REPORT:

Set-stability for nonlinear systems

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Preface

In this report we will give a treatment of important tools for analyzing stability in nonlinear dynamical systems. This inludes both stability of equilibria and the more general concept of set-stability. Some suggested applications for its use are time-varying systems, maneuvering systems (Skjetne et al.; 2004), adaptive systems, and observer designs.

A variety of references are used where the most important are Lin (1992); Lin et al. (1995); Sontag and Wang (1995a); Lin et al. (1996); Teel and Praly (2000); Teel (2002); Khalil (2002). Most of the definitions and theorems are taken from the mentioned references and organized in a consistent notation. The proper citations are clearly referenced. Using these references a few theorems have been developed further by the author as minor extensions to the existing theory. This concerns in particular Theorem 4.3, published in Skjetne et al. (2005), and Theorem 5.2, published in Skjetne et al. (2004).

Chapter 1

Ordinary differential equations

Consider the time-varying ordinary differential equation[1]

$$\dot{x} = f(x,t) \tag{1.1}$$

where for each $t \geq 0$ the vector $x(t) \in \mathbb{R}^n$ is the state.

To ensure existence and uniqueness of solutions, f is assumed to satisfy the following properties (Teel; 2002): For each starting point $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ and each compact set $\mathcal{X} \times \mathcal{T}$ containing (x_0, t_0) then:

- for all $(x,t) \in \mathcal{X} \times \mathcal{T}$, the function $f(\cdot, t)$ is continuous and $f(x, \cdot)$ is piecewise continuous,
- there exists $L > 0$ such that

$$|f(x,t) - f(y,t)| \leq L\,|x-y|\,, \qquad \forall (x,y,t) \in \mathcal{X} \times \mathcal{X} \times \mathcal{T},$$

- f is bounded on $\mathcal{X} \times \mathcal{T}$.

This will ensure that there exists $T > t_0 \geq 0$ such that there is one and only one solution of (1.1) on $[t_0, T]$. Often we simply assume that $f(\cdot,\cdot)$ is smooth which implies all the above conditions.

Let $x(t, t_0, x_0)$ denote the solution of (1.1) at time t with initial time and state $x(t_0) = x_0$ where $0 \leq t_0 < \infty$. If there is no ambiguity from the context, the solution is simply written as $x(t)$ with the initial state x_0 at time t_0. The solution is defined on some maximal interval of existence $(T_{\min}(x_0),\, T_{\max}(x_0))$ where $T_{\min}(x_0) < t_0 < T_{\max}(x_0)$. The system (1.1) is said to be *forward complete*

[1] Since the vector x in reality is a function of time, the notation $\dot{x}(t) = f(x(t), t)$ would perhaps be more precise than (1.1). However, to indicate that t in (1.1) is an explicit time-variation in the system, the notation without the time argument for the states is chosen.

if $T_{\max}(x_0) = +\infty$ for all x_0, *backward complete* if $T_{\min}(x_0) = -\infty$ for all x_0, and *complete* if it is both forward and backward complete (Lin et al.; 1996).

A solution is an absolutely continuous function satisfying $x(t_0, t_0, x_0) = x_0$ and:

- $x(\cdot, t_0, x_0)$ is differentiable a.e. on $(T_{\min}, T_{\max})$,
- $\frac{d}{dt}x(t, t_0, x_0) = f(x(t, t_0, x_0), t)$ is Lesbegue integrable on $(T_{\min}, T_{\max})$,
- $x(t, t_0, x_0) - x_0 = \int_{t_0}^{t} \frac{d}{dt}x(\tau, t_0, x_0)d\tau = \int_{t_0}^{t} f(x(\tau, t_0, x_0), \tau)d\tau$.

A convenient but crude way to ensure forward completeness is:

Proposition 1.1 *(Teel; 2002) Suppose the function $f : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ satisfies the above conditions for existence and uniqueness of solutions. Suppose also that $f(\cdot, \cdot)$ satisfies a global sector bound, that is, $\exists L \geq 0$ and $c \geq 0$ such that $\forall (x, t)$,*

$$|f(x, t)| \leq L\,|x| + c.$$

Then all solutions are defined for all $t \geq t_0$.

To prove this proposition, a differential version of the Gronwall-Bellman lemma is needed:

Lemma 1.2 *Let $y : \mathbb{R} \to \mathbb{R}$ be absolutely continuous and satisfy*

$$\dot{y}(t) \leq a(t)y(t) + b(t), \quad a.e.\ t \in [t_0, t_1] \tag{1.2}$$

where $a(t)$, $b(t)$ are continuously differentiable functions that satisfy $\dot{a}(t)b(t) - a(t)\dot{b}(t) = 0$ and $0 < a_0 \leq |a(t)| < \infty$ for some a_0 and $|b(t)| < \infty$, $\forall t \in [t_0, t_1]$. Then,

$$y(t) \leq \left(y(t_0) + \frac{b(t_0)}{a(t_0)}\right) \exp\left(\int_{t_0}^{t} a(s)ds\right) - \frac{b(t)}{a(t)}, \quad \forall t \in [t_0, t_1]. \tag{1.3}$$

If $b \equiv 0$ then the above constraints can be relaxed and $a(t)$ needs only be locally integrable to give

$$y(t) \leq y(t_0) \exp\left(\int_{t_0}^{t} a(s)ds\right), \quad \forall t \in [t_0, t_1]. \tag{1.4}$$

Furthermore, when a, b are constants, the result is

$$y(t) \leq \left(y(t_0) + \frac{b}{a}\right) \exp\left(a(t - t_0)\right) - \frac{b}{a}. \tag{1.5}$$

Proof. *Consider the differentiable function*

$$\eta(t) := \left(y(t) + \frac{b(t)}{a(t)}\right) \exp\left(-\int_{t_0}^{t} a(s)ds\right). \tag{1.6}$$

In view of (1.2) and the constraints, differentiation gives

$$\begin{aligned}
\dot{\eta}(t) &= \left(\dot{y}(t) + \frac{\dot{b}(t)a(t) - b(t)\dot{a}(t)}{a(t)^2}\right)\exp\left(-\int_{t_0}^{t} a(s)ds\right) \\
&\quad - a(t)\left(y(t) + \frac{b(t)}{a(t)}\right)\exp\left(-\int_{t_0}^{t} a(s)ds\right) \\
&= (\dot{y}(t) - a(t)y(t) - b(t))\exp\left(-\int_{t_0}^{t} a(s)ds\right) \\
&\leq 0.
\end{aligned} \tag{1.7}$$

This implies that $\eta(t) \leq \eta(t_0)$, $\forall t \in [t_0, t_1]$ so that when substituting the definition for $\eta(\cdot)$ and using that $\exp\left(-\int_{t_0}^{t} a(s)ds\right) > 0$ gives (1.3). When $b \equiv 0$, then the fraction b/a in (1.6) vanishes so that the same result follows by only a locally integrable function $a(\cdot)$. ■

Proof of Proposition 1.1: *Consider $y := |x| = \sqrt{x^\top x}$ which is continuously differentiable on $\mathbb{R}^n \backslash \{0\}$. Suppose that a solution $x(t, t_0, x_0)$ of $\dot{x} = f(x,t)$ escapes at the finite time $T > t_0$. Then, for each $M < \infty$ there exists $\tau \in [t_0, T)$ such that $|x(\tau, t_0, x_0)| > M$. Differentiating y with respect to time gives for each compact time interval $[t_1, t_2] \subset [t_0, T)$, with $x(t) \neq 0$ $\forall t \in [t_1, t_2]$,*

$$\dot{y}(t) = \frac{d}{dt}|x(t, t_1, x_1)| = \frac{x(t, t_1, x_1)^\top f(x(t, t_1, x_1), t)}{|x(t, t_1, x_1)|} \leq Ly(t) + c$$

where $x_1 := x(t_1, t_0, x_0)$. In view of Lemma 1.2 this implies that

$$|x(t, t_1, x_1)| \leq \left(|x_1| + \frac{c}{L}\right)e^{L(t-t_1)} - \frac{c}{L}, \qquad \forall t \in [t_1, t_2].$$

By picking $M > \left(|x_1| + \frac{c}{L}\right)e^{L(T-t_1)} - \frac{c}{L}$ this last inequality implies that no $\tau \in [t_0, T)$ can be found so that $|x(\tau, t_0, x_0)| > M$. By contradiction it follows that $T = \infty$.

