $\tau_h \geq 10.39$ [min].

(c) Consider the case with two buffer tanks with total residence time τ_h. Let the residence time in one of the tanks be x. Then

$$h_2(s) = \frac{1}{(1+xs)(1+(\tau_h - x)s)} = \frac{1}{1+\tau_h s + (\tau_h - x)xs^2}$$

and the high-frequency asymptote becomes $|h_2(j\omega)| \approx 1/[(\tau_h - x)x\omega^2]$ which is minimized by selecting $x = \tau_h/2$, that is, the tanks should be of equal size to get the best disturbance attentuation with a given total volume. For n tanks in series we get the same result by considering the high-frequency asymptote.

(d) It seems no any such reasons.

(e) It certainly may dampen disturbances, but probably tanks are better.

are also used in chemical processes to dampen

Exercise 5.11 What information about a plant is important for controller design, and in particular, in which frequency range is it important to know the model well? To answer this problem you may think about the following sub-problems:

(a) Explain what information about the plant is used for Ziegler-Nichols tuning of a SISO PID-controller.

(b) Is the steady-state plant gain $G(0)$ important for controller design? (As an example consider the plant $G(s) = \frac{1}{s+a}$ with $|a| \leq 1$ and design a P-controller $K(s) = K_c$ such that $\omega_c = 100$. How does the controller design and the closed-loop response depend on the steady-state gain $G(0) = 1/a$?)

Solution. *(a) ω_u and $|G(\omega_u)|$, that is, only information at frequency w_u is used.*

(b) As just noted the steady-state information is not needed for Ziegler-Nichols tuning, and it is generally not important for feedback design. This is illustrated by the example where $K_c = \sqrt{\omega_c^2 + a^2} = \sqrt{10000 + a^2} \approx 100$ for any value of $a \leq 1$. Thus response hardly effected by a. Main effect is on steady-state offset which is $\frac{1}{1+K_c/a} \approx a/K_c$ and may vary between 0 (for $a = 0$) and 0.01 (for $a = 1$).

Exercise 5.13 A heat exchanger is used to exchange heat between two streams; a coolant with flowrate q (1 ± 1 kg/s) is used to cool a hot stream with inlet temperature T_0 ($100 \pm 10°$C) to the outlet temperature T (which should be $60 \pm 10°$C). The measurement delay for T is 3s. The main disturbance is on T_0. The following model in terms of deviation variables is derived from heat balances

$$T(s) = \frac{8}{(60s+1)(12s+1)}q(s) + \frac{0.6(20s+1)}{(60s+1)(12s+1)}T_0(s) \tag{5.2}$$

where T and T_0 are in °C, q is in kg/s, and the unit for time is seconds. Derive the scaled model. Is the plant controllable with feedback control? (Solution: The delay poses no problem (performance), but the effect of the disturbance is a bit too large at high frequencies (input saturation), so the plant is not controllable).

Solution. *Since* $|G_d(j\omega)| < 1$, *especially* $|G_d(j\frac{1}{\theta})| < 1$, *the delay poses no problem. But input constraint does! Let* $y = T/10$, $u = q/1$ *and* $d = T_0/10$, *then the scaled model is:*

$$y(s) = \frac{0.8}{(60s+1)(12s+1)}u(s) + \frac{0.6(20s+1)}{(60s+1)(12s+1)}d(s)$$

So, $|G|/|G_d| = \frac{0.8}{0.6\sqrt{400\omega^2+1}} < 1$, *when* $\omega > \frac{\sqrt{(0.8/0.6)^2-1}}{20} = 0.0441$, *i.e. at high frequencies the disturbance is too large which will cause input saturation. Thus the plant is not controllable.*

6

LIMITATIONS ON PERFORMANCE IN MIMO SYSTEMS

Exercise 6.1 To further illustrate the above arguments, determine the sensitivity function S for the plant (6.20) using a simple diagonal controller $K = \frac{k}{s}I$. Use the approximation $e^{-\theta s} \approx 1 - \theta s$ to show that at low frequencies the elements of $S(s)$ are of magnitude $k/(2k+\theta)$. How large must k be to have acceptable performance (less than 10% offset at low frequencies)? What is the corresponding bandwidth? (Answer: Need $k > 8/\theta$. Bandwidth is equal to k.)

Solution. *Using $S = (I + GK)^{-1}$ and $e^{-\theta s} \approx 1 - \theta s$, it can be obtained that*

$$S = \frac{1}{s + k(2 + \theta k)} \begin{bmatrix} s + k & -k \\ -k(1 - \theta s) & s + k \end{bmatrix}$$

Thus at low frequencies ($s \approx 0$), the elements of S all have the magnitude of $1/(2+\theta k)$. (Not $k/(2k + \theta)$!) To maintain 10% offset at low frequencies, let $1/(2 + \theta k) < 1/10$. It leads to $k > 8/\theta$. The bandwidth is obtained by letting:

$$|\frac{-k(1 - j\theta\omega)}{j\omega + k(2 + \theta k)}| = 1.$$

This gives $k^2 + k^2\theta^2\omega^2 = \omega^2 + k^2(2+\theta k)^2$. Since $k > 8/\theta$, $\theta k \gg 2$, i.e. $k^2\theta^2 \approx (2+\theta k)^2$. Thus the equation for bandwidth can be simplified as $k^2 = \omega^2$, i.e. the bandwidth is equal to k.

Exercise 6.3 Consider the plant

$$G(s) = \begin{bmatrix} \alpha & 1 \\ \frac{1}{s+1} & \alpha \end{bmatrix} \tag{6.25}$$

(a) Find the zero and its output direction.
(b) Which values of α yield a RHP-zero, and which of these values is best/worst in terms of achievable performance?
(c) Suppose $\alpha = 0.1$. Which output is the most difficult to control? Illustrate your conclusion using Theorem 6.4.

Solution. *(a) $z = \frac{1}{\alpha^2} - 1$ because $\mathrm{Rank}G(z) = 1$. The output zero direction is obtained from $y_z^H G(z) = 0$. This gives $y_z = [\,-\alpha \quad 1\,]^T$.*
(b) We have a RHP-zero for $|\alpha| < 1$. Best for $\alpha = 0$ with zero at infinity; if control at steady-state is required then worst for $\alpha = 1$ with zero at $s = 0$.
(c) Output 2 is most difficult since the zero is mainly in that direction; we get strong interaction with $\beta = 20$ in (6.24) if we want to control y_2 perfectly.

Exercise 6.5 Show that the disturbance condition number may be interpreted as the ratio between the actual input for disturbance rejection and the input that would be needed if the same disturbance was aligned with the "best" plant direction.

Solution. *The actual input for disturbance rejection can be expressed as $u = -G^\dagger g_d d$. If the same disturbance is aligned with the "best" plant direction, then $G^\dagger g_d = \frac{\|g_d\|_2}{\overline{\sigma}(G)} v_1$ where v_1 is the singular vector corresponding to $\overline{\sigma}(G)$. Thus the radio of input norm is $\|G^\dagger g_d\| \cdot \overline{\sigma}(G)/\|g_d\|_2$ which is equal to the disturbance condition number in (6.31).*

Exercise 6.7 Analyze input-output controllability for

$$G(s) = \frac{1}{s^2+100}\begin{bmatrix} \frac{1}{0.01s+1} & 1 \\ \frac{s+0.1}{s+1} & 1 \end{bmatrix}$$

Compute the zeros and poles, plot the RGA as a function of frequency, etc.

