Support varieties for modules and complexes

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Introduction

Around 1960 it was shown that the group cohomology ring of a finite group \( G \) is a noetherian ring (see [Ev, Go, V]). Further results on the structure of the group cohomology rings was obtained in [Q]. In the seminal papers [Carl] and [Carl2] J. F. Carlson defined a support variety of any finitely generated module over \( G \) whose dimension is given by the complexity of the module. The complexity of a module was defined by J. Alperin in 1977 as the rate of growth of a minimal projective resolution of the given module. A stratification of the support variety of a module in terms of the elementary abelian subgroups of \( G \) was given in [AvrS]. These papers mark the genesis of the theory of support varieties in the various settings it has been considered to far. Here, the focus is on giving a review of a recent extension of the theory of support varieties for group rings to a theory defined for every finite dimensional algebra in [SS]. The support varieties in [SS] are defined using the Hochschild cohomology ring of \( \Lambda \). With this as a starting point the theory was advanced in [EHSST]. In particular, finiteness conditions are found in [EHSST] such that most of the essential properties and some of the applications of support varieties for group rings can be generalized. We show that the finiteness conditions given in [EHSST] are equivalent to the Hochschild cohomology ring being noetherian and the Ext-algebra of the direct sum of the simples being a finitely generated module over the Hochschild cohomology ring. When these properties are satisfied for a finite dimensional (selfinjective) algebra \( \Lambda \), a compact way of summarizing this paper is to say that the geometry of the Hochschild cohomology ring relates to the homological algebra of \( \Lambda \)-modules in a way very similar to that of the group ring case.

The presentation in this paper follows quite closely the one given in the lectures in the workshop. The paper is organized as follows. The first section is devoted to expanding the motivation for studying support varieties given above through a more detailed treatment of the group ring case and a brief encounter with the support varieties for complete intersections [Av, AvB]. Here we also recall some of the most essential results for the group ring case. The support varieties defined in [SS] use the Hochschild cohomology ring, so in the second section we give a short review of the elementary properties of the Hochschild cohomology ring of an algebra with sketching some of the proofs of these well-known facts. The support varieties

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we consider in this paper use the action of the Hochschild cohomology ring of an
algebra $\Lambda$ on the direct sum $\Ext^*_\Lambda(M,N) = \bigoplus_{i \geq 0} \Ext^i_\Lambda(M,N)$ for any pair $M$ and
$N$ of $\Lambda$-modules. This action and properties of it are discussed in the third section.
In the fourth section the definition of the support variety from $[SS]$ of a finitely
generated module over a finite dimensional algebra $\Lambda$ is given. Some elementary
properties from $[SS]$ are given and the need for further assumptions are pointed out.
The finiteness conditions that make the theory of support varieties work for us are
discussed in the fifth section. They are shown to be equivalent to the Hochschild
cohomology ring being noetherian and the Ext-algebra of the direct sum of the
simples being a finitely generated module over the Hochschild cohomology ring.
With these assumption some of the essential properties of the support varieties for
group rings are generalized. The sixth section contains the most important result
for our theory of support varieties from $[EHSST]$. It gives a construction for how
to cut down the support variety of a module by tensoring with certain bimodules.
As a consequence every closed homogeneous variety is shown to occur as the variety
of a module. Again, we recover an analogue of a result for group rings. The next
section is devoted to giving some applications of the theory (see $[EHSST]$). Here
we characterize periodic modules for selfinjective algebras satisfying our finiteness
conditions, show how extensions between two modules can be controlled through
how their varieties intersect, and show that the variety of an indecomposable module
is connected. All of these applications are generalizations of similar results for group
rings. This is also the case for the generalization of Webb’s theorem on the shape
of the stable connected Auslander-Reiten quiver of a selfinjective algebra where the
Nakayama functor has finite order on all indecomposable modules. In the eighth
section the relationship between complexity and representation type is discussed.
Koszul algebras where the degree zero part is a direct sum of copies of the field is
the theme of section nine (see $[BGMS, BGSS, GHMS]$). For a Koszul algebra $\Lambda$
the minimal projective resolution of the degree zero part $\Lambda_0$ of $\Lambda$ over $\Lambda$, is shown
to have a “comultiplicative” structure, which is used to find a minimal projective
resolution of $\Lambda$ over the enveloping algebra $\Lambda^e$ of $\Lambda$ (see $[GHMS]$). This is shown
to give closed formulas for the multiplication in the Koszul dual and the Hochschild
cohomology ring of $\Lambda$ (see $[BGSS]$). A result of D. Eisenbud ($[Ei]$) is reproved using
the theory developed in this paper. The section ends with pointing out that the
Koszul duals of Artin-Schelter regular noetherian Koszul algebras which are finitely
generated modules over their ordinary centre, satisfy our finiteness condition for
having a good theory of support varieties. Support varieties for complexes and a
general setup for support varieties are discussed in the tenth section. The paper
ends with a short discussion on some differences with the group ring case and some
indications of further investigations.

For a ring $\Lambda$ we denote by $\Mod \Lambda$ the category of all left $\Lambda$-modules and by
$\mod \Lambda$ the full subcategory consisting of all finitely presented modules.

1. Motivation

Our motivation in $[SS]$ for considering support varieties for finitely generated
modules over a finite dimensional algebra came primarily from observing that the
support varieties for modules over complete intersections introduced by L. Avramov
$[Av]$ (and later $[AvB]$) could be given through the action of the Hochschild co-
homology ring on Ext-algebras of the module (in the equicharacteristic case). In
addition it was well-known that the support varieties for finitely generated modules over group rings of finite groups as introduced by J. F. Carlson in [Carl] could also be similarly defined through the action of the Hochschild cohomology ring on the Ext-algebra of a module. As there is a homomorphism of graded rings from the Hochschild cohomology ring of any finite dimensional algebra to the Ext-algebra of any module, this led us to believe that there should be a similar fruitful theory of support varieties using the Hochschild cohomology ring instead of the group cohomology ring for more general classes of finite dimensional algebras.

Support varieties have not only been considered in the two settings mentioned above, but also for (i) restricted Lie algebras (infinite group schemes of height at most one) ([FP, J]), (ii) $r$-th Frobenius kernel of smooth algebraic groups (infinite group schemes of height at most $r$) ([SFB]), (iii) finite dimensional cocommutative Hopf algebras (finite group schemes) ([FS]), (iv) special infinite groups being approximated by finite elementary $p$-groups [Ben2], (v) Steenrod algebras and sequence of finite dimensional cocommutative Hopf subalgebras ([(NP)]) and (vi) in stable homotopy categories ([(HPS)]).

Next we review the setup and some of the most important results for the group ring case and the cases included in (i)–(iii). We concentrate on the group ring case as this has served as a model for our investigations.

Let $G$ be a finite group, and let $k$ be an algebraically closed field. Assume that the characteristic of $k$ divides the order of $G$. The support varieties for finitely generated modules over the group ring $kG$ are defined by J. F. Carlson in terms of the group cohomology ring $H^*(G, k) = \text{Ext}^*_G(k, k) = \Pi_{i \geq 0} \text{Ext}^i_G(k, k)$ in [Carl].

The multiplication is given by the Yoneda product, and under this product $H^*(G, k)$ is naturally a positively graded ring. Furthermore, it is well-known that $H^*(G, k)$ is a graded commutative ring; that is, $xy = (-1)^{|x||y|}yx$ for all homogeneous elements $x$ and $y$ in $H^*(G, k)$ where $|z|$ denotes the degree of a homogeneous element $z$ in $H^*(G, k)$. The support variety of a finitely generated module $M$ over $kG$ is defined via a ring homomorphism $\gamma_M : H^*(G, k) \to \text{Ext}^*_G(M, M) = \Pi_{i \geq 0} \text{Ext}^i_G(M, M)$. To see how this map is defined one needs to note the following. If $X$ and $Y$ are two left $kG$-modules, then $X \otimes_k Y$ is a left $kG$-module via $g(x \otimes y) = gx \otimes gy$ for elements $x$ in $X$ and $y$ in $Y$. Then $k \otimes_k Y \simeq Y$ as $kG$-modules, which we view as an identification. Then the homomorphism $\gamma_M$ is given by

$$\gamma_M(0 \to k \to E_1 \to \cdots \to E_n \to k \to 0) = 0 \to k \otimes_k M \to E_1 \otimes_k M \to \cdots \to E_n \otimes_k M \to k \otimes_k M \to 0$$

This induces a left and a right $H^*(G, k)$-module structure on $\text{Ext}^*_G(M, N)$ for left $kG$-modules $M$ and $N$ via the ring homomorphisms $\gamma_N$ and $\gamma_M$, respectively. The whole theory of support varieties is based on the following two facts, (i) the group cohomology ring $H^*(G, k)$ is a finitely generated $k$-algebra ([(Ev, Go, V)]), and (ii) the $H^*(G, k)$-module $\text{Ext}^*_G(M, N)$ is finitely generated for all finitely generated left $kG$-modules $M$ and $N$ ([(Ev)]). We shall later explain why this is of such importance.

For the definition of the support variety let $H^*(G, k)$ be equal to $H^{\text{even}}(G, k)$ when the characteristic of $k$ is different from 2 and equal to $H^*(G, k)$ otherwise. Since $H^*(G, k)$ is graded commutative, any homogeneous element of even degree commutes with any other element and in characteristic two $H^*(G, k)$ is commutative. So $H^*(G, k)$ is always a finitely generated commutative $k$-algebra. Then the support variety $V_G(M)$ of a finitely generated left $kG$-module $M$ is given as the
closed homogeneous subvariety

\[ V_G(M) = \text{MaxSpec}(H^\bullet(G, k)/\text{Ann}_{H^\bullet(G, k)} \text{Ext}^1_{kG}(M, M)) \]

of \( V_G = \text{MaxSpec} H^\bullet(G, k) \). Note that \( \text{Ann}_{H^\bullet(G, k)} \text{Ext}^1_{kG}(M, M) = \ker \gamma_M|_{H^\bullet(G, k)} \).

Let \( N^* \) and \( N^* \) be the ideals generated by the homogeneous nilpotent elements in \( H^\bullet(G, k) \) and \( H^\bullet(G, k) \), respectively. The maximal (prime) ideals in \( H^\bullet(G, k) \) and \( H^\bullet(G, k) \) are in one-to-one correspondence with the maximal (prime) ideals in \( H^\bullet(G, k)/N^* \) and \( H^\bullet(G, k)/N^* \), respectively. Since \( H^\bullet(G, k) \) is graded commutative, the square of all homogeneous elements of odd degree in \( H^\bullet(G, k) \) is zero in characteristic different from two. Therefore \( H^\bullet(G, k)/N^* \) and \( H^\bullet(G, k)/N^* \) are isomorphic \( k \)-algebras in all characteristics, so that one could also define the support varieties always using \( H^\bullet(G, k) \).