Some convenient classes of functions are next defined. These are instrumental in nonlinear control theory.

Definition 1.3 *A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\alpha(0) = 0$ is positive semidefinite if $\alpha(s) \geq 0$ for $s > 0$ and positive definite if $\alpha(s) > 0$ for $s > 0$. It belongs to class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, $\alpha(0) = 0$, and $\alpha(s_2) > \alpha(s_1)$, $\forall s_2 > s_1$, and it belongs to class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if in addition $\lim_{s\to\infty} \alpha(s) = \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if for each fixed $t \geq 0$, $\beta(\cdot, t) \in \mathcal{K}$, and for each fixed $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s, t) = 0$.*

An equilibrium point $x_e \in \mathbb{R}^n$ of (1.1) at $t = t_0$ is a point such that $f(x_e, t) = 0$, $\forall t \geq t_0$. Such an equilibrium can always be shifted to the origin, giving the following stability definitions:

Definition 1.4 *For the system (1.1), the origin* $x = 0$ *is:*

- Uniformly Stable *(US) if there exists* $\delta(\cdot) \in \mathcal{K}_\infty$ *such that for any* $\varepsilon > 0$,

$$|x_0| \leq \delta(\varepsilon), \; t \geq t_0 \geq 0 \;\Rightarrow\; |x(t, t_0, x_0)| \leq \varepsilon. \tag{1.8}$$

- Uniformly Globally Asymptotically Stable *(UGAS) if it is US and* Uniformly Attractive *(UA), that is, for each* $\varepsilon > 0$ *and* $r > 0$ *there exists* $T > t_0 \geq 0$ *such that*

$$|x_0| \leq r, \; t \geq T \;\Rightarrow\; |x(t, t_0, x_0)| \leq \varepsilon. \tag{1.9}$$

The following comparison principle (Lin et al.; 1996, Lemma 4.4) is also useful, especially in proving asymptotic stability by Lyapunov arguments and $\mathcal{KL}$-estimates:

Lemma 1.5 *For each continuous positive definite function* α *there exists a* $\mathcal{KL}$-*function* $\beta_\alpha(s, t)$ *with the following property: if* $y(\cdot)$ *is any (locally) absolutely continuous function defined for each* $t \geq t_0 \geq 0$ *and with* $y(t) \geq 0$, $\forall t \geq t_0$, *and* $y(\cdot)$ *satisfies the differential inequality*

$$\dot{y}(t) \leq -\alpha(y(t)), \qquad a.a.\; t \geq t_0 \tag{1.10}$$

with $y(t_0) = y_0 \geq 0$, *then it holds that*

$$y(t) \leq \beta_\alpha(y_0, \; t - t_0), \qquad \forall t \geq t_0. \tag{1.11}$$

Proof. *See Lin et al. (1996, Lemma 4.4).* ■

Chapter 2

Set-stability

Often we will consider attractors other than equilibrium points. Such attractors will be closed subsets $\mathcal{A}$ of the state space. They can be compact or noncompact sets. In order to measure the distance away from the set, the "distance to the set $\mathcal{A}$ function" is defined as

$$|x|_{\mathcal{A}} := d\,(x; \mathcal{A}) = \inf\,\{d(x, y) : \; y \in \mathcal{A}\} \qquad (2.1)$$

where the point-to-point distance function is here simply taken as the Euclidean distance $d(x, y) = |x - y|$. Stability of the set is then determined in terms of bounds on the distance function.

For instance, an equilibrium $x_e \in \mathbb{R}^n$ of the system $\dot{x} = f(x)$ is a point such that $f(x_e) = 0$. It is represented by the compact set

$$\mathcal{A} := \{x \in \mathbb{R}^n : \; x = x_e\}\,,$$

for which the distance function becomes $|x|_{\mathcal{A}} = \inf\,\{|x - y| : \; y = x_e\} = |x - x_e|$ showing that the distance function reduces to the traditional norm function. Another example is the ε-ball given by the compact set

$$\mathcal{A}_\varepsilon = \{x \in \mathbb{R}^n : \; |x| \leq \varepsilon\}\,,$$

for which the distance function becomes $|x|_{\mathcal{A}_\varepsilon} = \max\{0, \; |x| - \varepsilon\}$.

In this framework, as shown by Teel and Praly (2000), we can consider the explicit time dependence of $t \mapsto f(x, t)$ in (1.1) as a state with its own dynamics and analyze stability of an augmented system with respect to a noncompact set in which t is free. For clarity, for this purpose we use the variable p, that is, the extended-state dynamic system becomes

$$\dot{z} = \frac{d}{dt}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} f(x, p) \\ 1 \end{bmatrix} =: g(z) \qquad z_0 = \begin{bmatrix} x_0 \\ t_0 \end{bmatrix}. \qquad (2.2)$$

Correspondingly, the time variable for the new extended-state system will be denoted by t with initial time $t = 0$. Notice that, in particular, $p(t) = t + t_0$ for all $t \geq 0$ and consequently $f(\cdot, p(t))$ for $t \geq 0$ is equal to $f(\cdot, t)$ for $t \geq t_0 \geq 0$. According to Lin (1992, Lemma 5.1.1) it follows that $x(t, t_0, x_0)$ is a solution of (1.1) for $t \geq t_0 \geq 0$ if and only if $z(t, z_0) := \mathrm{col}\,(x(t + t_0, t_0, x_0), t + t_0)$ is a solution of (2.2) for $t \geq 0$.

Stability of the origin $x = 0$ for (1.1) is captured by stability of the set of points

$$\mathcal{A}' = \{(x, p) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} : \ x = 0\} \tag{2.3}$$

for which the distance-to-the-set function becomes $|z|_{\mathcal{A}'} = \inf\{|z - y| : \ y \in \mathcal{A}'\} = |x|$.

With this motivation in mind we can therefore, in general, use set-stability analysis for time-invariant ODEs

$$\dot{x} = f(x) \tag{2.4}$$

where $x(t, x_0) \in \mathbb{R}^n, \ \forall t \geq 0$, is the solution with initial condition $x_0 = x(0)$.

Definition 2.1 *A nonempty closed set $\mathcal{A} \subset \mathbb{R}^n$ is a forward invariant set for (2.4) if the system is forward complete and $\forall x_0 \in \mathcal{A}$ the solution $x(t, x_0) \in \mathcal{A}$, $\forall t \geq 0$.*

For noncompact sets there is a possibility that a solution may escape to infinity in finite time within the set. Forward completeness is therefore a requirement in stability analysis of such sets. The tool called *finite escape-time detectability through* $|\cdot|_{\mathcal{A}}$ is helpful:

Definition 2.2 *(Teel; 2002) The system (2.4) is finite escape-time detectable through $|\cdot|_{\mathcal{A}}$ if, whenever a solution's maximal interval of existence is bounded, that is, $x(t, x_0)$ is defined only on $[0, T)$ with T finite, then $\lim_{t \nearrow T} |x(t, x_0)|_{\mathcal{A}} = \infty$.*

This is equivalent to what is called the *unboundedness observability property* in Mazenc and Praly (1994) by defining the output $y = h(x) = |x|_{\mathcal{A}}$. Nevertheless, we will continue using finite escape-time detectability to ensure forward completeness of the system.