Solution. *The system has 6 poles: $\pm j10$, $\pm j10$, 1 and 100; 2 zeros: -9.537 and 9.437 (RHP-zero). The diagonal elements of the RGA are: $\lambda_{11} = \lambda_{22} = (1 - \frac{g_{12}g_{21}}{g_{11}g_{22}})^{-1}$ and off-diagonal elements of the RGA are: $\lambda_{12} = \lambda_{21} = 1 - \lambda_{11}$. The amplitudes of these elements are shown as functions of frequency in Fig. 6.7(a). It is shown in the figure that all elements of the RGA*

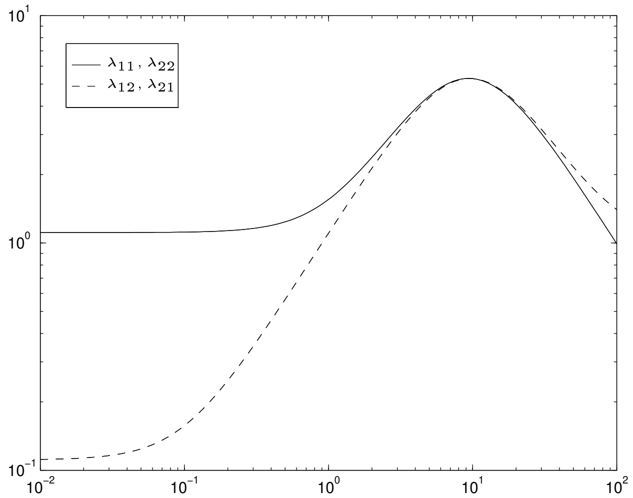


Fig. 6.7(a) The RGA of exercise 6.7

have a peak about 6 at $\omega = 10$. The singular values of the system are shown in Fig. 6.7(b). It is shown that the sv is less than 1 at all frequencies. Thus it is difficult to control this plant.

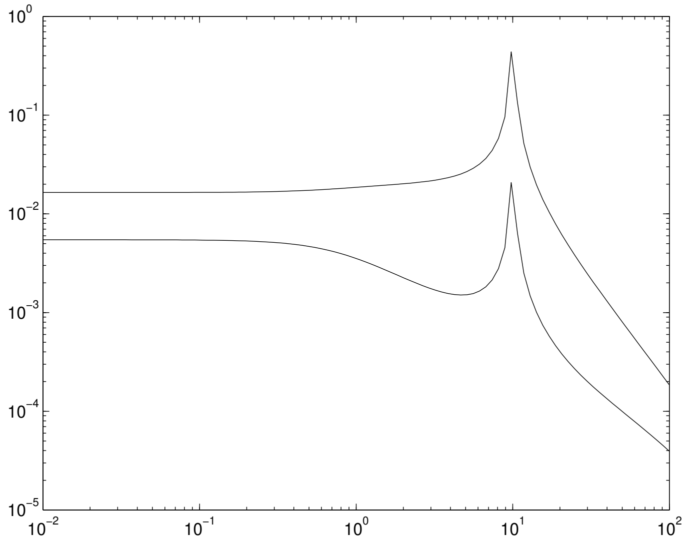


Fig. 6.7(b) The SV of exercise 6.7

Exercise 6.9 Let

$$A = \begin{bmatrix} -10 & 0 \\ 0 & -1 \end{bmatrix}, B = I, C = \begin{bmatrix} 10 & 1.1 \\ 10 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) Perform a controllability analysis of $G(s)$.
(b) Let $\dot{x} = Ax + Bu + d$ and consider a unit disturbance $d = [\, z_1 \quad z_2 \,]^T$. Which direction (value of z_1/z_2) gives a disturbance that is most difficult to reject (consider both RHP-zeros and input saturation)?
(c) Discuss decentralized control of the plant. How would you pair the variables?

Solution. *(a) The transfer function matrix is:*

$$G(s) = \begin{bmatrix} \frac{10}{s+10} & \frac{1.1}{s+1} \\ \frac{10}{s+10} & 1 \end{bmatrix}.$$

The system has two stable poles: -10 and -1 and one RHP-zero: $z = 0.1$. The RGA matrix is:

$$\mathrm{RGA} = \begin{bmatrix} \frac{10(s+1)}{10s-1} & \frac{-11}{10s-1} \\ \frac{-11}{10s-1} & \frac{10(s+1)}{10s-1} \end{bmatrix}.$$

Thus, the magnitudes of the RGA elements are large (about 10) at low frequencies and small at high frequency (approximating to a unit matrix). But the approximation is at $\omega \gg 1 \gg z$, i.e. "outside" the bandwidth which is limited by the RHP-zero, $z = 0.1$.

(b) At steady-state, $G(0) = \begin{bmatrix} 1 & 1.1 \\ 1 & 1 \end{bmatrix}$ and the most difficult output direction is $\underline{u}(0) = [\, -0.689 \quad 0.725 \,]^T$. Let $d = [\, 1 \quad k \,]^T$. Then, $g_d(0) = -BA^{-1}d = [\, 1 + 1.1k \quad 1 \,]^T$. So,

in the most difficult direction, $g_d(0)/\|g_d(0)\| = \underline{u}(0)$*, i.e.* $1 + 1.1k = -0.689/0.725$*. This gives* $k = -1.77$*.*
(c) At the steady state,

$$\mathrm{RGA}(0) = \begin{bmatrix} -10 & 11 \\ 11 & -10 \end{bmatrix}.$$

Thus the best pairing for decentralized control is: (y_1*,* u_2*) and (*y_2*,* u_1*).*

Exercise 6.11 Order the following three plants in terms of their expected ease of controllability

$$G_1(s) = \begin{bmatrix} 100 & 95 \\ 100 & 100 \end{bmatrix}, G_2(s) = \begin{bmatrix} 100e^{-s} & 95e^{-s} \\ 100 & 100 \end{bmatrix}, G_3(s) = \begin{bmatrix} 100 & 95e^{-s} \\ 100 & 100 \end{bmatrix}$$

Remember to also consider the sensitivity to input gain uncertainty.

Solution. G_1 *and* G_2 *have the same RGA matrix which is constant with* $\lambda_{11} = \lambda_{22} = 20$ *and* $\lambda_{12} = \lambda_{21} = 19$*. These large RGA values indicate both* G_1 *and* G_2 *are difficult to be controlled. Additionally* G_2 *has a delay in output 1 thus is worse than* G_1*. With the approximation of* $e^{-s} \approx 1 - s$ *the RGA of* G_3 *can be expressed as:* $\lambda_{11} = \lambda_{22} = \frac{100}{5+95s}$ *and* $\lambda_{11} = \lambda_{22} = \frac{95(s-1)}{5+95s}$*. When* $\omega \approx 1$ *the system is well-conditioned (the RGA*$\approx I$*). So* G_3 *is the best one, then* G_1 *and* G_2 *is the worst.*

Exercise 6.13 Analyze input-output controllability for

$$G(s) = \begin{bmatrix} 100 & 102 \\ 100 & 100 \end{bmatrix}, \quad g_{d1}(s) = \begin{bmatrix} \frac{10}{s+1} \\ \frac{10}{s+1} \end{bmatrix}; \quad g_{d2} = \begin{bmatrix} \frac{1}{s+1} \\ \frac{-1}{s+1} \end{bmatrix}$$

Which disturbance is the worst?

