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Some Comments on the GRAS Method

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ABSTRACT *Junius and Oosterhaven (2003) present a RAS matrix balancing variant that can incorporate negative elements in the balancing. There are, however, a couple of issues in the approach described – the first being the handling of zeros in the initial estimate, and the second being the formulation of their minimum-information principle. We present a corrected exposition of GRAS.*

KEY WORDS: RAS matrix balancing, GRAS, optimisation

1. Introduction

Junius and Oosterhaven (2003, hereafter J&O) describe a RAS matrix balancing variant called GRAS, which deals with an efficient and theoretically sound way of balancing a matrix containing negative entries, to prescribed row and column totals. In their paper, J&O derive GRAS from a minimum-information principle (see also Oosterhaven, 2005)

2. Problems in GRAS

We believe there are some problems in Junius and Oosterhaven's (2003) approach.

2.1. *Definition of z_{ij} for $a_{ij} = 0$*

J&O's definition of the variable $z_{ij} = x_{ij}/a_{ij}$ as the ratio of the adjusted table elements x_{ij} and the initial estimate a_{ij} does not work with their target function $t(\mathbf{A}, \mathbf{Z}) = \Sigma_{ij}|a_{ij}|z_{ij}$ $\ln(z_{ij})$. J&O set $z_{ij} \equiv 0$ for $a_{ij} = 0$. However, the target function cannot be evaluated for $z_{ij} = 0$ since ln(0) is not defined. That is, unless one defines inside the target function that $z_{ij} \ln(z_{ij}) \equiv 0$ for $z_{ij} = 0$. Assuming that the GRAS algorithm is to preserve zero

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values, one can assume that if $a_{ij} = 0$, then $x_{ij} = 0$ also. Therefore, it is better to set $z_{ij} \equiv lim_{\varepsilon\to0}(\varepsilon/\varepsilon) = 1$ for $a_{ij} = 0$.

2.2. *The Target Function* $t(\mathbf{A}, \mathbf{Z}) = \Sigma_{ij}|a_{ij}|z_{ij} \ln(z_{ij})$

J&O's target function $t(\mathbf{A}, \mathbf{Z}) = \Sigma_{ij}|a_{ij}|z_{ij} \ln(z_{ij})$ does not assume a minimum for $x_{ij} = a_{ij}$ as required for reflecting the distance between, or similarity of, **X** and **A**. Figure 1 illustrates this for the one-dimensional case $t(a,z) = |a|z\ln(z) = |a|(x/a)\ln(x/a)$ with $a = 2$, so that $t(x) = x \ln(x/2)$. Figure 1 (dashed curve) clearly shows clearly a minimum for $x = 2/e$, and not for $x = 2$. If however the target function is modified to $t'(\mathbf{A},\mathbf{Z}) = \Sigma_{ij}|a_{ij\text{-}}$ $'(\mathbf{A},\mathbf{Z}) = \Sigma_{ij}|a_{ij}|z_{ij} \ln(z_{ij}/e)$ with $z_{ij} = x_{ij}/a_{ij}$, condensing to $t(x) = x \ln(x/2e)$ in the one-dimensional case, a minimum occurs at $x = 2$ as required (solid curve). We will exploit this modification further on in the fourth section.

One can construct a case where J&O's GRAS balancing algorithm leads to a solution **X** that is inferior to the initial estimate in terms of their target function. This can be seen for the trivial case of starting with an initial estimate **A** that already satisfies all prescribed row and column totals, that is $\Sigma_j a_{ij} = u_i$, and $\Sigma_i a_{ij} = v_j$. In this case $z_{ij}^{\mathrm{A}} = 1\ \forall i, j$, and $t(\mathbf{A},\mathbf{Z}^{\mathbf{A}}) = 0$ (compare Figure 1). The initial estimate should be the optimal solution.

However, problems show up throughout J&O's exposition. First, J&O's solutions

$$z_{ij} = e^{\lambda i} e^{\tau j} e^{-1} \quad \text{for } a_{ij} \geq 0 \tag{1a}$$

$$z_{ij} = e^{-\lambda i} e^{-\tau j} e^{-1} \quad \text{for } a_{ij} < 0 \tag{1b}$$

cannot work, since in this trivial case *all* z_{ij}^{A} must equal 1, irrespective of the sign of a_{ij}. No selection of λ_i and τ_j can fulfil this.