Stability definitions using $\varepsilon - \delta$ neighborhoods as in Definition 1.4 is, as shown by Lin et al. (1996); Khalil (2002), equivalent to using class-$\mathcal{K}$ and class-$\mathcal{KL}$ estimates. For stability of sets we have:

Definition 2.3 *If the system (2.4) is forward complete, then for this system a closed, forward invariant set $\mathcal{A} \subset \mathbb{R}^n$ is:*

1. *Uniformly Globally Stable (UGS) if there exists a class-$\mathcal{K}_\infty$ function φ such that, $\forall x_0 \in \mathbb{R}^n$, the solution $x(t, x_0)$ satisfies*

$$|x(t, x_0)|_{\mathcal{A}} \leq \varphi\left(|x_0|_{\mathcal{A}}\right), \qquad \forall t \geq 0. \tag{2.5}$$

2. *Uniformly Globally Asymptotically Stable (UGAS) if there exists a class-$\mathcal{KL}$ function β such that, $\forall x_0 \in \mathbb{R}^n$, the solution $x(t, x_0)$ satisfies*

$$|x(t,x_0)|_{\mathcal{A}} \leq \beta\left(|x_0|_{\mathcal{A}}\,,\ t\right), \qquad \forall t \geq 0, \tag{2.6}$$

3. *Uniformly Globally Exponentially Stable (UGES) if there exist strictly positive real numbers $k > 0$ and $\lambda > 0$ such that, $\forall x_0 \in \mathbb{R}^n$, the solution $x(t, x_0)$ satisfies*

$$|x(t,x_0)|_{\mathcal{A}} \leq k\,|x_0|_{\mathcal{A}}\, e^{-\lambda t}, \qquad \forall t \geq 0. \tag{2.7}$$

When $\mathcal{A}$ is compact (for instance an equilibrium point), the forward completeness assumptions is redundant since in this case the system is finite escape-time detectable through $|\cdot|_{\mathcal{A}}$, and the above bounds therefore imply that solutions are bounded on the maximal interval of existence.

Definition 2.4 *A smooth Lyapunov function for (2.4) with respect to a nonempty, closed, forward invariant set $\mathcal{A} \subset \mathbb{R}^n$ is a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that satisfies:*

1. *there exist two $\mathcal{K}_\infty$-functions α_1 and α_2 such that for any $x \in \mathbb{R}^n$,*

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}), \tag{2.8}$$

2. *there exists a continuous and, at least, positive semidefinite function α_3 such that for any $x \in \mathbb{R}^n \backslash \mathcal{A}$,*

$$V^x(x) f(x) \leq -\alpha_3(|x|_{\mathcal{A}}). \tag{2.9}$$

Note that when $\mathcal{A}$ is compact, the existence of α_2 is a mere consequence of continuity of V. We now have:

Theorem 2.5 *Assume the system (2.4) is finite escape-time detectable through $|\cdot|_{\mathcal{A}}$. If there exists a smooth Lyapunov function for the system (2.4) with respect to a nonempty, closed, forward invariant set $\mathcal{A} \subset \mathbb{R}^n$, then $\mathcal{A}$ is UGS with respect to (2.4). Furthermore, if α_3 is strengthened to a positive definite function, then $\mathcal{A}$ is UGAS with respect to (2.4), and if $\alpha_i(|x|_{\mathcal{A}}) = c_i\,|x|_{\mathcal{A}}^r$ for $i = 1, 2, 3$, where c_1, c_2, c_3, r are strictly positive reals with $r \geq 1$, then $\mathcal{A}$ is UGES with respect to (2.4).*

Proof. *By integrating (2.9) along the solutions of $x(t, x_0)$ we get*

$$\begin{aligned}
V(x(t,x_0)) - V(x_0) &= \int_0^t \frac{d}{dt}\left\{V(x(\tau,x_0))\right\} d\tau \\
&= \int_0^t V^x(x(\tau,x_0)) f(x(\tau,x_0)) d\tau \\
&\leq -\int_0^t \alpha_3(|x(\tau,x_0)|_{\mathcal{A}}) d\tau \leq 0, \quad t \geq 0
\end{aligned}$$

showing that $V(x(t,x_0)) \leq V(x_0)$*, and consequently that*

$$|x(t,x_0)|_{\mathcal{A}} \leq \alpha_1^{-1}\left(V(x(t,x_0))\right) \leq \alpha_1^{-1}\left(V(x_0)\right) \leq \alpha_1^{-1}\left(\alpha_2\left(|x_0|_{\mathcal{A}}\right)\right) \tag{2.10}$$

for all t *in the maximal interval of existence* $[0,T)$*. Suppose the system escapes at a finite time* $T > 0$*. From the finite escape-time detectability property, this means that for each* $M < \infty$ *there exists* $t_1 \in [0,T)$ *such that* $|x(t_1,x_0)|_{\mathcal{A}} > M$*. Picking* $M > \alpha_1^{-1}\left(\alpha_2\left(|x_0|_{\mathcal{A}}\right)\right)$ *contradicts that (2.10) must hold* $\forall t \in [0,T)$*. Hence,* $T = \infty$ *and the system is forward complete. By defining* $\varphi(\cdot) := \alpha_1^{-1}\left(\alpha_2(\cdot)\right) \in \mathcal{K}_\infty$*, then (2.10) proves UGS according to (2.5). Suppose next that* $\alpha_3(\cdot)$ *is positive definite. From (2.8) and (2.9) we have*

$$\frac{d}{dt}\left\{V(x(t,x_0))\right\} \leq -\alpha(V(x(t,x_0)))$$

where $\alpha(\cdot) := \alpha_3(\alpha_2^{-1}(\cdot))$ *is positive definite. Let* $\beta_\alpha(\cdot,\cdot)$ *be the class-*$\mathcal{KL}$ *function, corresponding to* α*, from Lemma 1.5. This gives*

$$\begin{array}{rcl} V(x(t,x_0)) & \leq & \beta_\alpha(V(x_0),\ t), \qquad \forall t \geq 0 \\ & \Downarrow & \\ |x(t,x_0)|_{\mathcal{A}} & \leq & \alpha_1^{-1}\left(V(x(t,x_0))\right) \leq \alpha_1^{-1}\left(\beta_\alpha(V(x_0),\ t)\right) \\ & \leq & \alpha_1^{-1}\left(\beta_\alpha(\alpha_2(|x_0|_{\mathcal{A}}),\ t)\right) =: \beta\left(|x_0|_{\mathcal{A}},\ t\right), \quad \forall t \geq 0, \end{array}$$

where $\beta \in \mathcal{KL}$ *and UGAS follows from (2.6). In the last case we have that* $\alpha_i(|x|_{\mathcal{A}}) = c_i\,|x|_{\mathcal{A}}^r$ *for* $c_i > 0$ *and* $r \geq 1$*, and this gives*

$$\begin{array}{rcl} |x(t,x_0)|_{\mathcal{A}} & \leq & \sqrt[r]{\dfrac{1}{c_1}V(x(t,x_0))} \leq \sqrt[r]{\dfrac{1}{c_1}V(x_0)}e^{-\frac{c_3}{r}t} \\ & \leq & \sqrt[r]{\dfrac{c_2}{c_1}}\,|x_0|_{\mathcal{A}}\,e^{-\frac{c_3}{r}t}, \qquad \forall t \geq 0, \end{array}$$

which shows UGES according to (2.7). ■

Example 2.1 *Consider the linear system*

$$\begin{aligned} \dot{x} &= Ax + bu(t) && (2.11) \\ y &= c^\top x && (2.12) \end{aligned}$$

where $x \in \mathbb{R}^n$ *is the state,* $y \in \mathbb{R}^m$ *is the measured output,* A *is Hurwitz,* $(c^\top, A)$ *is an observable pair, and* $u(t)$ *is some known, bounded input function. In an observer design for this system we consider* $\hat{x} \in \mathbb{R}^n$ *as the observer state and stability of the noncompact set*

$$\mathcal{A} = \left\{(x,\hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n :\ \hat{x} = x\right\}. \tag{2.13}$$