Solution. *This is a ill-conditioned plant. The RGA,* $\lambda_{11} = -50$ *at all frequencies. Disturbance* d_1 *is in the same direction of* u_1*, thus is easy to be rejected. The disturbance condition numbers for the two disturbances are,* $\gamma_{d1} = 1.4213$ *and* $\gamma_{d2} = 202.01$*. Hence,* d_2 *is more difficult to be rejected than* d_1*.*

Exercise 6.15 Find the poles and zeros and analyze input-output controllability for

$$G(s) = \begin{bmatrix} c + (1/s) & 1/s \\ 1/s & c + (1/s) \end{bmatrix}$$

(c is a constant; A misprint in the book has been corrected)

Remark. A similar model form is encountered for distillation columns controlled with the DB-configuration. In which case the physical reason for the model being singular at steady-state is that the sum of the two manipulated inputs is fixed at steady-state, $D + B = F$.

Solution. *For the computations we select* $c = 1$*. The plant has two poles at* $s = 0$ *and one LHP-zero at* $s = 0$*, e.g. consider the realization*

$$G(s) = \frac{1}{s^2} \begin{bmatrix} cs + 1 & 1 \\ 1 & cs + 1 \end{bmatrix}$$

Here $G(s)$ *has a zero at* $s = 0$ *since the matrix* $\begin{bmatrix} cs+1 & 1 \\ 1 & cs+1 \end{bmatrix}$ *has a zero for* $s = 0$. *(alternatively, we may have a realization with only one pole at* $s = 0$ *e.g.* $A = 0$ *(scalar),* $B = [1 \quad 1]$, $C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $D = I$*)), but we then have no zero at* $s = 0$*). In any case, the plant is unstable and needs to be stabilized.*

The plant is singular at steady-state (with infinite RGA and condition number). This may seem to indicate that control is very difficult, but this is not the case. The reason is that the singularity is caused by the largest singular value being infinite at steady-state, whereas the smallest singular value is $|c| = 1$ *at* $s = 0$ *so it is non-zero (if the smallest singular value was zero at* $s = 0$ *then tight control at steady-state would be impossible).*

In summary, acceptable control for this plant is possible provided the bandwidth is sufficiently high (higher than about frequency $|c|$*).*

7

UNCERTAINTY AND ROBUSTNESS FOR SISO SYSTEMS

Exercise 7.1 Suppose that the nominal model of a plant is

$$G(s) = \frac{1}{s+1}$$

and the uncertainty in the model is parameterized by multiplicative uncertainty with the weight

$$w_I(s) = \frac{0.125s + 0.25}{(0.125/4)s + 1}$$

Call the resulting set Π. Now find the extreme parameter values in each of the plants (a)-(g) below so that each plant belongs to the set Π. All parameters are assumed to be positive. One approach is to plot $l_I(\omega) = |G^{-1}G' - 1|$ in (7.15) for each G' (G_a, G_b, etc.) and adjust the parameter in question until l_I just touches $|w_I(j\omega)|$.

(a) Neglected delay: Find the largest θ for $G_a = Ge^{-\theta s}$ (Answer: 0.13).

(b) Neglected lag: Find the largest τ for $G_b = G\frac{1}{\tau s+1}$ (Answer: 0.15).

(c) Uncertain pole: Find the range of a for $G_c = \frac{1}{s+a}$ (Answer: 0.8 to 1.33).

(d) Uncertain pole (time constant form): Find the range of T for $G_d = \frac{1}{Ts+1}$ (Answer: 0.7 to 1.5).

(e) Neglected resonance: Find the range of ζ for $G_e = G\frac{1}{(s/70)^2+2\zeta(s/70)+1}$ (Answer: 0.02 to 0.8).

(f) Neglected dynamics: Find the largest integer m for $G_f = G\left(\frac{1}{0.01s+1}\right)^m$ (Answer: 13).

(g) Neglected RHP-zero: Find the largest τ_z for $G_g = G\frac{-\tau_z s+1}{\tau_z s+1}$ (Answer: 0.07). These results imply that a control system which meets given stability and performance requirements for all plants in Π, is also guaranteed to satisfy the same requirements for the above plants G_a, $G_b, \ldots, G_g$.

(h) Repeat the above with a new nominal plant $G = 1/(s-1)$ (and with everything else the same except $G_d = 1/(Ts-1)$). (Answer: Same as above).

Solution.
(a) $l_I(\omega) = |1 - e^{-j\omega\theta}|$. When $\theta = 0.13$ $l_I(\omega)$ just touches $|w_I(j\omega)|$.

(b) $l_I(\omega) = |\frac{j\omega\tau}{j\omega\tau+1}|$. *When* $\tau = 0.16$ $l_I(\omega)$ *just touches* $|w_I(j\omega)|$.
(c) $l_I(\omega) = |\frac{1-a}{j\omega+a}|$. *When* $0.8 \le a \le 1.33$ $l_I(\omega) \le |w_I(j\omega)|$.
(d) $l_I(\omega) = |\frac{(1-T)j\omega}{j\omega T+1}|$. *When* $0.7 \le T \le 1.5$ $l_I(\omega) \le |w_I(j\omega)|$.
(e) $l_I(\omega) = |\frac{(j\omega/70)^2+j\omega 2\zeta/70}{(j\omega/70)^2+j\omega 2\zeta/70+1}|$. *When* $0.14 \le \zeta \le 5.6$ $l_I(\omega) \le |w_I(j\omega)|$. *Note the answer given in the book is not right. If* $G_f = G\frac{1}{(s/70)^2+2\zeta(s/10)+1}$, *i.e. the middle term is* $2\zeta(s/10)$ *not* $2\zeta(s/70)$ *than the answer given in the book is right.*
(f) $l_I(\omega) = |\left(\frac{1}{j0.01\omega+1}\right)^m - 1|$. *When* $m \le 13$ $l_I(\omega) \le |w_I(j\omega)|$.
(g) $l_I(\omega) = |\frac{-j2\tau_z\omega}{j\tau_z\omega+1}|$. *When* $\tau_z \le 0.07$, $l_I(\omega) \le |w_I(j\omega)|$.
(h) If we change $G_c = \frac{1}{s-a}$ *then* $l_I(\omega)$*'s for all plants* G_a *to* G_g *are the same as (a) to (g). Thus the answer is the same as above.*

Exercise 7.3 Represent the gain uncertainty in (7.43) as multiplicative complex uncertainty with nominal model $G = G_0$ (rather than $G = \bar{k}G_0$ used above).

(a) Find w_I and use the RS-condition $\|w_I T\|_\infty < 1$ to find $k_{\max,3}$. Note that no iteration is needed in this case since the nominal model and thus $T = T_0$ is independent of $k_{\max}$.

(b) One expects $k_{\max,3}$ to be even more conservative than $k_{\max,2}$ since this uncertainty description is not even tight when Δ is real. Show that this is indeed the case using the numerical values from Example 7.7.

Solution.
(a) Let $k_p G = G(1 - w_I\Delta)$, $kp \in [1, k_{\max,3}]$ *and* $|\Delta| \le 1$. *Then* $|1 - k_{\max,3}| = \|w_I\Delta\|_\infty$. *This leads to* $\|w_I\|_\infty = k_{\max,3} - 1$, *so* $k_{\max,3} = (1/\|T\|_\infty) + 1$.
(b) Numerically we can get $\|T_0\|_\infty = 1.79$. *So* $k_{\max,3} = 1.56$ *(no iteration needed in this case).*

Exercise 7.5 Also derive, from $|w_P S| + |w_I T| < 1$, the following necessary bounds for RP (which *must* be satisfied)

$$|L| > \frac{|w_P| - 1}{1 - |w_I|}, \quad (\text{for } |w_P| > 1 \text{ and } |w_I| < 1)$$

$$|L| < \frac{1 - |w_P|}{|w_I| - 1}, \quad (\text{for } |w_P| < 1 \text{ and } |w_I| > 1)$$

Hint: Use $|1 + L| \le 1 + |L|$.