Second, using J&O's balancing algorithm – i.e. their equations (20)–(25) – but starting with a perfect initial estimate, the result will actually move *away* from that perfect solution. This is because J&O's GRAS algorithm actually balances any initial estimate towards scaled row and column totals $\mathbf{u}^* = e\mathbf{u}$ and $\mathbf{v}^* = e\mathbf{v}$, instead of prescribed totals

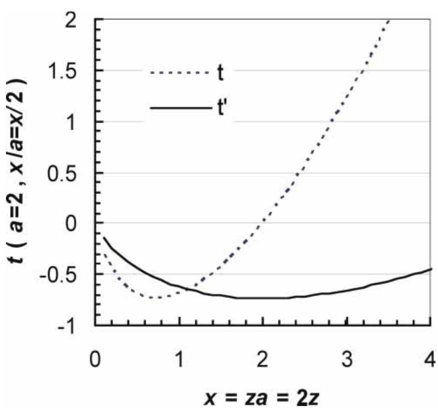

Figure 1. J&O's target function $t\,(\mathbf{A}, \mathbf{Z})$, plus a modified target function $t'(\mathbf{A}, \mathbf{Z})$ for a one-dimensional case

u and **v**. This scaling results directly out of the Lagrangean optimality condition for their target function. So, one would start with a perfect solution **A** satisfying totals **u** and **v**, but then unbalance it until it satisfies $\mathbf{u}^*$ and $\mathbf{v}^*$ – see J&O's solution conditions in their equations (9) and (10) – and then scale it down by a factor *e*. That GRAS solution **X** may minimize J&O's target function *t*, but is clearly inferior to the initial estimate **A**. This is also reflected in Figure 1. The following section will give a numerical example in order to demonstrate the magnitude of this effect.

3. A Numerical Example

Consider the following row-sum- and column-sum-constrained 2×2 sector table, and initial estimate:

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix}; \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix} \quad (2)$$
$$\begin{pmatrix} 1 & 5 \end{pmatrix}$$

In contrast, J&O's GRAS algorithm would start with **A** and scale this to satisfy row and column sums $\mathbf{u}^* = e\mathbf{u} = (2.72, 13.59)$ and $\mathbf{v}^* = e\mathbf{v} = (2.72, 13.59)$. **X** is 'adjusted away' from **A**, thus increasing the value of the target function $t(\mathbf{A}, \mathbf{Z}) = \Sigma_{ij}|a_{ij}|z_{ij}\ln(z_{ij})$ (see Table 1).

After downscaling by *e*, J&O's final solution is $\mathbf{X} = (-0.357, 1.357, 1.357, 3.643)$. This solution satisfies all row- and column-sum constraints, but its target function value of $t(\mathbf{A}, \mathbf{Z}) = -0.713$ shows that it is clearly inferior to $\mathbf{A} = (-1,2,2,3)$, simply because of its distance to **A**. A correct GRAS algorithm would stop during the very first step,

Table 1. J&O's GRAS algorithm for the constrained table in equation (2)

	Variables				J&O target function		Constraint adjustment			
Step	x_1	x_2	x_3	x_4	$t(\mathbf{A}, \mathbf{Z})$	Constraint	u_1	u_2	v_1	v_2
1	−0.602	2.000	3.321	3.000	−2.776	1	**2.72**	5.0	1.4	6.32
1	−0.602	5.437	3.321	8.155	−0.936	2	2.72	**13.59**	4.83	11.48
1	−0.904	3.622	3.321	8.155	−1.509	3	2.42	11.78	**2.72**	11.48
1	**−0.904**	**3.622**	**3.933**	**9.659**	**−0.774**	4	3.03	13.28	2.72	**13.59**
2	−0.965	3.622	3.683	9.659	−0.835	1	**2.72**	13.28	2.66	13.34
2	−0.965	3.707	3.683	9.884	−0.718	2	2.72	**13.59**	2.74	13.57
2	−0.970	3.688	3.683	9.884	−0.722	3	2.71	13.57	**2.72**	13.57
2	**−0.970**	**3.688**	**3.690**	**9.902**	**−0.713**	4	2.72	13.59	2.72	**13.59**
3	−0.970	3.688	3.689	9.902	−0.714	1	**2.72**	13.59	2.72	13.59
3	−0.970	3.689	3.689	9.903	−0.713	2	2.72	**13.59**	2.72	13.59
3	−0.970	3.689	3.689	9.903	−0.713	3	2.72	13.59	**2.72**	13.59
3	**−0.970**	**3.689**	**3.689**	**9.903**	**−0.713**	4	2.72	13.59	2.72	**13.59**

Note: One GRAS step is defined as one cycle through all four constraints (end of each cycle printed in bold in left part of Table). The variables x_i are scaled in order to satisfy row-sum- and column-sum-constraints u_i and v_i in turn (bold in right part of Table).

and return **A** as the solution, since **A** satisfies all four constraints. This is the subject of the following two sections.

4. A Consistent Solution

We believe that the target function must take the following form in order to avoid this unwanted effect (compare Bacharach, 1970, pp. 80, 85–86; see also Figure 1):

$$t'(\mathbf{A}, \mathbf{Z}) = \Sigma_{ij}|a_{ij}|z_{ij} \ln(z_{ij}/e) \qquad \text{with } z_{ij} = \frac{x_{ij}}{a_{ij}} \tag{3}$$