Using a traditional linear observer design, we propose

$$\dot{\hat{x}} = A\hat{x} + bu(t) + L\left(y - c^\top\hat{x}\right) \tag{2.14}$$

where the feedback gain L is designed so that $A - Lc^\top$ is Hurwitz.
For stability analysis we first calculate the distance function. Letting $(x, \hat{x})$ be fixed, we get

$$\begin{aligned} |(x,\hat{x})|_{\mathcal{A}} &= \inf_{(\xi,\hat{\xi})\in\mathcal{A}} \left| \left[\begin{array}{c} x-\xi \\ \hat{x}-\hat{\xi} \end{array} \right] \right| = \inf_{\xi} \left| \left[\begin{array}{c} x-\xi \\ \hat{x}-\xi \end{array} \right] \right| \\ &= \inf_{\xi} \sqrt{|x-\xi|^2 + |\hat{x}-\xi|^2} = \min_{\xi} J(\xi). \end{aligned} \tag{2.15}$$

The $\arg\min$ ξ^* *of* $J(\xi)$ *is found where* $\frac{\partial J}{\partial \xi} = 0$, *giving*

$$\frac{\partial J(\xi)}{\partial \xi} = \frac{-1}{2J(\xi)} \left[(x-\xi)^\top + (\hat{x}-\xi)^\top \right] = 0 \;\Rightarrow\; \xi^* = \frac{1}{2}\left(x + \hat{x}\right), \tag{2.16}$$

and substituted back into the definition, this gives

$$|(x,\hat{x})|_{\mathcal{A}} = \left| \left[\begin{array}{c} x-\xi^* \\ \hat{x}-\xi^* \end{array} \right] \right| = \frac{1}{2} \left| \left[\begin{array}{c} (x-\hat{x}) \\ -(x-\hat{x}) \end{array} \right] \right| = \frac{\sqrt{2}}{2} \left| x - \hat{x} \right|. \tag{2.17}$$

Since A is Hurwitz and $u(t)$ is bounded, the solution $x(t)$ is also bounded. This implies that the system is finite escape-time detectable through $|(\cdot,\cdot)|_{\mathcal{A}}$. Letting next $P = P^\top > 0$ solve $P\left(A - Lc^\top\right) + \left(A - Lc^\top\right)^\top P = -I$, we define the function

$$V(x,\hat{x}) := (x-\hat{x})^\top P (x-\hat{x}) \tag{2.18}$$

for which the time derivative is

$$\begin{aligned} \dot{V} &= (x-\hat{x})^\top P \left(\dot{x} - \dot{\hat{x}}\right) + \left(\dot{x} - \dot{\hat{x}}\right)^\top P (x-\hat{x}) \\ &= -(x-\hat{x})^\top (x-\hat{x}). \end{aligned} \tag{2.19}$$

Since $V(x,\hat{x})$ satisfies

$$2p_m \left|(x,\hat{x})\right|_{\mathcal{A}}^2 \le V(x,\hat{x}) \le 2p_M \left|(x,\hat{x})\right|_{\mathcal{A}}^2 \tag{2.20}$$

$$\dot{V} = -2\left|(x,\hat{x})\right|_{\mathcal{A}}^2 \tag{2.21}$$

where $p_m := \lambda_{\min}(P)$ and $p_M := \lambda_{\max}(P)$, it follows that V is a smooth Lyapunov function for the overall system, and Theorem 2.5 states that the set $\mathcal{A}$ is UGES.

Chapter 3

Set-stability for systems with inputs

Consider the system

$$\dot{x} = f(x, u) \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $\forall t \geq 0$, and the map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is smooth. The input u is a measurable, locally essentially bounded function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. The space of such functions is denoted $\mathcal{L}_\infty^m$ with the norm $||u_{[t_0,\infty)}|| := \operatorname{ess\,sup}\{u(t) : t \geq t_0 \geq 0\}$. We use $||u|| = ||u_{[0,\infty)}||$ and let $||u_{[0,t]}||$ be the signal norm over the truncated interval $[0, t]$. For each initial state $x_0 = x(0) \in \mathbb{R}^n$ and each $u \in \mathcal{L}_\infty^m$, let $x(t, x_0, u)$ denote the solution of (3.1) at time t. If there is no ambiguity from the context, the solution is simply written $x(t)$.

For a nonempty closed set $\mathcal{A} \subset \mathbb{R}^n$ we have:

Definition 3.1 *The set $\mathcal{A}$ is called a* 0*-invariant set for (3.1) if, for the associated "zero-input" system*

$$\dot{x} = f(x, 0) =: f_0(x), \tag{3.2}$$

it holds that for each $x_0 \in \mathcal{A}$ then $x(t, x_0, 0) \in \mathcal{A}$ for all $t \geq 0$.

Definition 3.2 *The system (3.1) is input-to-state stable (ISS) with respect to a closed,* 0*-invariant set $\mathcal{A}$ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for each $u \in \mathcal{L}_\infty^m$ and all initial states x_0, the solution $x(t, x_0, u)$ is defined for all $t \geq 0$ and satisfies*

$$|x(t, x_0, u)|_{\mathcal{A}} \leq \beta\left(|x_0|_{\mathcal{A}},\ t\right) + \gamma\left(\left|\left|u_{[0,t]}\right|\right|\right) \tag{3.3}$$

for each $t \geq 0$.

Definition 3.3 *A smooth ISS-Lyapunov function for the system (3.1) with respect to the closed set $\mathcal{A}$ is a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that satisfies:*

1. *there exist two class-$\mathcal{K}_\infty$ functions α_1 and α_2 such that for any $x \in \mathbb{R}^n$,*

$$\alpha_1\left(|x|_{\mathcal{A}}\right) \leq V(x) \leq \alpha_2\left(|x|_{\mathcal{A}}\right), \tag{3.4}$$

2. *there exist a class-$\mathcal{K}$ function α_3 and a $\mathcal{K}_\infty$-function χ such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,*

$$|x|_{\mathcal{A}} \geq \chi(|u|) \;\Rightarrow\; V^x(x)f(x,u) \leq -\alpha_3\left(|x|_{\mathcal{A}}\right). \tag{3.5}$$

For compact sets $\mathcal{A}$, an equivalent representation of (3.5) is:

2.' *There exist two class-$\mathcal{K}_\infty$ functions α_3 and α_4 such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,*

$$V^x(x)f(x,u) \leq -\alpha_3\left(|x|_{\mathcal{A}}\right) + \alpha_4\left(|u|\right). \tag{3.6}$$

Note that (3.6) implies (3.5) for both compact and noncompact sets:

$$\begin{array}{rcl} V^x(x)f(x,u) & \leq & -\alpha_3\left(|x|_{\mathcal{A}}\right) + \alpha_4\left(|u|\right) \\ & \Downarrow & \\ V^x(x)f(x,u) & \leq & -\varepsilon\alpha_3\left(|x|_{\mathcal{A}}\right) \end{array}$$

for all $|x|_{\mathcal{A}} \geq \alpha_3^{-1}\left(\frac{1}{1-\varepsilon}\alpha_4\left(|u|\right)\right) =: \chi(|u|)$ where $\varepsilon \in (0,1)$. The converse is a bit more technical (Sontag and Wang; 1995b, Remark 2.4) and not generically true for noncompact sets (Sontag and Wang; 1995a, Remark 2.9).