The dimension of the variety \( V_G(M) \) is as usual defined as the Krull dimension of the ring \( H^\bullet(G, k)/\text{Ann}_{H^\bullet(G, k)} \text{Ext}^1_{kG}(M, M) \). Since \( H^0(G, k) = \text{Hom}_{kG}(k, k) = k \), the ring \( H^\bullet(G, k) \) is a graded local ring with unique graded maximal ideal \( H^\bullet_{2^1}(G, k) \). Since \( \text{Ann}_{H^\bullet(G, k)} \text{Ext}^1_{kG}(M, M) \) is a graded ideal, this ideal is always contained in \( H^\bullet_{2^1}(G, k) \). Hence the smallest possible variety that can occur is the set \( \{H^\bullet_{2^1}(G, k)\} \), which for instance is the support variety of any projective module. A module \( M \) is said to have trivial variety if \( V_G(M) = \{H^\bullet_{2^1}(G, k)\} \).

The dimension of \( V_G(M) \) is related to the complexity of a module (Alperin, 1977), which is defined as follows. Given a finitely generated left \( k \)-module \( M \) and a minimal projective resolution \( \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0 \), the complexity of \( M \) is

\[ c(M) = \min\{b \in \mathbb{N}_0 \mid \exists a \in \mathbb{R}; \dim_k P_n \leq an^{b-1}, \forall n \gg 0\} \]

Next we collect some of the most essential properties of the support varieties for group rings. Here \( D = \text{Hom}_{kG}(-, k) \) is the usual duality of modules, and \( \Omega_{kG}(M) \) denotes the first syzygy of \( M \). Also recall that a module \( M \) is said to be \( \Omega \)-periodic if \( M \approx \Omega^n_{kG}(M) \) for some \( n \geq 1 \).

**Theorem 1.1.** Let \( M, M' \) and \( \{M_1, M_2, M_3\} \) be finitely generated left \( k \)-modules. Then the following assertions hold.

(a) \( \dim V_G(M) = c(M) \).

(b) \( M \) is projective if and only if \( V_G(M) \) is trivial.

(c) Suppose that \( M \) is indecomposable. Then \( M \) is \( \Omega \)-periodic if and only if \( \dim V_G(M) = 1 \).

(d) If \( 0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0 \) is an exact sequence, then \( V_G(M_r) \subseteq V_G(M_s) \cup V_G(M_t) \) whenever \( \{r, s, t\} = \{1, 2, 3\} \).

(e) \( V_G(M \oplus M') = V_G(M) \cup V_G(M') \).

(f) \( V_G(M) = V_G(D(M)) = V(\Omega_{kG}(M)) \).

(g) Given any graded ideal \( \mathfrak{a} \) in \( H^\bullet(G, k) \), there exists \( M \) in \( \text{mod } kG \) such that \( V_G(M) = V(\mathfrak{a}) = \{m \in \text{MaxSpec } H^\bullet(G, k) \mid \mathfrak{a} \subseteq \mathfrak{m}\} \).

(h) \( V_G(M \otimes_k N) = V_G(M) \cap V_G(N) \).

(i) If \( M_1 \) and \( M_2 \) are two finitely generated \( k \)-modules where \( V_G(M_1) \cap V_G(M_2) \) is trivial, then \( \text{Ext}^1_{kG}(M_1, M_2) = (0) \) for all \( i > 0 \).

We return to these properties in our more general setting and investigate which of them generalize.

We conclude this section by giving more details on how the support varieties defined for finite group rings and how they can be similarly defined through the
action of the Hochschild cohomology ring on the Ext-algebra of a module. We also
discuss the support varieties for complete intersections briefly.

First observe that we may identify \((kG)^e\) with \(k(G \times G)\) via the isomorphism
\(\psi: (kG)^e \rightarrow k(G \times G)\) given by \(\psi(g \otimes h) = g \otimes h^{-1}\) for \(g\) and \(h\) in \(G\). View \(G\) as a
subgroup of \(G \times G\) via the diagonal map (comultiplication) \(\Delta: G \rightarrow G \times G\) given by
\(\Delta(g) = g \otimes g\) for \(g\) in \(G\). Then the natural induction functor \(\text{Ind}^G_{\Delta G}: \text{mod}kG \rightarrow \text{mod}(kG)^e\) is an exact functor, and therefore it induces a natural homomorphism

\[ H^*(G, k) = \text{Ext}^*_G(k, k) \rightarrow \text{Ext}^*_G(kG, kG) \]

of graded rings. Here \(\text{Ext}^*_G(kG, kG)\) is by definition the Hochschild cohomology ring \(\text{HH}^*(kG)\) of \(kG\). For a \(kG\)-module \(M\) the ring homomorphism \(\gamma_M: H^*(G, k) \rightarrow \text{Ext}^*_G(M, M)\) is induced from the exact functor \(- \otimes_k k\). Similarly, the functor
\[- \otimes_{kG} M: \text{mod}(kG) \rightarrow \text{mod}kG\]
can be shown to induce a homomorphism

\[ \varphi_M = - \otimes_{kG} M: \text{HH}^*(kG) \rightarrow \text{Ext}^*_G(M, M) \]

of graded rings. Using these homomorphisms it is straightforward to show that the diagram

\[ \begin{array}{ccc}
H^*(G, k) = \text{Ext}^*_G(k, k) & \rightarrow & \text{Ext}^*_G(M, M) \\
\text{Ind}^G_{\Delta G} & & \\
\text{HH}^*(kG) = \text{Ext}^*_G(kG, kG) & \varphi_M = - \otimes_{kG} M & \text{Ext}^*_G(M, M)
\end{array} \]

is commutative for any left \(kG\)-module \(M\). Consider \(kG\) as the adjoint representation
over \(kG\); that is, the left module structure of \(kG\) is induced by the ring homomorphism \((\text{id} \otimes S)\Delta: kG \rightarrow (kG)^e\), where \(g \mapsto g \otimes g^{-1}\) for \(g\) in \(G\). Then it is
known that \(\text{HH}^*(kG) \simeq \text{Ext}^*_G(k, kG)\). Using that \(\text{Ext}^*_G(M, N)\) is a \(kG\) module over \(H^*(G, k)\) it is straightforward to show that \(\text{HH}^*(kG)\) is a \(kG\) module over \(H^*(G, k)\). Let
\(\text{HH}^*(kG)\) be defined similarly to \(H^*(G, k)\). The Hochschild cohomology ring of any algebra is graded commutative, so it follows that \(\text{HH}^*(kG)\) is a \(kG\) module over \(H^*(G, k)\). Therefore it is a \(kG\)-module. If we define a new support variety

\[ V_G^I(M) = \{ m \in \text{MaxSpec} \text{HH}^*(kG) \mid \text{Ann}_{\text{HH}^*(kG)} \text{Ext}^*_G(M, M) \subseteq m \}, \]

the above observations imply that there is a finite surjective map \(V_G^I(M) \rightarrow V_G(M)\) for any \(kG\)-module \(M\). Hence these varieties are closely related.

We end this section by a short review of the support varieties for complete
intersections \(R \ ((\mathbf{A}^n, \mathbf{A}^n[1]))\) in the situation \(R = \mathbb{K}[x_1, x_2, \ldots, x_n]/(a_1, \ldots, a_t)\), where \(\{a_1, \ldots, a_t\}\) is a regular sequence in the square of the maximal ideal of \(R\).

For finitely generated \(R\)-modules \(M\) and \(N\) there are operators \(\{\gamma_i\}_{i=1}^t\) on
\(\text{Ext}^*_R(M, N)\), which define an \(R[\chi_1, \ldots, \chi_t]\)-module structure on \(\text{Ext}^*_R(M, N)\). Under
this module structure \(\text{Ext}^*_R(M, N)\) is a \(R[\chi_1, \ldots, \chi_t]\)-module for all \(R\)-modules \(M\) and \(N\) \([\text{Gu}]\). Then the support variety of a \(R\)-module \(M\) is given by

\[ V(M) = \{ (b_1, \ldots, b_t) \in k^t \mid \forall \varphi \in \text{Ann}_{k[\chi_1, \ldots, \chi_t]}(\text{Ext}^*_R(M, k) \otimes_R k), \varphi(b_1, \ldots, b_t) = 0 \} \]
These operations have various descriptions. Using the description found in [Me],
the operators \( \{ \chi_i \}_{i=1}^t \) can be shown to correspond to elements in \( HH^2(R) \) ([SS]).
It follows that the varieties are defined by the action of a commutative noetherian graded subalgebra of the Hochschild cohomology ring \( HH^*(R) \), since all the operators have degree 2. Using the techniques developed later in this paper one can show that the variety of an indecomposable module is connected. It is unknown if there is a proof not using the approach of this paper.

2. The Hochschild cohomology ring

Since the support varieties for finitely generated modules given in [SS] over a (finite dimensional) algebra over a field are defined in terms of the Hochschild cohomology ring of the algebra, this section is devoted to giving a short review of some well-known properties of the Hochschild cohomology ring of an algebra. Some of the arguments presented here are taken from [BGSS].

Let \( \Lambda \) be an algebra over a field (commutative ring) \( k \), and let \( \Lambda^c = \Lambda \otimes_k \Lambda^{op} \) be the enveloping algebra of \( \Lambda \).

Recall that the Hochschild cohomology ring \( HH^*(\Lambda) \) of \( \Lambda \) is given by

\[
HH^*(\Lambda) = \text{Ext}_{\Lambda^c}^*(\Lambda, \Lambda) = \bigoplus_{i \geq 0} \text{Ext}_{\Lambda^c}^i(\Lambda, \Lambda),
\]

which is a graded ring via the Yoneda product. It is well-known and easy to see that \( HH^0(\Lambda) = \text{Hom}_{\Lambda^c}(\Lambda, \Lambda) = Z(\Lambda) \) is the centre of \( \Lambda \). Furthermore, \( HH^1(\Lambda) \) is isomorphic to the derivations \( \Lambda \to \Lambda \) modulo the inner derivations, and the second Hochschild cohomology group measures the infinitesimal deformations of the algebra (see [Ge]).