Solution. $|w_P S| + |w_I T| < 1 \Leftrightarrow |w_P| + |w_I L| < |1 + L| \le 1 + |L| \Rightarrow |w_P| - 1 < |L| - |w_I| \cdot |L| = (1 - |w_I|)|L|$. *So, if* $|w_I| < 1$ *and* $|w_P| > 1$ *then* $|L| > \frac{|w_P|-1}{1-|w_I|}$, *and if* $|w_p| < 1$ *and* $|w_I| > 1$ *then* $|L| < \frac{1-|w_P|}{|w_I|-1}$.

Exercise 7.7 Consider a "true" plant

$$G'(s) = \frac{3e^{-0.1s}}{(2s+1)(0.1s+1)^2}$$

(a) Derive and sketch the additive uncertainty weight when the nominal model is $G(s) = 3/(2s+1)$.

(b) Derive the corresponding robust stability condition.

(c) Apply this test for the controller $K(s) = k/s$ and find the values of k that yield stability. Is this condition tight?

Solution.
(a) Based on a plot we find that $e^{-0.3s}/(0.1s+1)^2 - 1 \approx 0.3s/(0.1s+1)$ *and* $w_A(s) = \frac{0.9s}{(2s+1)(0.1s+1)}$.
(b) Using (7.12), the RS condition (7.34) can be derived from (7.34): $|T| < |G|/|w_A|$.
(c) $L = GK = \frac{3k}{s(2s+1)}$ *and* $T = L/(1+L) = \frac{3k}{2s^2+s+3k}$. *So,* $|T| < |G|/|w_A|$ *leads to* $|\frac{3k}{2(j\omega)^2+j\omega+3k}| < |\frac{j0.1\omega+1}{j0.3\omega}|$. *This gives* $k \leq 1$. *This condition is tight.*

Exercise 7.9 Consider again the system in Figure 7.18. What kind of uncertainty might w_u and Δ_u represent?

Solution. w_u *may seem to represent some additive uncertainly, but actually it represents inverse multiplicative ("pole") uncertainty. (This may be seen by moving the point where the signal goes to* w_u *to just after the block for* w_p*). (Also: By comparing* w_u *in (7.68) with* w_{iI} *in (7.89) it can be seen that* r_u *may represent relative error on a time constant which is nominally* $\tau = 1$.*)*

Exercise

Exercise 7.11 Parametric gain uncertainty. *We showed in Example 7.1 how to represent scalar parametric gain uncertainty* $G_p(s) = k_p G_0(s)$ *where*

$$k_{\min} \leq k_p \leq k_{\max} \tag{7.1}$$

as multiplicative uncertainty $G_p = G(1 + w_I \Delta_I)$ *with nominal model* $G(s) = \bar{k} G_0(s)$ *and uncertainty weight* $w_I = r_k = (k_{\max} - k_{\min})/(k_{\max} + k_{\min})$. Δ_I *is here a real scalar,* $-1 \leq \Delta_I \leq 1$. *Alternatively, we can represent gain uncertainty as* inverse multiplicative uncertainty*:*

$$\Pi_{iI}: \quad G_p(s) = G(s)(1 + w_{iI}(s)\Delta_{iI})^{-1}; \quad -1 \leq \Delta_{iI} \leq 1 \tag{7.2}$$

with $w_{iI} = r_k$ *and* $G(s) = k_i G$ *where*

$$k_i = 2\frac{k_{\min} k_{\max}}{k_{\max} + k_{\min}} \tag{7.3}$$

(a) Derive (7.118) and (7.119). (Hint: The gain variation in (7.117) can be written exactly as $k_p = k_i/(1 - r_k \Delta)$.*)*
(b) Show that the form in (7.118) does not allow for $k_p = 0$.
(c) Discuss why (b) may be a possible advantage.

Solution.
(a) Let $k_p = k_i(1 + r_k\Delta)^{-1}$. If $Delta = 1$ then $k_p = k_{\min}$, and if $Delta = -1$ then $k_p = k_{\max}$. Solving the equations:

$$\begin{aligned} k_{\min}(1 + r_k) &= k_i, \\ k_{\max}(1 - r_k) &= k_i, \end{aligned}$$

gives that

$$\begin{aligned} r_k &= \frac{k_{\max} - k_{\min}}{k_{\max} + k_{\min}}, \\ k_i &= 2\frac{k_{\max} k_{\min}}{k_{\max} + k_{\min}}. \end{aligned}$$

Inserting k_p into $G_p(s) = k_p G_0(s)$ and letting $w_I = r_k$ leads to (7.118) and (7.119).
(b) Since $k_i \neq 0$, $k_p = 0$ implies $1 + r_k\Delta = \infty$. But this is impossible when $-1 \leq \Delta \leq 1$. Thus $k_p = 0$ is not allowed.
(c) Usually the gain CANNOT be zero physically, so using an uncertainty description where this is not possible may be an advantage. However, note that the inverse gain form may allow for k_p being infinite which is also impossible physically.

8

ROBUST STABILITY AND PERFORMANCE ANALYSIS

Exercise 8.1 *The uncertain plant in (8.7) may be represented in the additive uncertainty form $G_p = G + W_2\Delta_A W_1$ where $\Delta_A = \delta$ is a single scalar perturbation. Find W_1 and W_2.*

Solution.

$$W_2 = \begin{bmatrix} w \\ -w \end{bmatrix}, \qquad W_1 = [\,1 \quad -1\,].$$

Exercise 8.3 *Obtain H in Figure 8.6 for the uncertain plant in Figure 7.20(b).*

Solution.

$$\begin{array}{ll} H_{11} = \begin{bmatrix} w_1 & w_1 G \\ 0 & 0 \end{bmatrix}, & H_{12} = \begin{bmatrix} w_1 G \\ w_2 \end{bmatrix} \\ H_{21} = [\,I \quad G\,], & H_{22} = G \end{array}$$

Exercise 8.5 *Show in detail how P in (8.29) is derived.*

Solution. From Figure 8.7 it can be obtained that:

$$\begin{aligned} y_\Delta &= W_I u \\ z &= -W_P v \\ v &= -w - Gu - u_\Delta \end{aligned}$$

So,

$$\begin{bmatrix} y_\Delta \\ z \\ v \end{bmatrix} = \begin{bmatrix} 0 & 0 & W_I \\ W_P G & W_P & W_P G \\ -G & -I & -G \end{bmatrix} \begin{bmatrix} u_\Delta \\ w \\ u \end{bmatrix}.$$

Exercise 8.7 *Derive N in (8.32) from P in (8.29) using the lower LFT in (8.2). You will note that the algebra is quite tedious, and that it is much simpler to derive N directly from the block diagram as described above.*

Solution.

$$N = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

$$
\begin{aligned}
&= \begin{bmatrix} 0 & 0 \\ W_P G & W_P \end{bmatrix} + \begin{bmatrix} W_I \\ W_P G \end{bmatrix} K(I+GK)^{-1} \begin{bmatrix} -G & -I \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ W_P G & W_P \end{bmatrix} + \begin{bmatrix} -W_I KSG & -W_I K \\ -W_P GKSG & -W_P GKS \end{bmatrix} \\
&= \begin{bmatrix} -W_I KSG & -W_I KS \\ W_P G(I-KSG) & W_P(I-GKS) \end{bmatrix}
\end{aligned}
$$

where $S=(I+GK)^{-1}$. Using the identities $I-GKS=I-T=S=(I+GK)^{-1}$, and $I-KSG=(I+KG)^{-1}$, it finally yields the desired matrix N.