These preliminaries now lead to the ISS sufficiency theorem. Some powerful converse results are found in Sontag and Wang (2000).

Theorem 3.4 *Assume the closed set $\mathcal{A}$ is 0-invariant for (3.1). If the system (3.1) is finite escape-time detectable through $|\cdot|_{\mathcal{A}}$ and admits a smooth ISS-Lyapunov function with respect to $\mathcal{A}$, then it is ISS with respect to $\mathcal{A}$.*

Proof. *It follows from the bounds (3.4) and (3.5) that*

$$\forall\, |x(t)|_{\mathcal{A}} \geq \chi\left(||u||\right) \Rightarrow \frac{d}{dt}\left\{V(x(t))\right\} \leq -\alpha_3\left(\alpha_2^{-1}\left(V(x(t))\right)\right).$$

By Lemma 1.5 this shows that $V(x(t))$ and, consequently, $|x(t)|_{\mathcal{A}}$ are bounded on the maximal interval of existence. By the finite escape-time detectability through $|\cdot|_{\mathcal{A}}$ property it follows that the system is forward complete and the solutions exist for all $t \geq 0$. (The proof from here is the same as given in Sontag and Wang (1995b, Lemma 2.14).) Let α_i, $i = 1,2,3$ and χ be as in Definition 3.3, 1. and 2. For an inital state x_0 and input function u, let $x(t) = x(t, x_0, u)$ be the corresponding trajectory of (3.1). Define the set $\Omega := \{x : V(x) \leq \alpha_2\left(\chi(||u||)\right)\}$. If there exists $t_1 \geq 0$ such that $x(t_1) \in \Omega$, then $x(t) \in \Omega$ for all $t \geq t_1$. To prove this, assume otherwise. Then there exist some $t \geq t_1$ and some $\varepsilon > 0$ such that

$V(x(t)) > \alpha_2(\chi(||u||)) + \varepsilon$. Let $t_2 = \inf\{t \geq t_1 : V(x(t)) \geq \alpha_2(\chi(||u||)) + \varepsilon\}$. Then $|x(t_2)|_{\mathcal{A}} \geq \chi(||u||)$ such that

$$\dot{V}(x(t_2)) \leq -\alpha_3(|x(t_2)|_{\mathcal{A}}) \leq -\alpha_3\left(\alpha_2^{-1}(V(x(t_2)))\right) < 0.$$

Hence, there must exist $t \in (t_1, t_2)$ so that $\alpha_2(\chi(||u||)) + \varepsilon \leq V(x(t_2)) \leq V(x(t))$ which contradicts minimality of t_2.
To continue, let $t_3 = \inf\{t \geq 0 : x(t) \in \Omega\}$ where t_3 may be infinite. For all $t \geq t_3$ we have that $V(x(t)) \leq \alpha_2(\chi(||u||))$ so that

$$|x(t)|_{\mathcal{A}} \leq \alpha_1^{-1}(V(x(t))) \leq \alpha_1^{-1}(\alpha_2(\chi(||u||))) =: \gamma(||u||). \tag{3.7}$$

Moreover, for $0 \leq t < t_3$ then $x(t) \notin \Omega$ so that $|x(t)|_{\mathcal{A}} \geq \chi(||u||)$ and

$$\dot{V}(x(t)) \leq -\alpha_3(|x(t)|_{\mathcal{A}}) \leq -\alpha_3\left(\alpha_2^{-1}(V(x(t)))\right) =: -\alpha(V(x(t))).$$

Let β_α be the class-$\mathcal{KL}$ function from Lemma 1.5 such that

$$V(x(t)) \leq \beta_\alpha(V(x_0), t), \quad \forall t \in [0, t_3).$$

Define $\beta(s, t) := \alpha_1^{-1}(\beta_\alpha(\alpha_2(s), t)) \in \mathcal{KL}$. Then for all $0 \leq t < t_3$ it follows that

$$|x(t)|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}}, t). \tag{3.8}$$

Note that neither γ or β depends on the initial state x_0 or the input function u. Therefore, combining (3.7) and (3.8) gives

$$|x(t)|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}}, t) + \gamma(||u||) \tag{3.9}$$

for all $t \geq 0$, and (3.3) follows by causality. ■

Corollary 3.5 *Suppose the system (3.1) is ISS with respect to a closed, 0-invariant set $\mathcal{A}$. Then*

$$\lim_{t\to\infty} |u(t)| = 0 \Rightarrow \lim_{t\to\infty} |x(t)|_{\mathcal{A}} = 0. \tag{3.10}$$

Proof. *For each $\varepsilon > 0, r > 0$, and each input function u such that $\lim_{t\to\infty} u(t) = 0$, we need to show that there exists $T = T(\varepsilon, r, u) > 0$ such that*

$$|x_0|_{\mathcal{A}} \leq r,\, t \geq T \Rightarrow |x(t, x_0, u)|_{\mathcal{A}} \leq \varepsilon. \tag{3.11}$$

Existence and uniqueness of solutions for all forward time implies that for all $0 \leq t_1 \leq t$, a solution satisfies

$$x(t, x(0), u) = x(t, x(t_1, x(0), u), u) \quad a.e. \tag{3.12}$$

This is verified by integrating (3.1) to get

$$\begin{aligned} x\left(t, x(0), u\right) &= x(0) + \int_0^t f\left(x(s), u(s)\right) ds \\ &= x(0) + \int_0^{t_1} f\left(x(s), u(s)\right) ds + \int_{t_1}^t f\left(x(s), u(s)\right) ds \\ &= x(t_1, x(0), u) + \int_{t_1}^t f\left(x(s), u(s)\right) ds \\ &= x\left(t, x(t_1, x(0), u), u\right) \qquad a.e.\ t \geq t_1 \geq 0. \end{aligned}$$

Moreover, for each initial state x_0 and input function u, ISS guarantees a uniform bound $c = c\left(x_0, u\right) > 0$ such that

$$|x(t, x_0, u)|_{\mathcal{A}} \leq \beta\left(|x_0|_{\mathcal{A}}\,,\ t\right) + \gamma\left(||u||\right) \leq c\left(x_0, u\right). \tag{3.13}$$

Pick $t_1 \geq 0$ such that $\gamma\left(\left|\left|u_{[t_1,\infty)}\right|\right|\right) \leq \frac{\varepsilon}{2}$, and $T \geq t_1$ such that $\beta\left(c,\, t - t_1\right) \leq \frac{\varepsilon}{2}$ for all $t \geq T$. This gives

$$\begin{aligned} |x(t, x_0, u)|_{\mathcal{A}} &= \left|x\left(t, x(t_1, x_0, u), u\right)\right|_{\mathcal{A}} \\ &\leq \beta\left(|x(t_1)|_{\mathcal{A}}\,,\ t - t_1\right) + \gamma\left(\left|\left|u_{[t_1,\infty)}\right|\right|\right), \quad \forall t \geq t_1 \\ &\leq \beta\left(c\left(x_0, u\right),\, t - t_1\right) + \frac{\varepsilon}{2}, \qquad \forall t \geq t_1 \\ &\leq \varepsilon, \qquad \forall t \geq T \geq t_1 \end{aligned} \tag{3.14}$$

where T depends on ε, u, and x_0. ■

An application of the set-stability and ISS tools is illustrated by the following example.