The Hochschild cohomology ring was originally defined in [Ho] using the bar resolution \((B, d)\) of \( \Lambda \) over \( \Lambda^c \). It was shown in [CartE, IX, § 6] that it is isomorphic to \( \text{Ext}_{\Lambda^c}^*(\Lambda, \Lambda) \) whenever \( \Lambda \) is projective over \( k \). Recall that the bar resolution \((B, d)\) of \( \Lambda \) is given by \( B^n = \Lambda \otimes_k (\Lambda^{op})^n \) for \( n \geq 0 \), and \( d^n : B^n \to B^{n-1} \) is given by

\[
d^n(\lambda_0 \otimes \cdots \otimes \lambda_{n+1}) = \sum_{i=0}^n (-1)^i (\lambda_0 \otimes \cdots \otimes \lambda_i \lambda_{i+1} \otimes \cdots \otimes \lambda_{n+1})
\]

for \( n \geq 1 \). Then \( HH^*(\Lambda) \) was defined in [Ho] as the homology of \( \text{Hom}_{\Lambda^c}(B, \Lambda) \) and the multiplication was given by the cup product. To define the cup product one can use the following chain map. There is a chain map \( \Delta : B \to B \otimes_{\Lambda} B \) lifting the identity map given by

\[
\Delta(\lambda_0 \otimes \cdots \otimes \lambda_{n+1}) = \sum_{i=0}^n (\lambda_0 \otimes \cdots \otimes \lambda_i \otimes 1) \otimes (1 \otimes \lambda_{i+1} \otimes \cdots \otimes \lambda_{n+1})
\]

(see for example [Sa, 1.2]). Let \( \eta : B^m \to \Lambda \) and \( \eta' : B^n \to \Lambda \) be elements in \( HH^m(\Lambda) \) and \( HH^n(\Lambda) \) respectively. Then the cup product of \( \eta \) and \( \eta' \) is given as the composition of the maps

\[
B^m \otimes_{\Lambda} B \xrightarrow{\Delta \otimes id} B \otimes_{\Lambda} B \otimes_{\Lambda} B \xrightarrow{\nu \otimes id} B \otimes_{\Lambda} B \xrightarrow{id \otimes \eta'} B \otimes_{\Lambda} \Lambda \xrightarrow{\nu} \Lambda,
\]

where \( \nu : \Lambda \otimes_{\Lambda} \Lambda \to \Lambda \) is the multiplication map. Explicitly the cup product \( \eta \cup \eta' \) as a map \( B^{m+n} \to \Lambda \) is given by

\[
(\eta \cup \eta')(\lambda_0 \otimes \cdots \otimes \lambda_{m+n+1}) = \eta(\lambda_0 \otimes \cdots \otimes \lambda_m \otimes 1)\eta'(1 \otimes \lambda_{m+1} \otimes \cdots \otimes \lambda_{m+n+1}).
\]
The Yoneda product $\eta \ast \eta'$ of the two elements $\eta$ and $\eta'$ is given through a lifting of $\eta'$. It is straightforward to see that the composition $\eta'$ given by

$$B \xrightarrow{\Delta} B \otimes_B B \xrightarrow{\eta \otimes_B \eta'} \Lambda[n] \otimes_B B = B[n]$$

is a lifting of $\eta'$ (see [BGSS]). Then it is easy to check that

$$\eta \cup \eta' = \eta \eta' \mid_n = \eta \ast \eta'.$$

On the other hand if we let $\eta''_i : B^{n+i} \to B^i$ be given by

$$\eta''_i(\lambda_0 \otimes \cdots \otimes \lambda_{n+i+1}) = (-1)^m \eta'((\lambda_0 \otimes \cdots \otimes \lambda_{n+1}) \otimes \cdots \otimes \lambda_{n+i+1}),$$

it is easy to see that it is a lifting of $\eta'$, hence the diagram

$$\cdots \xrightarrow{\eta''_i} B^{n+2} \xrightarrow{\eta''_{n+1}} B^{n+1} \xrightarrow{\eta''_n} B^n \xrightarrow{\eta''_0} B^{n-1} \xrightarrow{\eta''_1} B \xrightarrow{\eta''_2} \cdots$$

commutes. Moreover, we have that

$$\eta \ast \eta'(\lambda_0 \otimes \cdots \otimes \lambda_{m+n+1}) = \eta''_m(\lambda_0 \otimes \cdots \otimes \lambda_{m+n+1})$$

$$= (-1)^{mn}(\eta'((\lambda_0 \otimes \cdots \otimes \lambda_{m+1}) \otimes \cdots \otimes \lambda_{n+i+1}))$$

$$= (-1)^{mn}(\eta'(\lambda_0 \otimes \cdots \otimes \lambda_{n+1}) \eta(1 \otimes \lambda_{n+1} \otimes \cdots \otimes \lambda_{n+i+1}))$$

$$= (-1)^{mn}(\eta' \cup \eta)(\lambda_0 \otimes \cdots \otimes \lambda_{m+n+1})$$

Hence this shows the following well-known facts about the Hochschild cohomology ring.

**Theorem 2.1.**

(a) The cup and the Yoneda product in $\text{HH}^*(\Lambda)$ coincide.

(b) $\text{HH}^*(\Lambda)$ is graded commutative ring; that is, $xy = (-1)^{|z||y|}yx$ for all homogeneous elements $x$ and $y$ in $\text{HH}^*(\Lambda)$, where $|z|$ denotes the degree of a homogeneous element $z$ in $\text{HH}^*(\Lambda)$.

In particular, when the characteristic of $k$ is different from two and $x$ is a homogeneous element of odd degree, then $x^2 = 0$.

In [SiWi] it is shown that any projective resolution $(P, \delta)$ of $\Lambda$ over $\Lambda^c$ gives rise to a “cup product”, which coincides with the cup product given by the bar resolution: Let $\Delta : P \to P \otimes_A P$ be a chain map lifting the identity map. They show that the “cup product” of two elements $\eta : P^m \to \Lambda$ and $\eta' : P^n \to \Lambda$ in $\text{HH}^*(\Lambda)$ defined as the composition of the maps

$$P \xrightarrow{\Delta} P \otimes_A P \xrightarrow{\eta \otimes \eta'} \Lambda \otimes_A \Lambda \xrightarrow{\nu} \Lambda$$

is the same as the ordinary cup product. We shall use this in Section 9 for Koszul algebras.

We close this section by noting that if $\Lambda$ decomposes as an algebra, then the Hochschild cohomology ring decomposes accordingly; that is, if $\Lambda = \Lambda_1 \amalg \Lambda_2$ then $\text{HH}^*(\Lambda) \simeq \text{HH}^*(\Lambda_1) \amalg \text{HH}^*(\Lambda_2)$. In this way, we can always assume without any loss of generality that the algebras we consider are indecomposable.
3. The action of the Hochschild cohomology ring

This section is devoted to defining the action of the Hochschild cohomology ring of an algebra \( A \) over a field on the direct sum of the extension groups \( \text{Ext}^i_A(M,N) = \Pi_{i \geq 0} \text{Ext}^i_A(M,N) \) for pairs of left \( A \)-modules \( M \) and \( N \). Here we refer to [SS] for details. This section gives the foundation for the support varieties we define in the next section.

Let \( A \) be an algebra over a field \( k \). As we have already mentioned in the motivation, the definition of the support variety of a module using the Hochschild cohomology ring is given in terms of a homomorphism \( \varphi_M : \text{HH}^*(A) \to \text{Ext}^*_A(M,M) \) of graded rings. Let us explain how this map is obtained. To this end note that for any projective left \( A \)-module \( P \), the module \( P \otimes_A M \) is a projective left \( A \)-module for any left \( A \)-module \( M \). Let

\[
\cdots \to P^n \to \cdots \to P^1 \to P^0 \to A \to 0
\]

be a \( A \)-projective resolution of \( A \). This sequence splits as a sequence of right \( A \)-modules. In particular, the syzygies \( \Omega_A^n(A) \) are projective as a right \( A \)-module for all \( i \geq 0 \). This and the above observation imply that

\[
\cdots \to P^n \otimes_A M \to \cdots \to P^1 \otimes_A M \to P^0 \otimes_A M \to M \to 0
\]

is a \( A \)-projective resolution of \( M \).

An element \( \eta \) in \( \text{HH}^n(A) \) can be considered as a map \( \eta : \Omega_A^n(A) \to A \). This induces a map

\[
\eta \otimes \text{id}_M : \Omega_A^n(A) \otimes_A M \to A \otimes_A M \cong M.
\]

Then we define \( \varphi_M(\eta) = \eta \otimes \text{id}_M \) as an element in \( \text{Ext}^n_A(M,M) \). We can also view the element \( \varphi_M(\eta) \) directly as an extension. Consider the following pushout diagram

\[
\begin{array}{ccccccc}
0 & \to & \Omega_A^n(A) & \to & P^{n-1} & \to & P^{n-2} & \to & \cdots & \to & P^0 & \to & A & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \Lambda & \to & M_\eta & \to & P^{n-2} & \to & \cdots & \to & P^0 & \to & A & \to & 0 \\
\end{array}
\]

As sequences of right \( A \)-modules, both rows are built up of split short exact sequences. Hence, applying the functor \(- \otimes_A M\) leaves the diagram exact. It follows from this that \( \varphi_M(\eta) \) is given by

\[
0 \to M \to M_\eta \otimes_A M \to P^{n-2} \otimes_A M \to \cdots \to P^0 \otimes_A M \to M \to 0
\]

Note that \( \text{Ext}^*_A(M,M) \) is not graded commutative as \( \text{HH}^*(A) \) is in general, since in particular \( \text{Hom}_A(M,M) \) is not commutative in general.

The Hochschild cohomology ring \( \text{HH}^*(A) \) acts on \( \text{Ext}^*_A(M,N) \) on the left and on the right via \( \varphi_N : \text{HH}^*(A) \to \text{Ext}^*_A(N,N) \) and \( \varphi_M : \text{HH}^*(A) \to \text{Ext}^*_A(M,M) \), respectively. Explicitly for an element \( \eta \) in \( \text{HH}^n(A) \) and \( \theta \) in \( \text{Ext}^*_A(M,N) \) we have

\[
\eta \cdot \theta = \varphi_N(\eta)\theta
\]

and

\[
\theta \cdot \eta = \varphi_M(\eta).
\]
These actions are closely related as we recall next. The proof of this result is elementary using the $3 \times 3$-splice lemma [MacL, Lemma VIII.3.1]: If
\[
\begin{array}{c}
0 \\ \\
0 & 0 & 0 \\
0 & A' & A & A'' & 0 \\
0 & B' & B & B'' & 0 \\
0 & C' & C & C'' & 0 \\
0 & 0 & 0 & 0
\end{array}
\]
is an exact commutative diagram, then
\[
(0 \to A' \to A \to A'' \to 0) = -(0 \to A' \to B' \to C \to C'' \to 0)
\]
as two-fold extensions.