Exercise 8.9 *Find P for the uncertain system in Figure 7.18.*

Solution.

$$
P = \begin{bmatrix} -w_u & -w_u & -w_u G \\ w_P & w_P & w_P G \\ -I & -I & -G \end{bmatrix}
$$

Exercise 8.11 *Find the interconnection matrix N for the uncertain system in Figure 7.18. What is M?*

Solution.

$$
N = \begin{bmatrix} -w_u(I+GK)^{-1} & -w_u(I+GK)^{-1} \\ w_P(I+GK)^{-1} & w_P(I+GK)^{-1} \end{bmatrix}
$$

So, $M=N_{11}=-w_u(I+GK)^{-1}$.

Exercise 8.13 $M\Delta$-structure for combined input and output uncertainties. *Consider the block diagram in Figure 8.8 where we have both input and output multiplicative uncertainty blocks. The set of possible plants is given by*

$$
G_p = (I + W_{2O}\Delta_O W_{1O})G(I + W_{2I}\Delta_I W_{1I}) \tag{8.1}
$$

where $\|\Delta_I\|_\infty \le 1$ and $\|\Delta_O\|_\infty \le 1$. Collect the perturbations into $\Delta = \mathrm{diag}\{\Delta_I, \Delta_O\}$ and rearrange Figure 8.8 into the $M\Delta$-structure in Figure 8.3 Show that

$$
M = \begin{bmatrix} W_{1I} & 0 \\ 0 & W_{1O} \end{bmatrix} \begin{bmatrix} -T_I & -KS \\ SG & -T \end{bmatrix} \begin{bmatrix} W_{2I} & 0 \\ 0 & W_{2O} \end{bmatrix} \tag{8.2}
$$

Solution. Since $N=P_{11}+P_{12}K(I+GK)^{-1}P_{21}$ and $M=N_{11}$, it yields:

$$
\begin{aligned}
M &= \begin{bmatrix} 0 & 0 \\ W_{1O} & W_{2I} \end{bmatrix} + \begin{bmatrix} W_{1I} \\ W_{1O} G \end{bmatrix} K(I+GK)^{-1} \begin{bmatrix} -GW_{2I} & -W_{2O} \end{bmatrix} \\
&= \begin{bmatrix} -W_{1I}K(I+GK)^{-1}GW_{2I} & -W_{1I}K(I+GK)^{-1}W_{20} \\ W_{1O}G(I+KG)^{-1}W_{2I} & -W_{1O}GK(I+GK)^{-1}W_{20} \end{bmatrix} \\
&= \begin{bmatrix} W_{1I} & 0 \\ 0 & W_{1O} \end{bmatrix} \begin{bmatrix} -T_I & -KS \\ SG & -T \end{bmatrix} \begin{bmatrix} W_{2I} & 0 \\ 0 & W_{2O} \end{bmatrix}
\end{aligned}
$$

Exercise 8.15 *Consider combined multiplicative and inverse multiplicative uncertainty at the output, $G_p = (I - \Delta_{iO} W_{iO})^{-1}(I + \Delta_O W_O)G$, where we choose to norm-bound the combined uncertainty, $\|\,[\,\Delta_{iO} \quad \Delta_O\,]\,\|_\infty \le 1$. Make a block diagram of the uncertain plant, and derive a necessary and sufficient condition for robust stability of the closed-loop system.*

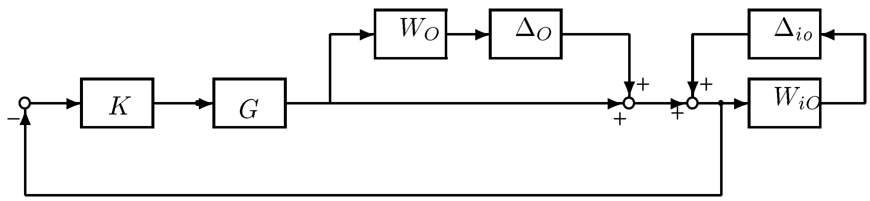


Fig. 8.15 Combined output uncertainties for Exercise 8.15

Solution. A block diagram of the uncertain plant is shown in Fig. 8.15. From the diagram it can be obtained that

$$M = \begin{bmatrix} W_{iO}S \\ W_O T \end{bmatrix}.$$

According to (8.64) the RS condition is:

$$\|M\|_\infty < 1.$$

Exercise 8.17 *For M in (8.98) and a diagonal Δ show that $\mu(M) = |a| + |b|$ using the upper bound $\mu(M) \leq \min_D \bar{\sigma}(DMD^{-1})$ (which is exact in this case since D has two "blocks").*

Solution. Use $D = \mathrm{diag}\{d, 1\}$. Since DMD^{-1} is a singular matrix we have from (A.36) that

$$\bar{\sigma}(DMD^{-1}) = \bar{\sigma}\begin{bmatrix} a & da \\ \frac{1}{d}b & b \end{bmatrix} = \sqrt{|a|^2 + |da|^2 + |b/d|^2 + |b|^2} \tag{8.3}$$

which we want to minimize with respect to d. The solution is $d = \sqrt{|a|/|b|}$ which gives $\mu(M) = \sqrt{|a|^2 + 2|ab| + |b|^2} = |a| + |b|$.

Exercise 8.19 *Let a, b, c and d be complex scalars. Show that for*

$$\Delta = \mathrm{diag}\{\delta_1, \delta_2\}: \quad \mu\begin{bmatrix} ab & ad \\ bc & cd \end{bmatrix} = \mu\begin{bmatrix} ab & ab \\ cd & cd \end{bmatrix} = |ab| + |cd| \tag{8.4}$$

Does this hold when Δ is scalar times identity, or when Δ is full? (Answers: Yes and No).

Solution. Using (8.86), and letting $U = \mathrm{diag}\{e^{j\phi}, 1\}$ we can get

$$\begin{bmatrix} ab & ad \\ bc & cd \end{bmatrix} U = \begin{bmatrix} abe^{j\phi} & ad \\ bce^{j\phi} & cd \end{bmatrix},$$

which is singular. Thus

$$\begin{aligned}
\mu\begin{bmatrix} ab & ad \\ bc & cd \end{bmatrix} &= \max_U \rho(\begin{bmatrix} ab & ad \\ bc & cd \end{bmatrix} U) \\
&= \max_\phi |\mathrm{tr}(\begin{bmatrix} abe^{j\phi} & ad \\ bce^{j\phi} & cd \end{bmatrix})| \\
&= \max_\phi |abe^{j\phi} + cd| \\
&= |ab| + |cd|.
\end{aligned}$$

When $\Delta = \delta I, U = e^{j\phi} I$. So the above result is still right. But when Δ is full, U is a general unitary matrix. Hence $\begin{bmatrix} ab & ad \\ bc & cd \end{bmatrix} U$ may not be singular. So the answer for the last question is NO.

The second equality has been proven in Example 8.7.

Exercise 8.21 *If (8.94) were true for any structure of Δ then it would imply $\rho(AB) \leq \bar{\sigma}(A)\rho(B)$. Show by a counterexample that this is not true.*

Solution. Select $A = \begin{bmatrix} 1 & 0 \\ 10 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$. Then $\rho(AB) = 102$, $\bar{\sigma}(A) = 10$, and $\rho(B) = 1$, so $\rho(AB) > \bar{\sigma}(A)\rho(B)$.

Exercise 8.23 *Consider the plant $G(s)$ in (8.107) which is ill-conditioned with $\gamma(G) = 70.8$ at all frequencies (but note that the RGA-elements of G are all about 0.5). With an inverse-based controller $K(s) = \frac{0.7}{s} G(s)^{-1}$, compute μ for RP with both diagonal and full-block input uncertainty using the weights in (8.132). The value of μ is much smaller in the former case.*

Solution. The μ curves for diagonal and full-block input uncertainty are shown in Fig. 8.23. The value of μ for diagonal uncertainty is much smaller than that of the case discussed in Section 8.11.3.