Example 3.1 ***Claim:*** *The noncompact set*

$$\mathcal{A} = \left\{(x, t) :\ x = x_d(t)\right\}$$

is UGAS with respect to the scalar system

$$\dot{x} = -\left(x^3 - x_d(t)^3\right) + \dot{x}_d(t) =: f(x, t)$$

where the desired state $x_d(t)$ is bounded and absolutely continuous, and $|\dot{x}_d(t)| \leq M$, a.a. $t \geq 0$.
Proof: *Forward completeness is established by the auxiliary function $W := \frac{1}{2}x^2$ having a derivative $\dot{W} = -x^4 + x\delta(t) \leq -\varepsilon\, |x|^4\,,\ \forall\, |x| \geq \sqrt[3]{\frac{\delta_0}{1-\varepsilon}}$ where δ_0 is a bound on $\delta(t) := x_d(t)^3 + \dot{x}_d(t)$ and $\varepsilon \in (0, 1)$. This shows input-to-state stability (ISS) of the system with δ as input, and consequently that $x(t)$ and $f(x(t), t)$ are bounded for all $t \geq 0$.*
For the distance function we have that

$$|(x, t)|_{\mathcal{A}} = \inf_{(y,\tau)\in\mathcal{A}} \left|\left| \begin{bmatrix} x - y \\ t - \tau \end{bmatrix} \right|\right| = \inf_{\tau} \left|\left| \begin{bmatrix} x - x_d(\tau) \\ t - \tau \end{bmatrix} \right|\right| \leq \left|x - x_d(t)\right|.$$

The absolute continuity of $x_d(t)$ together with boundedness of $\dot{x}_d(t)$ implies that $x_d(t)$ is globally Lipschitz such that $|x_d(t) - x_d(\tau)| \leq M\,|t-\tau|$ holds. Let τ^* be the (optimal) value that satisfies the above infimum. Then

$$\begin{aligned}
|x - x_d(t)| &= |x - x_d(\tau^*) + x_d(\tau^*) - x_d(t)| \\
&\leq |x - x_d(\tau^*)| + |x_d(\tau^*) - x_d(t)| \\
&\leq |x - x_d(\tau^*)| + M\,|t-\tau^*| \leq \max\{1, M\} \left| \left[\begin{array}{c} x - x_d(\tau^*) \\ t - \tau^* \end{array} \right] \right|_1 \\
&\leq \sqrt{2}\max\{1, M\}\, |(x,t)|_{\mathcal{A}}\,.
\end{aligned}$$

Defining $k := \sqrt{2}\max\{1, M\}$ the result is the equivalence relation

$$\frac{1}{k}\,|x - x_d(t)| \leq |(x,t)|_{\mathcal{A}} \leq |x - x_d(t)|\,.$$

Let a smooth Lyapunov function be $V(x,t) := \frac{1}{2}\left(x - x_d(t)\right)^2$. This has the bounding functions, according to (2.8) and (2.9), defined as:

$$\begin{aligned}
\alpha_1\left(|(x,t)|_{\mathcal{A}}\right) := \frac{1}{2}\,|(x,t)|_{\mathcal{A}}^2 \leq V(x,t) \leq \frac{k^2}{2}\,|(x,t)|_{\mathcal{A}}^2 =: \alpha_2\left(|(x,t)|_{\mathcal{A}}\right) \\
V^x(x,t)f(x,t) + V^t(x,t) = -\left(x - x_d(t)\right)\left(x^3 - x_d(t)^3\right) =: -\alpha_3(|(x,t)|_{\mathcal{A}}).
\end{aligned}$$

Recall the property $(x-y)(d(x)-d(y)) > 0$, $\forall x \neq y$, of a monotonically strictly increasing function $d(x)$. Using this with $d(x) = x^3$ shows that α_3 is a positive definite function, and $\mathcal{A}$ is therefore UGAS according to Theorem 2.5.

Chapter 4

Convergence analysis

Many systems have stronger properties than only stability. A UGS system may also have internal signals that converge to some value, often to zero. For such convergence analysis the most commonly used result is Barbalat's Lemma (Barbălat; 1959):

Lemma 4.1 (Barbălat) *Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a uniformly continuous function on $[0, \infty)$. Suppose that $\lim_{t\to\infty} \int_0^t \phi(\tau)d\tau$ exists and is finite. Then*

$$\phi(t) \to 0 \text{ as } t \to \infty.$$

Proof. *Khalil (2002, Lemma 8.2).* ■

A corollary is the following:

Corollary 4.2 *If a function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfies $\phi, \dot{\phi} \in \mathcal{L}_\infty$ and $\phi \in \mathcal{L}_p$ for some $p \in [1, \infty)$, then $\phi(t) \to 0$ as $t \to \infty$.*

Blending Lyapunov's direct method and Barbalat's Lemma gives the theorem due to LaSalle (1968) and Yoshizawa (1966). This is next extended in terms of stability of closed, forward invariant sets:

Theorem 4.3 (LaSalle-Yoshizawa) *Let a closed set $\mathcal{A} \subset \mathbb{R}^n$ be a forward invariant set for (2.4). Suppose for each $K \in [0, \infty)$ there exists $L \in [0, \infty)$ such that $|x|_{\mathcal{A}} \leq K \Rightarrow |f(x)| \leq L$. Then, if there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that*

$$\alpha_1 \left(|x|_{\mathcal{A}}\right) \leq V(x) \leq \alpha_2 \left(|x|_{\mathcal{A}}\right) \tag{4.1}$$

$$\dot{V} = V^x(x)f(x) \leq -\alpha_3 \left(|x|_{\mathcal{A}}\right) \leq 0, \tag{4.2}$$

$\forall x \in \mathbb{R}^n$, where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and α_3 is a continuous positive semidefinite function, then $\mathcal{A}$ is UGS with respect to (2.4) and

$$\lim_{t\to\infty} \alpha_3 \left(|x(t, x_0)|_{\mathcal{A}}\right) = 0. \tag{4.3}$$

If α_3 is strengthened to continuous positive definite, then $\mathcal{A}$ is UGAS with respect to (2.4).