**Theorem 3.1** ([SS, Theorem 1.1]). Let $\theta$ be in $\operatorname{Ext}^n_\Lambda(M, N)$, and let $\eta$ be in $\operatorname{HH}^m(\Lambda)$. Then
\[
\eta \cdot \theta = (-1)^{mn} \theta \cdot \eta.
\]

This has the following immediate consequence, which we use later for Koszul algebras in Section 9.

**Corollary 3.2.** Let $\Gamma_M = \operatorname{Ext}^1_\Lambda(M, M)$. Then
\[
\operatorname{Im} \varphi_M \subseteq Z_{gr}(\Gamma_M),
\]
where $Z_{gr}(\Gamma_M) = \langle z \in \Gamma_M \mid z\gamma = (-1)^{|z||\gamma|}z \gamma, \text{ for all homogeneous elements } \gamma \in \Gamma_M \rangle$ is the graded centre of $\Gamma_M$.

In general the image of $\varphi_M$ is strictly contained in $Z_{gr}(\Gamma_M)$ for a module $M$ ([GSS2, Example 7.6]). But we later see that for a Koszul algebra $\Lambda$ and $M$ equal to $\Lambda$ modulo the graded Jacobson radical, say $r$, the image of $\varphi_{\Lambda/r}$ is equal to $Z_{gr}(\Gamma_{\Lambda/r})$. For a monomial algebra $\Lambda$ the image of $\varphi_{\Lambda/r}$, where $r$ is the Jacobson radical, is not equal to $Z_{gr}(\Gamma_{\Lambda/r})$ in general ([GSS2, Example 7.6]). But modulo homogeneous nilpotent elements $\varphi_{\Lambda/r}$ is eventually surjective ([GSS2, Proposition 7.2]).

### 4. Support varieties for modules

In this section we recall the definition of the support variety of a finitely generated module over a finite dimensional algebra $\Lambda$ in terms of the Hochschild cohomology ring of $\Lambda$ given in [SS]. We give some the elementary properties of these support varieties. Further properties and details can be found in [SS].

Throughout this section let $\Lambda$ be a finite dimensional indecomposable algebra over an algebraically closed field $k$. Denote by $r$ the Jacobson radical of $\Lambda$.

The definition of the support variety of a module is always relative to a chosen commutative noetherian graded subalgebra $H$ of $\operatorname{HH}^*(\Lambda)$, where we always assume...
that $H^0 = HH^0(\Lambda) = Z(\Lambda)$. Fix such a subalgebra $H$ throughout this section. For a pair of left $\Lambda$-modules $M$ and $N$ both the left and the right annihilator of $\text{Ext}^*_\Lambda(M, N)$ as an $H$-module are graded ideals, and by Theorem 3.1 they clearly are equal. We simply denote this ideal by $\text{Ann}_H \text{Ext}^*_\Lambda(M, N)$. First we define the support variety of a pair of modules $M$ and $N$.

**Definition 4.1.** For $M$ and $N$ in mod $\Lambda$ define the *support variety* $V_H(M, N)$ of the pair $(M, N)$ as the subvariety

$$V_H(M, N) = \text{MaxSpec}(H/\text{Ann}_H \text{Ext}^*_\Lambda(M, N))$$

of $\text{MaxSpec} H$.

Due to the assumptions on $H$ the variety $\text{MaxSpec} H$ and the subvarieties $V_H(M, N)$ are all closed homogeneous affine varieties.

The first elementary properties of the support varieties that we want to point out are the following. In the proof of this result it is heavily used that any finitely generated left $\Lambda$-module can by filtered in finitely generated semisimple modules.

**Proposition 4.2 ([SS, Proposition 3.1]).** Let $M$ and $N$ be finitely generated left $\Lambda$-modules.

(a) $V_H(M, N) \subseteq V_H(M, \Lambda/\tau) \cap V_H(\Lambda/\tau, N)$

(b) $V_H(M, N) \subseteq V_H(M, M) \cap V_H(N, N) \supseteq V_H(N, M)$

This result has the following immediate consequence, which suggests the proper definition of the support variety of a single module.

**Theorem 4.3 ([SS, Theorem 3.2]).** $V_H(M, \Lambda/\tau) = V_H(M, M) = V_H(\Lambda/\tau, M)$ for all $M$ in mod $\Lambda$.

**Proof.** By Proposition 4.2 (a) we have that $V_H(M, M) \subseteq V_H(M, \Lambda/\tau)$. Using Proposition 4.2 (b) we infer that $V_H(M, \Lambda/\tau) \subseteq V_H(M, M)$. It follows that $V_H(M, M) = V_H(M, \Lambda/\tau)$. Similarly we show that $V_H(M, M) = V_H(\Lambda/\tau, M)$. This completes the proof. \qed

Then the following is a natural definition of the support variety of a module relative to the chosen graded subalgebra $H$.

**Definition 4.4.** For $M$ in mod $\Lambda$ define the *support variety of* $M$ as

$$V_H(M) = V_H(M, \Lambda/\tau).$$

The definition of the support varieties depend on the chosen subalgebra $H$. We shall see in the next section that under the finiteness conditions we impose, the definition do not depend too strongly on the choice of $H$.

Since we are assuming that $\Lambda$ is an indecomposable algebra, the centre $Z(\Lambda)$ is a local artinian ring. Therefore the ideal $m_{gr} = (\text{rad} H^0, H^{\leq 1})$ is the unique maximal graded ideal in $H$. For any nonzero finitely generated left $\Lambda$-module $M$ the graded ideal $\text{Ann}_H \text{Ext}^*_\Lambda(M, \Lambda/\tau)$ is a proper graded ideal in $H$, so that it is contained in $m_{gr}$. Hence the ideal $m_{gr}$ is always contained in $V_H(M)$ for a finitely generated nonzero module $M$. As in the group ring case this leads to the following definition.

**Definition 4.5.** A finitely generated left $\Lambda$-module $M$ has *trivial* variety if $V_H(M) = \{ m_{gr} \}$.
For a projective module $P$ we have $\text{Ext}^n_P(P, \Lambda/\tau) = \text{Hom}_\Lambda(P, \Lambda/\tau)$. From this it is immediate that $\text{Ann}_H \text{Ext}^n_P(P, \Lambda/\tau)$ contains the ideal $H^{n+1}$. Since $\text{rad} H^0 = \text{rad} Z(\Lambda)$ is nilpotent, it is contained in any maximal ideal in $H$. This shows that the variety of a projective module is trivial. Even more is true and further properties generalize from the group ring case as we recall next.

**Theorem 4.6 ([SS, Proposition 3.4 and 3.5, Theorem 3.7]).** Let $M$ be a finitely generated left $\Lambda$-module.

(a) If $M$ is projective or injective, or $\text{Ext}^i(M, M) = (0)$ for all $i \gg 0$, then the variety of $M$ is trivial.

(b) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence in $\text{mod} \Lambda$, then $V_H(M_r) \subseteq V_H(M_s) \cup V_H(M_t)$ whenever $\{r, s, t\} = \{1, 2, 3\}$.

(c) $V_H(M \amalg M') = V_H(M) \cup V_H(M')$.

(d) $V_H(M) = V_H(D(M)) = V_H(\Omega^1_M(M))$ (the last equality holds whenever $\Omega^1_M(M) \neq (0)$).

(e) If $\Lambda$ is selfinjective, then all modules in a connected stable component of the Auslander-Reiten quiver have the same variety.

The last statement of the above theorem shows that the support variety of a module is a coarse invariant for selfinjective algebras, as all modules in a connected stable component of the Auslander-Reiten quiver are identified. However, we use this to our advantage later in Theorem 8.1 where we construct infinitely many indecomposable modules with different varieties; that is, in particular infinitely many indecomposable modules in infinitely many different connected stable components of the Auslander-Reiten quiver.

All the properties of the support varieties for group rings listed in Theorem 1.1 are not satisfied by the support varieties we have defined for finitely generated modules over a finite dimensional algebra with the set of assumptions we have imposed so far. The next example shows this and makes it clear that we are in dire need of further assumptions to achieve this.

**Example 4.7.** Let $\Lambda = k \left( \begin{array}{cc} \alpha & 2 \\ \beta \end{array} \right)$ for a field $k$. Then one can show that $\Gamma_{\Lambda/\tau} = \text{Ext}^1_\Lambda(\Lambda/\tau, \Lambda/\tau) = k \left( \begin{array}{cc} \alpha & 2 \\ \beta \end{array} \right)$. By direct computations we find that $Z_{\text{sp}}(\Gamma_{\Lambda/\tau}) = k$. The image of the map $\varphi_{\Lambda/\tau}: \text{HH}^*(\Lambda) \to \text{Ext}^1_\Lambda(\Lambda/\tau, \Lambda/\tau)$ is contained in $Z_{\text{sp}}(\Gamma_{\Lambda/\tau})$. This implies that $\text{HH}^{n+1}(\Lambda) \subseteq \text{Ker} \varphi_{\Lambda/\tau}$ and $\text{HH}^{n-1}(\Lambda) \subseteq \text{Ann}_H \text{Ext}^1_\Lambda(M, \Lambda/\tau)$ for any left $\Lambda$-module $M$. We infer from this that the variety of any finitely generated left $\Lambda$-module is trivial for any choice of a commutative noetherian graded subalgebra of $\text{HH}^*(\Lambda)$. In particular the non-projective simple $\Omega$-periodic module $S_2$ corresponding to vertex 2 has trivial variety, showing that a generalization of Theorem 1.1 is not true without further assumptions.

5. **Finiteness conditions**

As Example 4.7 shows, further assumptions are needed in order to obtain a theory with good properties. In this section we introduce some finiteness conditions $\text{Fg1}$ and $\text{Fg2}$ that we show are sufficient to recover and generalize some of the results from the group ring case. The results discussed in this section is mainly taken from [EHSST] with some exceptions from [SS, GSS2].
Let $\Lambda$ be a finite dimensional algebra over an algebraically closed field $k$. The first of these conditions we have already encountered, namely

**Fg1.** $H$ is a commutative noetherian graded subalgebra of $\text{HH}^*(\Lambda)$ with $H^0 = \mathbb{Z}(\Lambda)$.

In the following let $H$ be a graded subalgebra of $\text{HH}^*(\Lambda)$ satisfying **Fg1**. In order to introduce the second finiteness condition **Fg2**, it is useful to recall the definition of complexity of a module.

**Definition 5.1.** Let $\cdots \to P^n \to P^{n-1} \to \cdots \to P^0 \to M \to 0$ be a minimal projective resolution of $M$ in mod $\Lambda$. Then the complexity of $M$ is

$$c_\Lambda(M) = \min \{ b \in \mathbb{N}_0 \mid \exists a \in \mathbb{R}; \dim_k P^n \leq an^{b-1}, \forall n \geq 0 \}.$$  

This definition leads immediately to the following well-known observations.