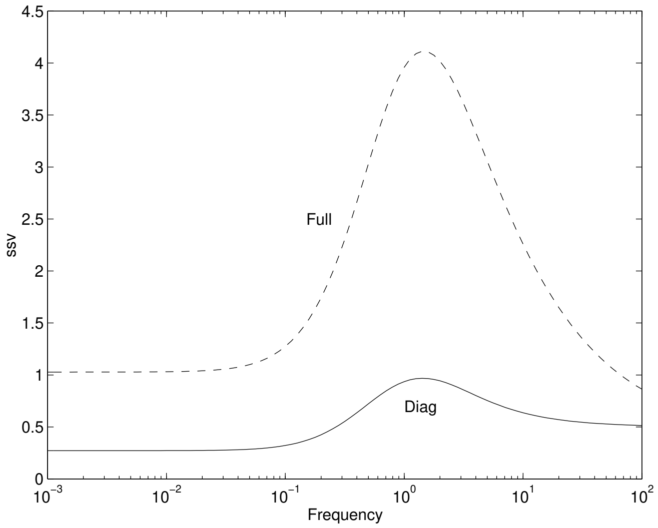


Fig. 8.23. μ curves for diagonal and full uncertainty in Exercise 8.23

Exercise 8.25 *Explain why the optimal μ-value would be the same if in the model (8.143) we changed the time constant of 75 [min] to another value. Note that the μ-iteration itself would be affected.*

Solution. The optimal μ-curve as a function of frequency is flat, i.e. the optimal μ is almost a constant (μ-value) over a wide rang of frequencies. In the model (8.143), the change of the time constant can be treated as a frequency scaling change, while the optimal μ-value is independent of frequency. Thus it would be the same.

9

CONTROLLER DESIGN

Exercise 9.1 *Show that the closed-loop objectives 1 to 6 can be approximated by the open-loop objectives 1 to 6 at the specified frequency ranges.*

Solution.

1. When $\bar{\sigma}(GK) \gg 1$, $S = (I + GK)^{-1} \approx (GK)^{-1}$. So the close-loop objective 1, making $\bar{\sigma}(S)$ small, can be approximated by the open-loop objective 1, making $\bar{\sigma}(GK)$ large.
2. When $\bar{\sigma}(GK) \ll 1$, $T = GK(I+GK)^{-1} \approx GK$. So the closed-loop objective 2, making $\bar{\sigma}(T)$ small, can be approximated by the open-loop objective 2, making $\bar{\sigma}(GK)$ small.
3. $\bar{\sigma}(GK) \gg 1 \Rightarrow T = GK(I + GK)^{-1} \approx I$. So $\bar{\sigma}(GK)$ small $\Rightarrow \bar{\sigma}(T) \approx \underline{\sigma}(T) \approx 1$.
4. $\bar{\sigma}(GK) \ll 1 \Rightarrow KS = K(I + GK)^{-1} \approx K$. Thus, $\bar{\sigma}(K)$ small $\Rightarrow \bar{\sigma}(KS)$ small.
5. See 4.
6. $\bar{\sigma}(GK) \ll 1 \Rightarrow T = GK(I + GK)^{-1} \approx GK$. Thus, $\bar{\sigma}(GK)$ small $\Rightarrow \bar{\sigma}(T)$ small.

Exercise 9.3 *In Figure 9.5, a reference input r is introduced into the LQG controller-plant configuration and the controller is rearranged to illustrate its two degrees-of-freedom structure. Show that the transfer function of the controller $K(s)$ linking $[\, r^T \quad y^T \,]^T$ to u is*

$$\begin{aligned} K(s) &= [\, K_1(s) \quad K_2(s) \,] \\ &= -K_r(sI - A + BK_r + K_f C)^{-1} [\, B \quad K_f \,] + [\, I \quad 0 \,] \end{aligned}$$

If the LQG controller is stable, for which there is no guarantee, then $K_1(s)$ can be implemented as a pre-filter and $K_2(s)$ as the feedback controller. However, since the degrees (number of states) of $K_1(s)$, $K_2(s)$ and $K_{LQG}(s)$ are in general the same, there is an increase in complexity on implementing $K_1(s)$ and $K_2(s)$ separately and so this is not recommended.

Solution. From Exercise 9.2 and Figure 9.5, we have:

$$\begin{aligned} \dot{\widehat{x}} &= A\widehat{x} + Bu + K_f(y - \widehat{y}) \\ \widehat{y} &= C\widehat{x} \\ \widehat{u} &= r - K_r\widehat{x} \end{aligned}$$

So, $s\widehat{x} = A\widehat{x} + B(r - K_r)\widehat{x} - K_f C\widehat{x} + K_f y$. This yields

$$\widehat{x} = (sI - A + BK_r + K_f C)^{-1} [\, B \quad K_f \,] \begin{bmatrix} r \\ y \end{bmatrix}.$$

Hence,

$$u = r - K_r \widehat{x} = K \begin{bmatrix} r \\ y \end{bmatrix},$$

where

$$K = -K_r(sI - A + BK_r + K_fC)^{-1} [\, B \quad K_f \,] + [\, I \quad 0 \,].$$

Exercise 9.5 *For the cost function*

$$\left\| \begin{bmatrix} W_1 S \\ W_2 T \\ W_3 KS \end{bmatrix} \right\|_\infty \tag{9.1}$$

formulate a standard problem, draw the corresponding control configuration and give expressions for the generalized plant P.

Solution. The control configuration is shown in Fig. 9.5. From the figure, the generalized plant

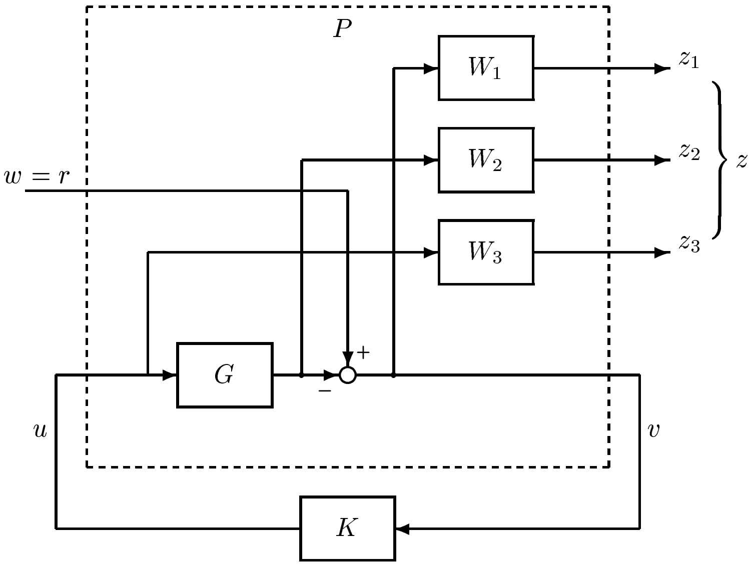


Fig. 9.5 Control configuration for $S/T/KS$ optimization problem

P can be derived as:

$$P_{11} = \begin{bmatrix} W_1 \\ 0 \\ 0 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} -W_1 G \\ W_2 G \\ W_3 \end{bmatrix},$$
$$P_{21} = I, \qquad P_{22} = -G.$$

Exercise 9.7 *Design an $\mathcal{H}_\infty$ loop-shaping controller for the disturbance process in (9.75) using the weight W_1 in (9.76), i.e. generate plots corresponding to those in Figure 9.18. Next, repeat the design with $W_1 = 2(s+3)/s$ (which results in an initial G_s which would yield closed-loop instability with $K_c = 1$). Compute the gain and phase margins and compare* *the disturbance and reference responses. In both cases find ω_c and use (2.37) to compute the maximum delay that can be tolerated in the plant before instability arises.*

Solution. The loop shapes, disturbance response and reference response of the system using the $\mathcal{H}_\infty$ loop-shaping controller designed with $W_1 = 2(s+3)/s$ are shown in Fig. 9.7. It is shown that the initial G_s would yield closed-loop instability with $K_c = 1$ (dashed-line). The gain margin of this design is 2.95, phase margin is $44.2°$. $\omega_c = 15.8$ rad/s. Thus the maximum delay is PM$/\omega_c = 0.049$ s which can be tolerated in the plant before instability.