Proof. *Integration of (4.2) and using the bounds in (4.1) imply that for each $x_0 \in \mathbb{R}^n$, $\exists K \geq 0$ such that*

$$
\begin{aligned}
|x(t,x_0)|_{\mathcal{A}} &\leq \alpha_1^{-1}\left(V(x(t,x_0))\right) \leq \alpha_1^{-1}\left(V(x_0)\right) \\
&\leq \alpha_1^{-1}\left(\alpha_2\left(|x_0|_{\mathcal{A}}\right)\right) = \varphi\left(|x_0|_{\mathcal{A}}\right) \leq K,
\end{aligned} \tag{4.4}
$$

holds for all t in the maximal interval of existence $[0,T)$, where $\varphi(\cdot) := \alpha_1^{-1}(\alpha_2(\cdot)) \in \mathcal{K}_\infty$ is independent of T. The bound (4.4) implies by assumption that there exists $L \geq 0$ such that $|f(x(t,x_0))| \leq L$, $\forall t \in [0,T)$. Integration along the solutions of (2.4) then yields

$$
|x(t,x_0) - x_0| \leq \int_0^t |f(x(s,x_0))|\, ds \leq \int_0^t L ds \leq Lt,
$$

$\forall t \in [0,T)$, thus excluding finite escape time so that $T = \infty$. UGS (and UGAS in the case α_3 is positive definite) follows then directly by Theorem 2.5. Since V is nonincreasing and bounded from below by zero, it has a limit V_∞ as $t \to \infty$. Integrating (4.2) gives

$$
\begin{aligned}
\lim_{t\to\infty} \int_0^t \alpha_3\left(|x(s,x_0)|_{\mathcal{A}}\right) ds &\leq -\lim_{t\to\infty} \int_0^t \dot{V}(x(s,x_0)) ds \\
&= \lim_{t\to\infty} \left\{V(x_0) - V(x(t,x_0))\right\} \\
&= V(x_0) - V_\infty
\end{aligned}
$$

which shows that the first integral exists and is finite. We next show that $t \mapsto \alpha_3\left(|x(t,x_0)|_{\mathcal{A}}\right)$ is uniformly continuous on $\mathbb{R}_{\geq 0}$. For each $\varepsilon > 0$ we let $\delta := \varepsilon / L$, and for any $t_1, t_2 \in \mathbb{R}_{\geq 0}$ with $|t_2 - t_1| \leq \delta$ we get

$$
|x(t_2,x_0) - x(t_1,x_0)| \leq \int_{t_1}^{t_2} |f(x(s,x_0))|\, ds \leq L\,|t_2 - t_1| \leq \varepsilon
$$

which shows that the solution $x(t,x_0)$ is uniformly continuous. Next,

$$
\begin{aligned}
\left||x|_{\mathcal{A}} - |y|_{\mathcal{A}}\right| &= \left|\inf_{v\in\mathcal{A}} |x - v| - \inf_{w\in\mathcal{A}} |y - w|\right| \\
&\leq \left||x - s| - |y - s|\right|, \qquad s \in \mathcal{A} \\
&\leq |x - y|, \qquad \forall x, y \in \mathbb{R}^n,
\end{aligned}
$$

shows that $|\cdot|_{\mathcal{A}}$ is globally Lipschitz with Lipschitz constant equal to unity, and consequently, $|\cdot|_{\mathcal{A}}$ is uniformly continuous. Finally, since α_3 is continuous, it is uniformly continuous on the compact set $\{s \in \mathbb{R}_{\geq 0} : s \leq K\}$. Putting this together we conclude that $t \mapsto \alpha_3\left(|x(t,x_0)|_{\mathcal{A}}\right)$ is uniformly continuous, and $\lim_{t\to\infty} \alpha_3\left(|x(t,x_0)|_{\mathcal{A}}\right) = 0$ follows from Lemma 4.1. ■

Another important tool for convergence analysis is the invariance principle that can be used to prove convergence to an equilibrium in the case when the Lyapunov function only yields a negative semidefinite time derivative. One version is due to Krasovskii (1959), while another is given by LaSalle (1960) (see Rouche et al. (1977, Theorem 1.3, pp. 50-51) and Khalil (2002, Theorem 4.4, p. 128)). Since these theorems either require periodic systems or solutions that live in compact sets, they are usually not applicable to general time-varying systems. We omit these results here.

A powerful theorem that is applicable for time-varying systems and stability analysis of noncompact sets, is the theorem of Matrosov (1962). A version of this theorem, applicable to closed, forward invariant sets, is stated here as presented by Teel et al. (2002):

Theorem 4.4 (Matrosov) *Suppose the system (2.4) is finite escape-time detectable through* $|\cdot|_{\mathcal{A}}$, *and* $f(x)$ *is continuous. If there exist:*

- *a locally Lipschitz function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$,
- *a continuous function* $U : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *that for each pair of strictly positive real numbres* $\delta \leq \Delta$, *is uniformly continuous on*

$$\mathcal{H}_{\mathcal{A}}(\delta, \Delta) := \{x \in \mathbb{R}^n : \ \delta \leq |x|_{\mathcal{A}} \leq \Delta\}$$

- *class-*$\mathcal{K}_\infty$ *functions* α_1 *and* α_2,

such that

1. $\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$ for all $x \in \mathbb{R}^n$,
2. $V^x(x)f(x) \leq -U(x) \leq 0$ for a.a. $x \in \mathbb{R}^n$,

and, for each pair of strictly real numbers $\delta \leq \Delta$,

- a $\mathcal{C}^1$ function $W : \mathbb{R}^n \to \mathbb{R}$,
- strictly positive real numbers ε_1, ε_2, and ψ

such that

3. $\max\{|W(x)|, |f(x)W(x)|\} \leq \psi$ for all $x \in \mathcal{H}_{\mathcal{A}}(\delta, \Delta)$,
4. $x \in \mathcal{H}_{\mathcal{A}}(\delta, \Delta) \cap \{\xi \in \mathbb{R}^n : \ U(\xi) \leq \varepsilon_1\} \Rightarrow |W^x(x)f(x)| \geq \varepsilon_2$,

then, for the system (2.4), the set $\mathcal{A}$ *is UGAS.*

See Teel et al. (2002) for the proof. Another useful extension of Matrosov's Theorem is the version by Loría et al. (2002) where a family of auxiliary functions V_i, $i \in \{1, \ldots, j\}$, are used, instead of a single function W as above, to provide UGAS of the origin of a time-varying system. Consider the system (1.1) and suppose that the origin $x = 0$ is an equilibrium. Then:

Theorem 4.5 *The origin of the system (1.1) is UGAS under the following assumptions:*

1. *The origin of the system (1.1) is UGS.*

2. *There exist integers* $j, m > 0$ *and for each* $\Delta > 0$ *there exist*

   - *a number* $\mu > 0$,
   - *locally Lipschitz continuous functions* $V_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, \ldots, j\}$,
   - *a (continuous) function* $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, $i \in \{1, \ldots, j\}$,
   - *continuous functions* $Y_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i \in \{1, \ldots, j\}$,

   such that, for a.a. $(x, t) \in \mathcal{B}^n(\Delta) \times \mathbb{R}$,

   $$\begin{array}{rcl} \max\{|V_i(x,t)|, |\phi(x,t)|\} & \leq & \mu \\ V_i^x(x,t)f(x,t) + V_i^t(x,t) & \leq & Y_i(x, \phi(x,t)) \end{array}$$

   where $\mathcal{B}^n(r) := \{x \in \mathbb{R}^n : \ |x| \leq r\}$.

3. *For each integer* $k \in \{1, \ldots, j\}$ *we have that*

   $$\begin{gathered} \{(z, \psi) \in \mathcal{B}^n(\Delta) \times \mathcal{B}^m(\mu), \ Y_i(z, \psi) = 0, \ \forall i \in \{1, \ldots, k-1\}\} \\ \Downarrow \\ \{Y_k(z, \psi) \leq 0\}. \end{gathered}$$

4. *We have that*

   $$\begin{gathered} \{(z, \psi) \in \mathcal{B}^n(\Delta) \times \mathcal{B}^m(\mu), \ Y_i(z, \psi) = 0, \ \forall i \in \{1, \ldots, j\}\} \\ \Downarrow \\ \{z = 0\}. \end{gathered}$$

See Loría et al. (2002) for the proof.