**Remark 5.2.**

1. If a module $M$ has complexity zero, then $M$ has finite projective dimension.
2. If a module $M$ satisfies $c_\Lambda(M) \leq 1$, then $M$ has bounded Betti numbers, $\beta_n = \dim_k P^n$ for $n \geq 0$.

In the group ring case as well as in the other cases where (cohomological) support varieties have been considered, the finite generation of the Ext-modules like $\text{Ext}^*_G(M, M)$ play a very important role. The situation for us is similar. We impose the condition that $\text{Ext}^*_\Lambda(M, \Lambda/\tau)$ is a finitely generated module over $H$. This has the following consequence, where the proof of this result is similar to the one for group rings. Moreover, observe that we recover the description of the dimension of the variety as the complexity of the module from the group ring case.

**Proposition 5.3 ([EHSST, Proposition 1.1]).** Let $M$ be a finitely generated left $\Lambda$-module. Then the following assertions hold.

(a) If $\text{Ext}^*_\Lambda(M, \Lambda/\tau)$ is a finitely generated $H$-module, then $\dim V_H(M) = c_\Lambda(M) < \infty$.

(b) If $\text{Ext}^*_\Lambda(\Lambda/\tau, M)$ is a finitely generated $H$-module, then $\dim V_H(M) = c_\Lambda(D(M)) < \infty$.

As an immediate consequence of this result we obtain the following, which clearly restricts the class of algebras where we can expect a rich theory for our support varieties. Here it is crucial to note that the support variety both of a projective and an injective module is trivial.

**Proposition 5.4 ([EHSST, Proposition 1.2]).** Let $M$ be a finitely generated left $\Lambda$-module. Then we have the following.

(a) If $\text{Ext}^*_\Lambda(M, \Lambda/\tau)$ is a finitely generated $H$-module and $V_H(M)$ is trivial, then $M$ has finite projective dimension.

(b) If $\text{Ext}^*_\Lambda(\Lambda/\tau, M)$ is a finitely generated $H$-module and $V_H(M)$ is trivial, then $M$ has finite injective dimension.

(c) If $\text{Ext}^*_\Lambda(D(\Lambda^\text{op}), \Lambda/\tau)$ and $\text{Ext}^*_\Lambda(\Lambda/\tau, \Lambda)$ are finitely generated $H$-modules, then $\Lambda$ is a Gorenstein algebra.

Again using long exact sequences of Ext-groups and that any finitely generated left $\Lambda$-module can be filtered in semisimple $\Lambda$-modules, we obtain the equivalence of (a) and (b) in the next result. For the rest we need to recall that
all the homogeneous elements of odd degree in HH of the field generated by the homogeneous nilpotent elements in HH element, since HH can choose these elements. It is easy to see that HH is a finitely generated module over HH show that HH is a finitely generated module over HH.

This naturally guides us to the last and the second of our finiteness conditions:

\textbf{Proposition 5.6.} If \( \Lambda \) and \( H \) satisfy \( \text{Fg1} \) and \( \text{Fg2} \), then \( \text{Ext}_A^*(\Lambda/\tau, \Lambda/\tau) \) and \( \text{HH}^*(\Lambda) \) are noetherian rings.

For a local commutative noetherian ring \((R, m)\), the ring \( \text{Ext}_A^*(R/m, R/m) \) being noetherian is equivalent to \( R \) being a complete intersection ([BH, Gu]). The property that \( \text{Ext}_A^*(\Lambda/\tau, \Lambda/\tau) \) is noetherian, or that the Hochschild cohomology ring \( \text{HH}^*(\Lambda) \) is noetherian, is equivalent to \( \text{Ext}_A^*(\Lambda/\tau, \Lambda/\tau) \) being a finitely generated module over \( \text{HH}^*(\Lambda) \), are natural analogues of complete intersections for finite dimensional algebras (suggested by L. Avramov). Next we show that the existence of a commutative noetherian graded subalgebra \( H \) of \( \text{HH}^*(\Lambda) \) such that \( \Lambda \) and \( H \) satisfy \( \text{Fg1} \) and \( \text{Fg2} \) is equivalent to the latter “definition” of a complete intersection.

\textbf{Proposition 5.7.} Let \( \Lambda \) be a finite dimensional algebra. Then there exists a commutative noetherian graded subalgebra \( H \) of \( \text{HH}^*(\Lambda) \) such that \( \Lambda \) and \( H \) satisfy \( \text{Fg1} \) and \( \text{Fg2} \) if and only if \( \text{HH}^*(\Lambda) \) is a noetherian ring and \( \text{Ext}_A^*(\Lambda/\tau, \Lambda/\tau) \) is a finitely generated module over \( \text{HH}^*(\Lambda) \).

\textbf{Proof.} Suppose that there exists a commutative noetherian graded subalgebra \( H \) of \( \text{HH}^*(\Lambda) \) such that \( \Lambda \) and \( H \) satisfy \( \text{Fg1} \) and \( \text{Fg2} \). Then we have already pointed out above that \( \text{HH}^*(\Lambda) \) is a noetherian ring and \( \text{Ext}_A^*(\Lambda/\tau, \Lambda/\tau) \) is a finitely generated module over \( \text{HH}^*(\Lambda) \).

Conversely, suppose that \( \text{HH}^*(\Lambda) \) is a noetherian ring and that \( \text{Ext}_A^*(\Lambda/\tau, \Lambda/\tau) \) is a finitely generated module over \( \text{HH}^*(\Lambda) \). As in the commutative case one can show that \( \text{HH}^*(\Lambda) = \text{HH}^{\text{even}}(\Lambda)[\eta_1, \ldots, \eta_n] \) for some finite set of homogeneous elements \( \{\eta_1, \ldots, \eta_n\} \). Suppose \( \{\eta_1, \ldots, \eta_k\} \) are the elements of odd degree among these elements. It is easy to see that \( \text{HH}^{\text{even}}(\Lambda) \) is a commutative noetherian ring and that \( \text{HH}^*(\Lambda) \) is generated by \( \{\eta_1, \ldots, \eta_k\} \) as a module over \( \text{HH}^{\text{even}}(\Lambda) \). So we can choose \( H = \text{HH}^{\text{even}}(\Lambda) \) and \( \Lambda \) and \( H \) satisfy \( \text{Fg1} \) and \( \text{Fg2} \).

Any maximal (prime) ideal in \( \text{HH}^*(\Lambda) \) contains any homogeneous nilpotent element, since \( \text{HH}^*(\Lambda) \) is graded commutative. Denote by \( \mathcal{N} \) the ideal in \( \text{HH}^*(\Lambda) \) generated by the homogeneous nilpotent elements in \( \text{HH}^*(\Lambda) \). If the characteristic of the field \( k \) is two, then \( \text{HH}^*(\Lambda) \) is a commutative ring. Otherwise, \( \mathcal{N} \) contains all the homogeneous elements of odd degree in \( \text{HH}^*(\Lambda) \), so that the factor ring
$\text{HH}^\ast(\Lambda)/\mathcal{N}$ is always a commutative ring. So by the above remarks the underlying geometric object for the support varieties is $\text{HH}^\ast(\Lambda)/\mathcal{N}$ when $\text{Fg1}$ and $\text{Fg2}$ are satisfied: (i) there is a one-to-one correspondence between the maximal ideals in $\text{HH}^\ast(\Lambda)$ and $\text{HH}^\ast(\Lambda)/\mathcal{N}$, and (ii) $\text{HH}^\ast(\Lambda)/\mathcal{N}$ is a finitely generated module over any graded subalgebra $\mathcal{H}$ defining the support varieties (in particular $\text{HH}^\ast(\Lambda)/\mathcal{N}$ is integral over the image of $\mathcal{H}$ in $\text{HH}^\ast(\Lambda)/\mathcal{N}$).

By the above remarks it interesting to find a description of $\mathcal{N}$. An approximation of $\mathcal{N}$ is given by the following result.

**Proposition 5.8** ([SS, Proposition 4.6], [GSS2, Lemma 7.4]).

(a) The ideal $\text{Ker}\varphi_{\Lambda/\mathcal{N}}$ is nilpotent with nilpotency index at most the Loewy length of $\Lambda$.

(b) $\sqrt{\text{Ker}\varphi_{\Lambda/\mathcal{N}}} = \mathcal{N}$.

This leads to the following natural conjecture.

**Conjecture** ([SS]). Let $\Lambda$ be a finite dimensional algebra over a field $k$. Then $\text{HH}^\ast(\Lambda)/\mathcal{N}$ is a finitely generated $k$-algebra.

This conjecture is known to be true for (i) any block of a group ring of a finite group [Ev, Go, V], (ii) any block of a finite dimensional cocommutative Hopf algebra [FS], (iii) finite dimensional selfinjective algebras of finite representation type over an algebraically closed field [GSS1, Sc], and (iv) a finite dimensional monomial algebra [GSS2].

Let us return to the conditions $\text{Fg1}$ and $\text{Fg2}$. They are known to be true for (i) any block of a group ring of a finite group [Ev, Go, V], (ii) any block of a finite dimensional cocommutative Hopf algebra [FS] and (iii) any exterior algebra. In the commutative setting it is true for a complete intersection [Gu].

We conclude this section with a summary of our findings.

**Theorem 5.9.** If $\Lambda$ and $\mathcal{H}$ satisfy $\text{Fg1}$ and $\text{Fg2}$, then

(a) $\Lambda$ is Gorenstein.

(b) The following are equivalent.

(i) $V_H(M)$ is trivial.

(ii) $M$ has finite projective dimension.

(iii) $M$ has finite injective dimension.

(c) $\dim V_H(M) = c_\Lambda(M) < \infty$ for all $M$ in mod $\Lambda$.

Note that if $\Lambda$ is selfinjective and $\Lambda$ and $\mathcal{H}$ satisfy $\text{Fg1}$ and $\text{Fg2}$, then the variety of a module $M$ is trivial if and only if $M$ is projective. Hence we recover the corresponding result for group rings.

### 6. A pivotal result

For group rings $kG$ of a finite group $G$ the construction of special modules $L_\zeta$ for any homogeneous non-zero element $\zeta$ in $H^\ast(G, k)$ with $V_G(L_\zeta) = V_G(\langle \zeta \rangle)$ is very central for many results and constructions. This section is devoted to finding a similar construction in our situation, and showing that we have analogous results at least when $\text{Fg1}$ and $\text{Fg2}$ are satisfied. Under these assumptions we in particular prove that any closed homogeneous variety occurs as the variety of a module. The results we discuss in this section can be found in [EHSST].
We start by recalling the construction of the modules $L_\zeta$ for the group ring situation (see [Carl3] or [Ben, § 5.9]). Let $G$ be a finite group, and let $\zeta$ be a homogeneous non-zero element in $H^*(G, k)$. The element $\zeta$ can be viewed as a map $\zeta : \Omega^n_{KG}(k) \rightarrow k$. This gives rise to the exact sequence

$$0 \rightarrow L_\zeta \rightarrow \Omega^n_{KG}(k) \rightarrow k \rightarrow 0,$$

which defines the module $L_\zeta$. The construction in our setting corresponds in some way to $\Omega^n_{\Lambda}(L_\zeta)$ as we explain next.