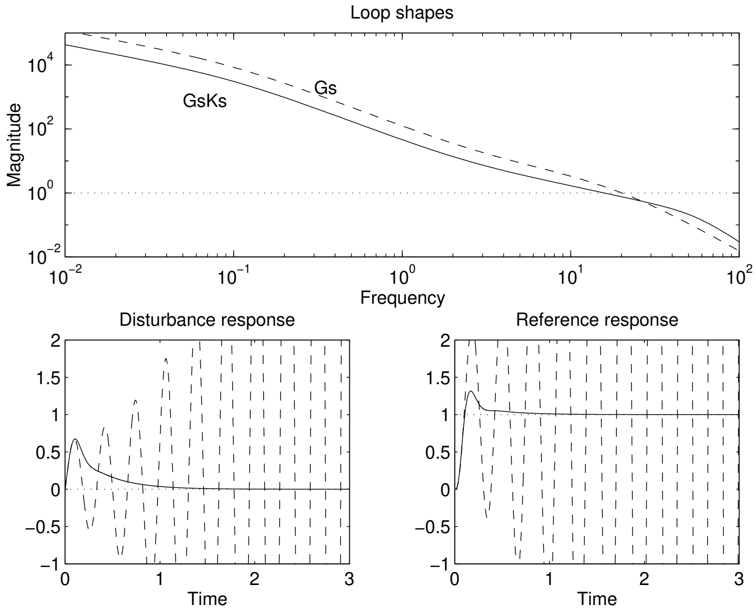


Fig. 9.7 Loop shaping design for Exercise 9.7

Exercise 9.9 *Show that the Hanus form of the weight W_1 in (9.109) simplifies to (9.109) when there is no saturation i.e. when $u_a = u$.*

Solution. Let $u = G_1 u_s + G_2 u_a$, where

$$G_1 \stackrel{s}{=} \left[\begin{array}{c|c} A_w - B_w D_w^{-1} C_w & 0 \\ \hline C_w & D_w \end{array}\right] = D_w$$

$$G_2 \stackrel{s}{=} \left[\begin{array}{c|c} A_w - B_w D_w^{-1} C_w & B_w D_w^{-1} \\ \hline C_w & 0 \end{array}\right]$$

If $u_a = u$, then $u = (I - G_2)^{-1} G_1 u_s$. We have,

$$I - G_2 \stackrel{s}{=} \left[\begin{array}{c|c} A_w - B_w D_w^{-1} C_w & -B_w D_w^{-1} \\ \hline C_w & I \end{array}\right]$$

and (see (4.27))

$$(I - G_2)^{-1} \overset{s}{=} \left[\begin{array}{c|c} A_w & -B_w D_w^{-1} \\ \hline -C_w & I \end{array}\right]$$

Thus,

$$(I - G_2)^{-1} G_1 \overset{s}{=} \left[\begin{array}{c|c} A_w & -B_w D_w^{-1} D_w \\ \hline -C_w & D_w \end{array}\right] = \overset{s}{=} \left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \end{array}\right]$$

10

CONTROL STRUCTURE DESIGN

Exercise 10.1 Conventional cascade control. *With reference to the special (but common) case of conventional cascade control shown in Figure 10.4, ? conclude that the use of extra measurements is useful under the following circumstances:*

(a) The disturbance d_2 is significant and G_1 is non-minimum phase.
(b) The plant G_2 has considerable uncertainty associated with it – e.g. a poorly known nonlinear behaviour – and the inner loop serves to remove the uncertainty.

In terms of design they recommended that K_2 is first designed to minimize the effect of d_2 on y_1 (with $K_1 = 0$) and then K_1 is designed to minimize the effect of d_1 on y_1. We want to derive conclusions (a) and (b) from an input-output controllability analysis, and also, (c) explain why we may choose to use cascade control if we want to use simple controllers (even with $d_2 = 0$).

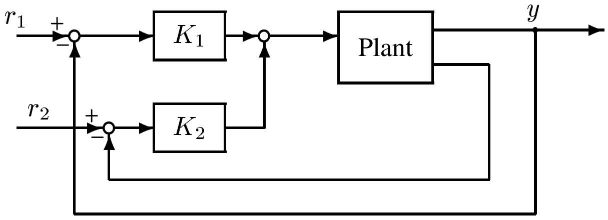
Outline of solution: *(a) Note that if G_1 is minimum phase, then the input-output controllability of G_2 and G_1G_2 are in theory the same, and for rejecting d_2 there is no fundamental advantage in measuring y_1 rather than y_2. (b) The inner loop $L_2 = G_2K_2$ removes the uncertainty if it is sufficiently fast (high gain feedback) and yields a transfer function $(I+L_2)^{-1}L_2$ close to I at frequencies where K_1 is active. (c) In most cases, such as when PID controllers are used, the practical bandwidth is limited by the frequency w_u where the phase of the plant is -180° (see section 5.12), so an inner cascade loop may yield faster control (for rejecting d_1 and tracking r_1) if the phase of G_2 is less than that of G_1G_2.*

Solution. See the above outline of solution.

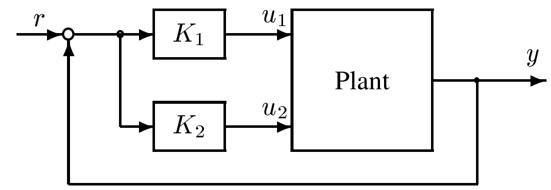
Exercise 10.3 *Draw the block diagrams for the two centralized (parallel) implementations corresponding to Figure 10.3 (in the book).*

Solution. See Figure 10.3 (in this solution manual)

Exercise 10.5 Process control application. *A practical case of a control system like the one in Figure 10.5 is in the use of a pre-heater to keep the reactor temperature y_1 at a given value r_1. In this case y_2 may be the outlet temperature from the pre-heater, u_2 the bypass flow (which should be reset to r_3, say 10% of the total flow), and u_3 the flow of heating medium (steam). Make a process flowsheet with instrumentation lines (not a block diagram) for this heater/reactor process.*



(c) Extra measurements y_2



(d) Extra inputs u_2

Fig. 10.3 Centralized implementations

Solution. See Fig. 10.5.

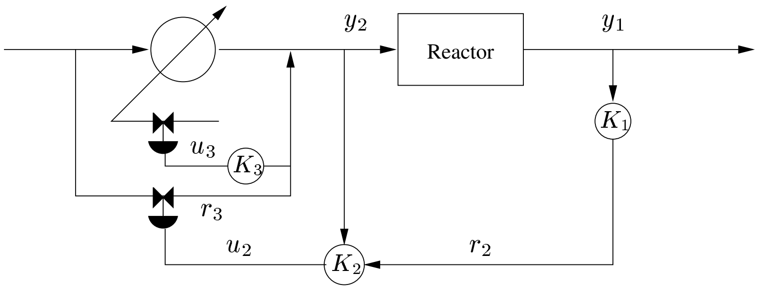


Fig. 10.5. Flowsheet for heater-reactor process.