Chapter 5

Partial set-stability for interconnected systems

Consider the interconnected system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u_1) \\ \dot{x}_2 &= f_2(x_1, x_2, u_2) \end{aligned} \tag{5.1}$$

where $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$ are the states, $u_1(t) \in \mathcal{U}_1 \subset \mathbb{R}^{m_1}$ and $u_2(t) \in \mathcal{U}_2 \subset \mathbb{R}^{m_2}$ are inputs where $\mathcal{U}_1$,$\mathcal{U}_2$ are compact sets, and the vector fields f_1, f_2 are smooth. We investigate stability of the set

$$\mathcal{A} := \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : |x_1|_{\mathcal{A}_1} = 0\}, \tag{5.2}$$

where $\mathcal{A}_1 \subset \mathbb{R}^{n_1}$ is a compact set (for instance an equilibrim point $x_1 = 0$). In this case, we get that $|x|_{\mathcal{A}} = |x_1|_{\mathcal{A}_1}$ where $x := \mathrm{col}(x_1, x_2)$.

The next lemma will be used to guarantee forward completeness:

Lemma 5.1 *If for each compact set $\mathcal{X} \subset \mathbb{R}^{n_1}$ there exist $L > 0$ and $c > 0$ such that:*

$$|f_2(\xi, x_2, v)| \leq L\,|x_2| + c, \qquad \forall x_2 \in \mathbb{R}^{n_2}, \tag{5.3}$$

uniformly for all $(\xi, v) \in \mathcal{X} \times \mathcal{U}_2$, that is, f_2 satisfies a sector growth condition in x_2, then the system (5.1) is finite escape-time detectable through $|\cdot|_{\mathcal{A}}$.

Proof. *We need to show that for each $x_{20} = x_2(0)$, each bounded function $x_1(\cdot) \in \mathcal{X}$, and each input function $u_2(\cdot) \in \mathcal{U}_2$, then the solution $x_2(t) = x_2(t, x_{20}, x_1, u_2)$ exists for all $t \geq 0$. Define $y(x_2) := |x_2| = \sqrt{x_2^\top x_2}$ which is continuously differentiable on $\mathbb{R}^{n_2}\backslash\{0\}$. Time-differentiation gives*

$$\frac{d}{dt} y\,(x_2(t)) = \frac{1}{|x_2(t)|} x_2(t)^\top f_2(x_1(t), x_2(t), u_2(t)) \leq Ly\,(x_2(t)) + c \tag{5.4}$$

which in view of Lemma 1.2 (and the proof of Proposition 1.1) shows that $y(x_2(t))$ *(and therefore* $x_2(t)$*) is bounded on the maximal interval of existence* $[0, T)$*. Assume that* $x_2(t)$ *has a finite escape-time at* $T < \infty$*. Then, for each* $M < \infty$ *there exists* $\tau \in [0, T)$ *such that* $|x_2(\tau)| > M$*. However, this contradicts boundedness of* $x_2(t)$ *on* $[0, T)$*, and the solution* $x_2(t)$ *must exist for all* $t \geq 0$*. As a result, the solution of (5.1) can only escape to infinity if* $x_1(t)$ *grows unbounded, but this must necessarily be detected through* $|x|_{\mathcal{A}} = |x_1|_{\mathcal{A}_1}$*.* ■

This gives the following stability result for (5.2) with respect to (5.1):

Theorem 5.2 *Assume that the sector bound (5.3) in Lemma 5.1 holds for (5.1). If, in addition, there exist a smooth function* $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ *and* $\mathcal{K}_\infty$*-functions* α_i*,* $i = 1, \ldots, 4$*, such that*

$$\alpha_1\left(|x_1|_{\mathcal{A}_1}\right) \leq V(x_1, x_2) \leq \alpha_2\left(|x_1|_{\mathcal{A}_1}\right) \tag{5.5}$$

and

$$\begin{aligned} &V^{x_1}(x_1, x_2) f_1\left(x_1, x_2, u_1\right) \\ &\quad + V^{x_2}(x_1, x_2) f_2\left(x_1, x_2, u_2\right) \leq -\alpha_3\left(|x_1|_{\mathcal{A}_1}\right) + \alpha_4\left(|u|\right) \end{aligned} \tag{5.6}$$

hold, where $u := \mathrm{col}(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$*, then the system (5.1) is ISS with respect to the closed,* 0*-invariant set (5.2). In the case when* $u_1 = 0$ *and* $u_2 = 0$ *then the closed, forward invariant set (5.2) is UGAS with respect to (5.1), and if* $\alpha_i(|x|_{\mathcal{A}_1}) = c_i\, |x|^r_{\mathcal{A}_1}$ *for* $i = 1, 2, 3$*, where* c_1, c_2, c_3, r *are strictly positive reals with* $r \geq 1$*, then (5.2) is UGES with respect to (5.1).*

Proof. *Since*

$$\begin{aligned} \tfrac{d}{dt} V(x_1(t), x_2(t)) &\leq -\alpha\left(V(x_1(t), x_2(t))\right) + \alpha_4\left(|u(t)|\right) \\ &\leq -\tfrac{1}{2}\alpha\left(V(x_1(t), x_2(t))\right), \end{aligned}$$

for all $V(x_1(t), x_2(t)) \geq \alpha^{-1}\left(2\alpha_4\left(|u(t)|\right)\right)$ *where* $\alpha = \alpha_3 \circ \alpha_1^{-1} \in \mathcal{K}_\infty$ *and* u *is bounded, then* $V(x_1(t), x_2(t))$*, and consequently* $|x_1(t)|_{\mathcal{A}_1}$*, is bounded on the maximal interval of existence* $[0, T)$*. Since* $\mathcal{A}_1$ *is compact this implies that* $x_1(t)$ *is bounded on* $[0, T)$*. By Lemma 5.1 this means that the system is finite escape-time detectable through* $|\cdot|_{\mathcal{A}}$*, and forward completeness follows. The fact that* $\mathcal{A}$ *is* 0*-invariant for (5.1) follows from the above Lyapunov bounds with* $u(t) \equiv 0$*. Recall Definition 3.3 and Theorem 3.4. Since* $|x|_{\mathcal{A}} = |x_1|_{\mathcal{A}_1}$*, the function* V *is a smooth ISS-Lyapunov function for (5.1) with respect to* $\mathcal{A}$*, and this proves ISS. UGAS in the case when* $u_1 = 0$ *and* $u_2 = 0$ *follows from the definition of ISS, and UGES further follows from Theorem 2.5.* ■

Remark 5.1 *By defining the output* $y = h(x_1, x_2) := |x_1|_{\mathcal{A}_1}$ *for (5.1), then ISS for the set* $\mathcal{A}$ *is equivalently characterized by the concept called State-Independent*

Input-to-Output Stability (SIIOS) as defined by Sontag and Wang (2000). Indeed, the smooth function V in (5.5) and (5.6) becomes a SIIOS-Lyapunov function for (5.1), and this can be used to deduce that

$$|y(t, x_0, u)| = |x_1(t, x_0, u)|_{\mathcal{A}_1} \leq \beta\left(|x_{10}|_{\mathcal{A}_1}, t\right) + \gamma\left(\|u\|\right) \tag{5.7}$$

where $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$. Since $|x|_{\mathcal{A}} = |x_1|_{\mathcal{A}_1}$ ISS of the system (5.1) with respect to the closed, 0-invariant set (5.2) follows from (5.7). The converse also holds as shown in Sontag and Wang (2000).

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