Let $\Lambda$ be an algebra over a field $k$. Given $\eta$ in $HH^n(\Lambda)$ consider the pushout diagram

$$
\begin{array}{cccc}
0 & \rightarrow & \Omega^n_\Lambda(\Lambda) & \rightarrow & p^{n-1} \rightarrow & \Omega^{n-1}_\Lambda(\Lambda) & \rightarrow & 0 \\
& \downarrow{\eta} & & \downarrow{=} & & \downarrow{=} & & \\
0 & \rightarrow & \Lambda & \rightarrow & M_\eta & \rightarrow & \Omega^{n-1}_\Lambda(\Lambda) & \rightarrow & 0
\end{array}
$$

where we as before observe that the lower row is split exact as a sequence of right $\Lambda$-modules. This implies that the sequence

$$0 \rightarrow M \rightarrow M_\eta \otimes \Lambda M \rightarrow \Omega^{n-1}_\Lambda(\Lambda) \otimes \Lambda M \rightarrow 0$$

is exact. As for group rings, one can show that $\eta^2$ annihilates $\text{Ext}_\Lambda^n(M_\eta \otimes \Lambda M, \Lambda/\theta)$ ([EHSST, Proposition 3.1]). Remember that $\Omega^{n-1}_\Lambda(\Lambda) \otimes \Lambda M \simeq \Omega^{n-1}_\Lambda(M) \oplus P$ for some projective module $P$, so that $V_H(M_\eta \otimes \Lambda M) \subseteq V_H(M) \cap V(\langle \eta \rangle)$. In fact equality holds when $\Lambda$ and $H$ satisfy $\text{Fg1}$ and $\text{Fg2}$. It is not known if this equality holds when $\text{Fg2}$ is not satisfied.

**Theorem 6.1 ([EHSST, Proposition 3.3]).** Let $\Lambda$ be a finite dimensional algebra. Assume that $\Lambda$ and $H$ satisfy $\text{Fg1}$ and $\text{Fg2}$. Then

$$V_H(M_\eta \otimes \Lambda M) = V(\langle \eta \rangle) \cap V_H(M)$$

for all $M$ in mod $\Lambda$.

The proof of the fact that $V_G(L_\zeta) = V(\langle \zeta \rangle)$ in the group ring case makes use of rank varieties (see [Carl3] or [Ben, Proposition 5.9.1]). We do not have any analogue of rank varieties in general in our setting. The proof of Theorem 6.1 uses localization techniques and Tate cohomology in the setting of Gorenstein algebras. In fact our proof gives a new proof also for the group ring case.

As an immediate consequence of the above result we get that any closed homogeneous variety occurs as the variety of a module. We only have to note that the ideal defining such a variety is generated by a finite set of homogeneous generators $\{\eta_1, \ldots, \eta_r\}$ and then the module doing the job is $M_{\eta_1} \otimes \Lambda \cdots \otimes \Lambda M_{\eta_r} \otimes \Lambda \theta$. Note that this module need not to be indecomposable.

**Theorem 6.2 ([EHSST, Theorem 3.4]).** Assume that $\Lambda$ and $H$ satisfy $\text{Fg1}$ and $\text{Fg2}$. If $\mathfrak{a}$ is a graded ideal in $H$, then there exists a $\Lambda$-module $M$ such that $V_H(M) = V(\mathfrak{a})$.

Another consequence of Theorem 6.1 is that it provides a way of cutting down the variety of a module. In particular, if $\eta$ is not in $\sqrt{\ker \varphi_M}$, then $\text{cx}_\Lambda(M_\eta \otimes \Lambda M) < \text{cx}_\Lambda(M)$. 
7. Applications

Throughout this section assume that $\Lambda$ is a selfinjective finite dimensional algebra over an algebraically closed field $k$. Here we point out that several more of the results in modular representation theory involving support varieties generalize to the setting of this section when $\text{Fg1}$ and $\text{Fg2}$ are satisfied. This includes a characterization of periodic modules (Theorem 1.1 (b)), relationship with extensions (Theorem 1.1 (i)) and connectedness of the variety of an indecomposable module ([Carl3]) and Webb’s theorem for stable components of the Auslander-Reiten quiver [W]. Some of the results in this section have analogues when $\Lambda$ is Gorenstein. We refer to [EHSST] for details on the results presented in this section.

7.1. Periodic modules. The first application we discuss is a characterization of $\Omega$-periodic modules. Recall that a module $M$ is $\Omega$-periodic if there exists a positive integer $n$ such that $M \cong M^n$. The smallest such positive integer is called the period of the module.

**Theorem 7.1 ([EHSST, Proposition 4.2, Theorem 4.3]).** Suppose that $\Lambda$ is selfinjective, and that $\Lambda$ and $H$ satisfy $\text{Fg1}$ and $\text{Fg2}$. Let $M$ be a finitely generated left $\Lambda$-module.

(a) If $M$ is an indecomposable periodic module, then $\dim V_H(M) = 1$.
(b) If $\dim V_H(M) = 1$, then $M$ is a direct sum of a periodic module and a projective module.

Assume that we are in the setting of the above theorem. If $\dim V_H(M) = 1$, then there exists a homogeneous element $\eta$ in $H$ such that

$$\dim(H/\langle \eta, \text{Ann}_H \rangle \text{Ext}^*_{\Lambda}(M, A/\eta))) < \dim V_H(M) = \dim H/\text{Ann}_H \text{Ext}^*_{\Lambda}(M, A/\eta).$$

Hence $\dim V_H(M_{\eta} \otimes_{\Lambda} M) = 0$ by Theorem 6.1 and $M_{\eta} \otimes_{\Lambda} M$ is projective by Theorem 5.9 (ii). Recall the construction of the exact sequence

$$0 \to M \to M_{\eta} \otimes_{\Lambda} M \to P_{n-2} \otimes_{\Lambda} M \to \cdots \to P_0 \otimes_{\Lambda} M \to M \to 0,$$

where in this case all middle terms are projective. Hence the period of $M$ divides the degree of $\eta$. This can be extended and refined to give the following.

**Proposition 7.2 ([EHSST, Proposition 4.4]).** Suppose that $\Lambda$ is selfinjective, and that $\Lambda$ and $H$ satisfy $\text{Fg1}$ and $\text{Fg2}$. Let $M$ be an indecomposable periodic $\Lambda$-module. If $H$ is generated as a $k$-algebra by homogeneous elements $\eta_1, \ldots, \eta_t$ over $H^0$, then the period of $M$ divides one of the degrees of $\eta_i$.

Given a periodic module $M$ and an element $\eta$ in $H^*(\Lambda)$ having the property that $M_{\eta} \otimes_{\Lambda} M$ is projective, $\eta$ is said to generate the periodicity of $M$. In [Ber] examples are given to show that there are periodic modules over finite dimensional algebras, where the periodicity is not generated by any element in $H^*(\Lambda)$.

7.2. Extensions and connectedness. The group of homomorphisms and extensions between modules are important invariants in representation theory. In this subsection we show that knowledge about how the support varieties of two modules intersect lets us deduce properties of their extensions.

**Proposition 7.3 ([EHSST, Proposition 6.1]).** Suppose that $\Lambda$ is selfinjective, and that $\Lambda$ and $H$ satisfy $\text{Fg1}$ and $\text{Fg2}$. Let $M$ and $N$ be two $\Lambda$-modules in $\text{mod} \Lambda$. Suppose that $V_H(M) \cap V_H(N)$ is trivial. Then $\text{Ext}^i_{\Lambda}(M, N) = (0)$ for all $i > 0$. 

This result can be used to prove that the support variety of an indecomposable module \( M \) is connected; that is, \( V_H(M) \) can not be written as the union of two homogeneous non-trivial varieties with trivial intersection.

**Theorem 7.4 ([EHSST, Theorem 6.3]).** Suppose that \( A \) is selfinjective, and that \( A \) and \( H \) satisfy \( \text{FG1} \) and \( \text{FG2} \). Let \( M \) be in \( \text{mod} A \). If \( V_H(M) = V_1 \cup V_2 \) for some homogeneous non-trivial varieties \( V_1 \) and \( V_2 \) with \( V_1 \cap V_2 \) trivial, then \( M \approx M_1 \oplus M_2 \) with \( V_H(M_1) = V_1 \) and \( V_H(M_2) = V_2 \).

The converse of this result is clearly not true, namely, if the support variety of a module is connected, then it need not to be indecomposable. For example, choose modules \( M \) and \( N \) such that \( \{ m_n \} \neq V_H(N) \subsetneq V_H(M) \) with \( V_H(M) \) connected. Then \( V_H(M \oplus N) = V_H(M) \) by Theorem 4.6 (c).

### 7.3. Webb’s theorem

As an invariant of a finite dimensional algebra the Auslander-Reiten quiver has had a profound impact on representation theory (see [ARS]). For wild representation type any classification of all indecomposable modules is beyond reach, so that only coarser classifications are at best possible. This is the case for group rings of finite groups, where the shapes of the stable components of the Auslander-Reiten quiver are classified. This subsection is devoted to discussing this classification, Webb’s theorem [W], and a generalization to our setting.

First we recall Webb’s theorem.

**Theorem 7.5.** Let \( G \) be a finite group, and suppose the characteristic of the algebraically closed field \( k \) divides the order of \( G \). Let \( \Delta \) be a connected component \((\mathbb{Z} \Delta/\Theta)\) of the stable Auslander-Reiten of \( \text{mod} kG \). Then \( \Delta \) is one of the diagrams in the following list.

1. A Dynkin diagram, \( \tilde{A}_n, \tilde{D}_n, \tilde{E}_{6,7,8} \)
2. A Euclidean diagram, \( \tilde{A}_n, \tilde{D}_n, \tilde{E}_{6,7,8}, \tilde{A}_{12} (= \tilde{A}_{2,2}) \)
3. \( \tilde{A}_\infty \): \( \cdots \rightarrow A \rightarrow \cdots \rightarrow \cdots \rightarrow \cdots \)
4. \( \tilde{D}_\infty \): \( \cdots \rightarrow A \rightarrow \cdots \rightarrow \cdots \rightarrow \cdots \)
5. \( \tilde{E}_\infty \): \( \cdots \rightarrow A \rightarrow \cdots \rightarrow \cdots \rightarrow \cdots \)

Our aim is to generalize Webb’s theorem to selfinjective algebras \( A \) satisfying \( \text{FG1} \) and \( \text{FG2} \) with periodic Nakayama functor \( \nu = D\text{Hom}_A(-, A) \). We shall do this by using subadditive and additive functions; notions which we recall next. We have adapted the definitions to fit our setting in that the valuation on the Auslander-Reiten quiver is always on the form \((a, a)\) in our case.