Exercise 10.7 *Assume that the* 4×4 *matrix in (A.82) represents the steady-state model of a plant. Show that* 20 *of the* 24 *possible pairings can be eliminated by requiring DIC.*

Solution. Applying (10.57) to (A.82), where the RGA is:

$$\Lambda(A_2) = \begin{bmatrix} 6.16 & -0.69 & -7.94 & 3.48 \\ -1.77 & 0.10 & 3.16 & -0.49 \\ -6.60 & 1.73 & 8.55 & -2.69 \\ 3.21 & -0.14 & -2.77 & 0.70 \end{bmatrix}$$

it can be seen that outputs 1 and 4 can only be paired with inputs 1 and 4 in 2 possible combinations, and that output 2 and 3 can only be paired with inputs 2 and 3 also in 2 possible combinations. Thus only 4 pairings give positive RGA required by DIC and other 20 pairings can be eliminated.

11

MODEL REDUCTION

Exercise 11.1 *The steady-state gain of a full order balanced system* (A, B, C, D) *is* $D - CA^{-1}B$*. Show, by algebraic manipulation, that this is also equal to* $D_r - C_r A_r^{-1} B_r$*, the steady-state gain of the balanced residualization given by (11.7)–(11.10).*

Solution. Using (A.8) and noting Y in (A.8) is equal to A_r in (11.7), we have

$$\begin{aligned}
& D - CA^{-1}B \\
= \quad & D - [\, C_1 \quad C_2 \,] \begin{bmatrix} A_r^{-1} & -A_r^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}A_r^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}A_r^{-1}A_{12}A_{22}^{-1} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\
= \quad & D - [\, C_r A_r^{-1} \quad C_2 A_{22}^{-1} - C_r A_r^{-1} A_{12} A_{22}^{-1} \,] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\
= \quad & D - C_2 A_{22}^{-1} B_2 - C_r A_r^{-1} (B_1 - A_{12} A_{22}^{-1} B_2) \\
= \quad & D_r - C_r A_r^{-1} B_r
\end{aligned}$$

Exercise 11.3 *Is Theorem 11.3 true, if we replace balanced truncation by balanced residualization?*

Solution. Yes, it is still true. Balanced truncation and balanced residualization are related by the bilinear transformation $s \rightarrow s^{-1}$. If $(N(s), M(s))$ is a normalized left-coprime factorization of $G(s)$ then $(N(s^{-1}), M(s^{-1})$ is a normalized left-coprime factorization of $G(s^{-1})$. Applying Theorem 11.3 to $(N(s^{-1}), M(s^{-1})$ is equivalent to using balanced residualization with $(N(s), M(s))$.

12

CASE STUDIES

Exercise 12.1 *Repeat the μ-optimal design based on DK-iteration in Section 8.12.4 using the model (12.19).*

Solution. Apply the MATLAB program given in Table 8.2 to G in (12.9). After 10 iterations, the resulting controller with 25 states (denoted K_{10} in the following) gives a peak -value of 0.8847 (see Fig. 12.1(a)). The final -curves for RS, NP and RP with controller K_{10} are shown in Fig. 12.1(b). It is shown that all requirements are well satisfied. The time response of y_1 and y_2 to a filtered setpoint change in y_1, $r_1 = 1/(5s+1)$ is shown in Fig. 12.1(c). both for the nominal case (solid line) and for 20% input gain uncertainty (dashed line)

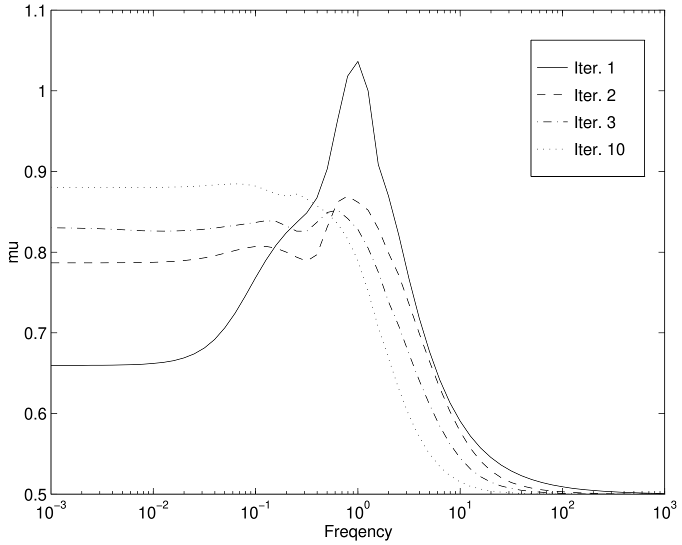


Fig. 12.1(a). Change in during DK-iteration.

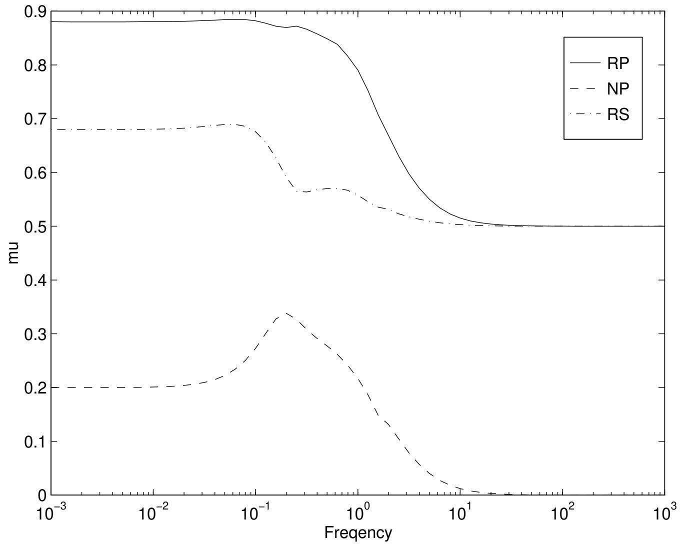


Fig. 12.1(b). -plots with -“optimal” controller K_{10}.

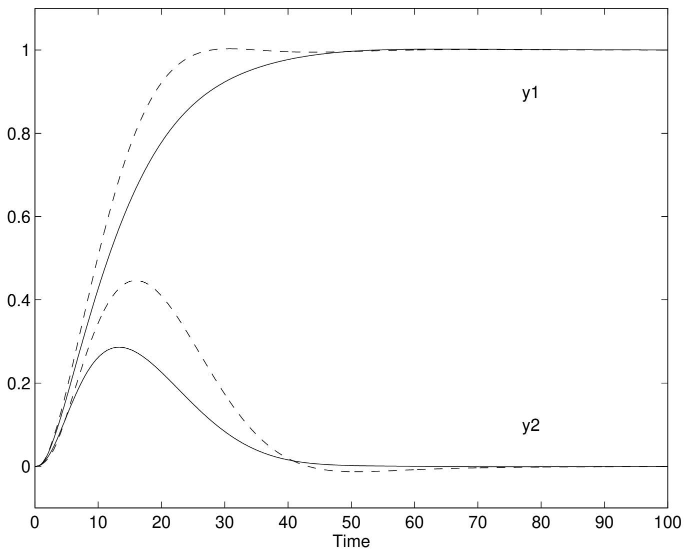


Fig. 12.1(c). Setpoint response for -“optimal” controller K_{10}. Solid line: nominal plant. Dashed line: “worst-case” plant.