The derivative of the GRAS Lagrangean

$$\mathcal{L}(\mathbf{Z}, \lambda, \tau) = \sum_{(i,j)\in P} a_{ij}z_{ij} \ln(z_{ij}/e) - \sum_{(i,j)\in N} a_{ij}z_{ij} \ln(z_{ij}/e) + \sum_i \lambda_i \left[u_i - \sum_j a_{ij}z_{ij}\right] + \sum_i \tau_j \left[v_j - \sum_i a_{ij}z_{ij}\right]$$

is then

$$\frac{\partial \mathcal{L}}{\partial z_{ij}} = a_{ij}\ln\frac{z_{ij}}{e} + a_{ij} - \lambda_i a_{ij} - \tau_j a_{ij} \quad \text{for } a_{ij} \geq 0 \tag{4a}$$

$$\frac{\partial \mathcal{L}}{\partial z_{ij}} = -a_{ij}\ln\frac{z_{ij}}{e} - a_{ij} - \lambda_i a_{ij} - \tau_j a_{ij} \quad \text{for } a_{ij} < 0. \tag{4b}$$

Imposing the optimality condition yields

$$\ln z_{ij} - \lambda_i - \tau_j = 0 \Longleftrightarrow z_{ij} = e^{\lambda_i}e^{\tau_j} \quad \text{for } a_{ij} \geq 0 \tag{5a}$$

$$-\ln z_{ij} - \lambda_i - \tau_j = 0 \Longleftrightarrow z_{ij} = e^{-\lambda_i}e^{-\tau_j} \quad \text{for } a_{ij} < 0 \tag{5b}$$

Calling $r_i = e^{\lambda_i}$ and $s_j = e^{\tau_j}$, the target matrix is then

$$x_{ij} = r_i a_{ij} s_j \quad \text{for } a_{ij} \geq 0 \tag{6a}$$

$$x_{ij} = r_i^{-1} a_{ij} s_j^{-1} \quad \text{for } a_{ij} < 0 \tag{6b}$$

with the diagonal matrices $\hat{\mathbf{r}}$ and $\hat{\mathbf{s}}$ solving

$$(\hat{\mathbf{r}}\mathbf{P}\hat{\mathbf{s}} - \hat{\mathbf{r}}^{-1}\mathbf{N}\hat{\mathbf{s}}^{-1})\mathbf{i} = \mathbf{u} \tag{7a}$$

and

$$\mathbf{i}(\hat{\mathbf{r}}\mathbf{P}\hat{\mathbf{s}} - \hat{\mathbf{r}}^{-1}\mathbf{N}\hat{\mathbf{s}}^{-1}) = \mathbf{v} \tag{7b}$$

instead of J&O's solution, which leads to $\mathbf{u}^* = e\mathbf{u}$ and $\mathbf{v}^* = e\mathbf{v}$. In equations (7a) and (7b), $\mathbf{P}$ and $\mathbf{N}$ are a decomposition of $\mathbf{A} = \mathbf{P} - \mathbf{N}$ into a matrix $\mathbf{P}$ containing the positive elements of $\mathbf{A}$, and $\mathbf{N}$ containing the absolutes of the negative elements of $\mathbf{A}$.

5. Iterative Formulation

The iterative solution of equations (7a) and (7b) becomes:

Step 1. Choose $\mathbf{r}(0) = \mathbf{i}$ (for justification, see J&O, p. 93)
Step 2. Calculate $\hat{\mathbf{s}}(1) = \hat{\boldsymbol{\sigma}}\hat{\mathbf{r}}(0)$ using equation (8a) below
Step 3. Calculate $\hat{\mathbf{r}}(1) = \hat{\boldsymbol{\rho}}\hat{\mathbf{s}}(1)$ using equation (8b) below
Step 4. Continue for $\hat{\mathbf{s}}(k) = \hat{\boldsymbol{\sigma}}\hat{\mathbf{r}}(k-1)$ and $\hat{\mathbf{r}}(k) = \hat{\boldsymbol{\rho}}\hat{\mathbf{s}}(k)$, for $k = 1,2,..,n$
Step n. Exit when $\hat{\mathbf{s}}(n) - \hat{\mathbf{s}}(n-1) < \varepsilon$ for some chosen $\varepsilon > 0$
Step $n+1$. $\mathbf{X} = \hat{\mathbf{r}}(n)\mathbf{P}\hat{\mathbf{s}}(n) - \hat{\mathbf{r}}(n)^{-1}\mathbf{N}\hat{\mathbf{s}}(n)^{-1}$

utilising

$$\rho_i(s) = \frac{u_i + \sqrt{u_i^2 + 4p_i(s)n_i(s)}}{2p_i(s)} \tag{8a}$$

and

$$\sigma_j(r) = \frac{v_j + \sqrt{v_j^2 + 4p_j(r)n_j(r)}}{2p_j(r)} \tag{8b}$$

where

$$p_i(s) = \sum_j p_{ij}s_j,\ p_j(s) = \sum_i p_{ij}s_i,\ n_i(s) = \sum_j \frac{n_{ij}}{s_j},\ \text{and } n_j(s) = \sum_i \frac{n_{ij}}{s_i} \tag{9}$$

6. Conclusion

Using the target function t' as in equation (3), the problems noted in the first section do not arise. First, for $\lambda_i = \tau_j = 0$, all $z_{ij}^{\mathbf{A}}$ equal 1, irrespective of the sign of a_{ij}. Second, using J&O's balancing procedure with t', starting with a perfect initial estimate will terminate the algorithm.

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