Let \( \Delta = (\Delta_0, \Delta_1, a) \) be a valued graph without multiple edges and loops, where \( a: \Delta_1 \rightarrow \mathbb{N} \) is the valuation. A function \( d: \Delta_0 \rightarrow \mathbb{N} \) is called *subadditive* if

\[
2d(i) \geq \sum_{\alpha} a(\alpha) d(j)
\]

for all \( i \) in \( \Delta_0 \). The function \( d \) is called *additive* if equality holds for all \( i \) in \( \Delta_0 \). For further details see [HPR].

The role of these functions is explained by the following powerful theorem.
Theorem 7.6 ([HPR]). Let $\Delta$ be a graph with no multiple edges and no loops, and let $d$ be a subadditive function on $\Delta$.

(a) $\Delta$ is one of the diagrams in the following list.

(i) a Dynkin diagram
(ii) a Euclidean diagram
(iii) $A_\infty$: \\
(iv) $A_\infty$: \\
(v) $B_\infty$: \\
(vi) $C_\infty$: \\
(vii) $D_\infty$: \\

(b) If $d$ is not additive, then $\Delta$ is Dynkin or $A_\infty$.

(c) If $d$ is unbounded, then $\Delta$ is $A_\infty$.

So we need to construct a subadditive function related to the stable components of the Auslander-Reiten quiver. We follow the method developed in [ES].

Let $C$ be a component in the stable Auslander-Reiten quiver of $H$. If $W \simeq \tau W$ and $\text{Ext}_A^1(W, M) \neq (0)$ for all $M$ in $C$, then dim $\text{Ext}_A^1(W, -)$ is known to give a subadditive function on the orbit graph $\Delta$ of the component.

For the construction of such modules $W$ we observe the following. Suppose that $\Lambda$ is selfinjective. Then by Theorem 4.6 (e) all modules in a fixed connected component of the stable Auslander-Reiten quiver have the same variety. The Auslander-Reiten translate for a selfinjective algebra $\Lambda$ is given by $\tau = \nu \Omega^2\Lambda(-) = \Omega^2\Lambda(-)\nu$. We are interested in finding $\tau$-periodic modules of period one, but the support varieties only detect $\Omega$-periodic modules. One way out of this is to be in a situation where the class of $\tau$-periodic and $\Omega$-periodic modules coincide. If the Nakayama functor has finite order on objects; that is, $\nu^{t_M}(M) \simeq M$ for all $M$ and for some $t_M$, then this is the case.

Suppose now that we are given a component $C$ in the stable Auslander-Reiten quiver with $V$ being the variety of modules in $C$. Without loss of generality we assume that $V = V(\mathfrak{a})$ for some graded ideal $\mathfrak{a}$ in $H$. Since $H$ is noetherian, we can choose a finite set of homogeneous elements $\{\eta_1, \eta_2, \ldots, \eta_t\}$ in $H$ such that $\mathfrak{a} \subseteq \langle \eta_1, \eta_2, \ldots, \eta_t \rangle$ and $\dim H/\langle \eta_1, \eta_2, \ldots, \eta_t \rangle = 1$. Let

$W' = (M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t})^* \otimes_\Lambda \Lambda/\mathfrak{r}$.

Then

$V_H((M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t})^* \otimes_\Lambda \Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) = \text{dim} \text{Ext}_A^1(W, M)$

using that $M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t}$ is a $\Lambda$-bimodule projective on either side as a $\Lambda$-module. By Theorem 6.1 the variety of this module is $\mathfrak{a}$ when $\text{Fg1}$ and $\text{Fg2}$ are satisfied. It follows from Theorem 7.1 that $W'$ is an $\Omega$-periodic module, and therefore gives rise to a $\tau$-periodic module $W$. Again using that $M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t}$ is projective as a $\Lambda$-module on either side, we infer that

$\text{Ext}_A^1((M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t})^* \otimes_\Lambda \Lambda/\mathfrak{r}, M) \simeq \text{Ext}_A^1(\Lambda/\mathfrak{r}, M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda M) \neq (0)$.
since the variety of $M_{\eta_1} \otimes_{\Lambda} \cdots \otimes_{\Lambda} M_{\eta_t} \otimes_{\Lambda} M$ is $V((\eta_1, \ldots, \eta_t))$ and of dimension one, so that it is non-projective (non-injective). These arguments essentially prove the following.

**Theorem 7.7** ([EHSST, Theorem 4.6]). Let $\Lambda$ be a finite dimensional selfinjective algebra over an algebraically closed field $k$, and assume that $\Lambda$ and $H$ satisfy $\text{Fg1}$ and $\text{Fg2}$. Suppose that the Nakayama functor is of finite order on any indecomposable module in $\mod \Lambda$. Then for a connected component $C(\mathbb{Z}/\Delta/\Theta)$ of the stable Auslander-Reiten quiver of $\Lambda$, the graph $\Delta$ is one of the following:

(i) a finite Dynkin diagram, $A_n, D_n, E_{6,7,8}$.
(ii) a Euclidean diagram, $A_n, D_n, E_{6,7,8}, A_{12}$.
(iii) an infinite Dynkin diagram of the type $A_\infty, D_\infty, A_{\infty}^\infty$.

Since we are assuming that the field $k$ is algebraically closed, it is well-known that the valuation on any arrow in the Auslander-Reiten quiver is of the form $(x, x)$, so that we may exclude the cases $B_\infty$ and $C_\infty$ for those occurring in the Happel-Preiser-Ringel theorem. It is known already that all the remaining graphs can occur for group rings of finite groups, so that they all definitely occur for finite dimensional selfinjective algebras.

Now we explain how this result can be applied to finite dimensional cocommutative Hopf algebras. The next result is the key to this application.

**Theorem 7.8** ([FMS, Lemma 1.5]). The Nakayama automorphism for a finite dimensional Hopf algebra $\Lambda$ has finite order dividing $2 \dim k \Lambda$.

This implies that the classes of $\tau$-periodic and $\Omega$-periodic modules coincide for finite dimensional Hopf algebras. To apply Theorem 7.5 we need the conditions $\text{Fg1}$ and $\text{Fg2}$ to be true, which is the case for a cocommutative finite dimensional Hopf algebra by [FS]. Therefore we have the following corollary.

**Corollary 7.9** ([EHSST, Corollary 4.7]). Webb’s theorem (Theorem 7.5) holds for finite dimensional cocommutative Hopf algebras.

### 8. Representation type and complexity

Also in this section let $\Lambda$ be a finite dimensional selfinjective algebra over an algebraically closed field. Heller observed in [He] that if $\Lambda$ is of finite representation type then the complexity of any module $M$ in $\mod \Lambda$ is at most 1. However the converse is not true, since there exist selfinjective preprojective algebras of wild representation type where all indecomposable non-projective modules are periodic. If $\Lambda$ is of tame representation type, then it is not known in general if there is any bound on the complexity of all indecomposable modules in $\mod \Lambda$. For some classes of tame selfinjective algebras it is known that the complexity of all modules in $\mod \Lambda$ is at most 2 (see [F]). Even if this were always true, the converse would still not be true by the same counterexamples as above. However, we point out that if $\Lambda$ is selfinjective and there exists an $H$ in $\text{HH}^*(\Lambda)$ with $\dim H \geq 2$ such that $\text{Fg1}$ and $\text{Fg2}$ are satisfied, then $\Lambda$ is of infinite representation type. The proof is based on the fact that every closed homogeneous variety occurs as the variety of a module. Since $\dim H \geq 2$, this guarantees that there exist an infinite number of different closed homogeneous varieties and therefore an infinite number of modules. Also, as all modules in a connected component of the Auslander-Reiten quiver have the
same variety, they would also lie in different connected components. More precisely, we have the following.

**Theorem 8.1 ([EHSST, Proposition 5.1]).** Suppose that $\Lambda$ is selfinjective and that $\Lambda$ and $H$ satisfy $\text{Fg}_1$ and $\text{Fg}_2$ with $\dim H \geq 2$. Then $\Lambda$ is of infinite representation type and $\Lambda$ has an infinite number of indecomposable periodic modules lying in infinitely many different connected components of the stable Auslander-Reiten quiver.

9. Koszul algebras

This section is devoted to studying support varieties for finite dimensional selfinjective Koszul algebras. As computing the Hochschild cohomology ring and the Koszul dual of a Koszul algebra are necessary, we first review the definitions involved, useful resolutions and general results for Koszul algebras. We end with reproving a result of D. Eisenbud, and pointing out that the Koszul duals of Artin-Schelter regular Koszul algebras, which are finite over their centre, do have a theory of support varieties. Most of the results we focus our attention on can be founded in [BGMS, BGSS, GHMS].

First we recall some elementary facts about Koszul algebras. Let $\Lambda = \oplus_{i \geq 0} \Lambda_i$ be a graded $k$-algebra with $\dim_k \Lambda_i < \infty$ for all $i \geq 0$ and $\Lambda_0$ semisimple ($\simeq k^n$). Since some of the results we shall use are formulated for right modules, we consider right modules in this section unless otherwise explicitly stated. So here the enveloping algebra $E(\Lambda) = \text{Ext}_\Lambda^1(\Lambda_0, \Lambda_0)$ is given by $\Lambda^{\text{op}} \otimes_k \Lambda$.

Recall that given two graded $\Lambda$-modules $M$ and $N$, a map $f: M = \oplus_{i \in \mathbb{Z}} M_i \to N = \oplus_{i \in \mathbb{Z}} N_i$ is a $\Lambda$-homomorphism of degree 0 if $f(M_i) \subseteq N_i$.

**Definition 9.1.**

(i) A graded module $M$ is called linear if it has a minimal graded projective resolution

$$\ldots \rightarrow P^n \xrightarrow{d^n} P^{n-1} \rightarrow \ldots \rightarrow P^0 \xrightarrow{d^0} M \rightarrow 0$$

where each projective $P^n$ is finitely generated in degree $n$ and $d^n$ is of degree zero for all $n \geq 0$.

(ii) An algebra $\Lambda$ is a Koszul algebra if $\Lambda_0$ is a linear module.

The following are known facts about Koszul algebras.

**Theorem 9.2.** Let $\Lambda$ be a Koszul algebra with $\Lambda_0 \simeq k^n$.

(a) $\Lambda$ is a quadratic algebra generated in degrees 0 and 1. In particular, if $\Lambda = kQ/I$, then $I = \langle I_2 \rangle$ where $I_2 = I \cap \Lambda_2$.

(b) The algebra $E(\Lambda) = \text{Ext}_\Lambda^1(\Lambda_0, \Lambda_0)$ is Koszul, and is called the Koszul dual of $\Lambda$.

(c) If $\Lambda = kQ/I$, then $E(\Lambda) = kQ^{\text{op}}/I_2^\perp$, where $I_2^\perp$ is given as follows. If $\alpha$ is an arrow in $Q$, denote by $\alpha^*$ the corresponding arrow in the opposite quiver $Q^{\text{op}}$. View $\alpha^*$ as a linear map $\alpha^*: k\{\beta\}_{\beta \in Q_1} \rightarrow k$ by letting $\alpha^*(\beta) = \delta_{\alpha, \beta}$. This induces linear maps $kQ_1^{\text{op}} \otimes_k kQ^1 \rightarrow k$. Then

$$I_2^\perp = \{ x \in kQ_1^{\text{op}} \otimes_k kQ_1^{\text{op}} | x(a) = 0, \forall a \in I_2 \}.$$

The following are classical examples of Koszul algebras.

**Example 9.3.** The polynomial ring $k[x_0, x_1, \ldots, x_n]$ is a Koszul algebra (the Koszul complex is the required resolution of $\Lambda_0 = k$).
The exterior algebra $\Lambda = k\langle x_0, x_1, \ldots, x_n \rangle / \langle \{x_i x_j + x_j x_i \}_{i \neq j}, \{x_i^2\}_i \rangle$ is a Koszul algebra.

Using the above result it is easy to see that these algebras are Koszul duals of each other.

**9.1. Resolutions.** Let $\Lambda = kQ/I$ be a Koszul algebra. A detailed knowledge of a minimal projective resolution of $\Lambda_0$ over $\Lambda$ and $\Lambda$ over $\Lambda'$ allows for a direct calculation of the Koszul dual and the Hochschild cohomology ring of $\Lambda$. We shall see that additional structure is available in the Koszul case, so that closed formulas for the multiplication are possible both for the Koszul dual and the Hochschild cohomology ring.

As pointed out earlier in this section we consider right modules, and multiplication of arrows in the path algebra follows the convention that if $a \cdot \frac{a}{b} \rightarrow b$, then $ab \neq 0$ while $ba = 0$.

The following result from [GSZ] is true for a finite dimensional algebra and for a graded algebra of the form $kQ/I$. In the theorem $R$ denotes the path algebra $kQ$.

**Theorem 9.4 ([GSZ]).** There exist $\{f^n_i\}_{i=0}^n$ in $R$ for all $n \geq 0$, such that a minimal right graded projective resolution

$$
(L,e) : \cdots \to L^n \xrightarrow{e^n} L^{n-1} \to \cdots \to L^1 \xrightarrow{e^1} L^0 \to \Lambda_0 \to 0
$$

of $\Lambda_0$ can be given in terms of a filtration of right ideals

$$
\cdots \subseteq \Pi_{i=0}^n f^n_i R \subseteq \Pi_{i=0}^{n-1} f^{n-1}_i R \subseteq \cdots \subseteq \Pi_{i=0}^{n_1} f^1_i R \subseteq \Pi_{i=0}^n f^0_i R = R
$$

in $R$. In particular,

$$
L^n = \Pi_{i=0}^n f^n_i R / \Pi_{i=0}^n f^n_i I
$$

and the differential is induced by the inclusion $\Pi_{i=0}^n f^n_i R \subseteq \Pi_{i=0}^{n-1} f^{n-1}_i R$.

**Example 9.5.** Let $\Lambda = k\langle x, y \rangle / \langle x^2, xy + yx, y^2 \rangle$ be the exterior algebra on a two dimensional vector space. Then the number of elements in each set $\{f_i^n\}$ is $n + 1$ for all $n$, and they are

- $f^n_0 = 1$
- $f^n_0 = x$, $f^n_1 = y$
- $f^n_2 = x^2$, $f^n_2 = xy + yx$, $f^n_2 = y^2$

for $n = 0, 1, 2$, and for an arbitrary $n$

$$
f_i^n = \sum \{\text{all words in } x \text{ and } y \text{ of length } n \text{ with } i \text{ 's}\}
= f_i^{n-1} y + f_i^n x
$$

It follows from this that $L^0 \simeq \Lambda$, $L^1 \simeq \Lambda^2$, $L^2 \simeq \Lambda^3$ and in general $L^n \simeq \Lambda^{n+1}$. The start of the projective resolution of $\Lambda_0 = k$ is therefore given by

$$
\cdots \to \Lambda^3 \xrightarrow{(x \ y \ 0)} \Lambda^2 \xrightarrow{(x \ y)} \Lambda \to k \to 0
$$
In particular, note that
\[ f_0^2 = f_0^1 f_0^1 \]
\[ f_1^2 = f_0^1 y + f_1^1 x = f_0^1 f_1^1 + f_1^1 f_0^1 \]
\[ f_2^2 = f_1^1 f_1^1 \]
\[ \vdots \]
\[ f_n^n = \sum_{j = \max\{0, t_i - n\}}^{\min\{t_i, i\}} f_j^1 f_{i-j}^{n-t}, \text{ for all } t = 0, 1, \ldots, n \]

The phenomenon that we observed in the above example is in fact true in general. This is a very powerful result as it gives the multiplicative structure both in the Koszul dual and the Hochschild cohomology ring of a Koszul algebra.

**Proposition 9.6 ([GHMS, Theorem 1.1]).** Let \( \Lambda = kQ/I \) be a Koszul algebra, and let \( \{f^n_i\}_{i=0}^n \) be as above defining a minimal projective resolution of \( \Lambda_0 \) over \( \Lambda \) as a right module.

For \( 0 \leq r \leq n \) and \( i \) with \( 0 \leq i \leq t_n \), there exist elements \( c_{pq}(n, i, r) \) in \( k \) such that
\[ f_i^n = \sum_{p=0}^{t_r} \sum_{q=0}^{t_{n-r}} c_{pq}(n, i, r) f_p^1 f_q^{n-r} \]
for all \( n \geq 1 \), all \( i \) in \( \{0, 1, \ldots, t_n\} \) and all \( r \) in \( \{0, 1, \ldots, n\} \).

As of now we do not know if there is a generalization to any classes of non-Koszul algebras. But for a Koszul algebra one can show that the formulas
\[ f_i^n = \sum_{p=0}^{t_1} \sum_{q=0}^{t_{n-1}} c_{pq}(n, i, 1) f_p^1 f_q^{n-1} \]
\[ = \sum_{p=0}^{t_{n-1}} \sum_{q=0}^{t_1} c_{pq}(n, i, n-1) f_p^{n-1} f_q^1 \]
give the differential in the minimal projective resolution of \( \Lambda_0 \) as a left and as a right \( \Lambda \)-module, respectively (see [GHMS]).


Given the minimal projective resolution of \( \Lambda_0 \) over \( \Lambda \) as a right module, it is easy to find a basis of each \( \text{Ext}^i_{\Lambda}(\Lambda_0, \Lambda_0) \) as a vector space over \( k \). Using the “comultiplicative” structure of the resolution \( L \) given in Proposition 9.6 one can show that the chosen vector space basis for \( E(\Lambda) \) has the following structure constants. For this result we use the following notation. If \( p \) is a path in \( Q \) such that \( p = upv \) for some vertices \( u \) and \( v \), then the origin \( o(p) = u \) and the terminus \( t(p) = v \).

**Theorem 9.7 ([BGSS, Theorem 3.1]).** Let \( \Lambda = kQ/I \) be a Koszul algebra. Let \( \{f_i^n\}_{i=0}^n \) be as above defining a minimal projective resolution \((L, e)\) of \( \Lambda_0 \).

Let \( \hat{f}_i^n : L^n \to \Lambda_0 \) for \( i = 0, 1, \ldots, t_n \) be given by \( \hat{f}_i^n (f_j^n) = \delta_{ij} t_i \), for all \( j = 0, 1, \ldots, t_n \). The set \( \{\hat{f}_i^n\}_{i=0}^n \) represents a \( k \)-basis of \( \text{Ext}^i_{\Lambda}(\Lambda_0, \Lambda_0) \).
Then
\[
\widehat{f}_i^m \widehat{f}_j^n = \sum_{l=0}^{t_m+n} c_{j,i}(m+n, l, n) \widehat{f}_l^{m+n}.
\]

In order to find the multiplicative structure of the Hochschild cohomology ring of a Koszul algebra \( \Lambda \) we first find a minimal projective resolution of \( \Lambda \) over \( \Lambda^e \).

**Theorem 9.8** ([GHMS, Theorem 2.1]). Let \( \Lambda = kQ/I \) be a Koszul algebra, and let \( \{f^n\}_{n=0}^{t_m} \) be as above defining a minimal projective resolution for \( \Lambda_0 \) as a right \( \Lambda \)-module. A minimal projective resolution \((P, \delta)\) of \( \Lambda \) over \( \Lambda^e \) is given by
\[
P^n = \Pi_{i=0}^n \Lambda \sigma(f_i^n) \otimes_k t(f_i^n) \Lambda
\]
for \( n \geq 0 \), where the \( j \)-th component of the differential \( \delta^n : P^n \to P^{n-1} \) applied to the \( i \)-th generator \( \sigma(f_i^n) \otimes t(f_i^n) \) is given by
\[
\sum_{p=0}^{t_1} c_{pq}(n, i, l) \overline{f}_p \sigma(f_j^{n-1}) \otimes t(f_j^{n-1}) + (-1)^n \sum_{q=0}^{t_1} c_{jq}(n, i, n-1) \sigma(f_j^{n-1}) \otimes t(f_j^{n-1}) \overline{f}_q
\]
for \( j = 0, 1, \ldots, t_{n-1} \) and \( n \geq 1 \), and \( \delta^0 : \Pi_{i=0}^n \Lambda e_i \otimes_k e_i \Lambda \to \Lambda \) is the multiplication map.

Let \( \varepsilon_i^n = \sigma(f_i^n) \otimes_k t(f_i^n) \). If we can find a chain map \( \Delta : P \to P \otimes \mathbb{P} \) lifting the identity, the multiplication in \( \text{HH}^*(\Lambda) \) is given by the cup product induced by \( \Delta \) as we pointed out in Section 2. In [BGSS] it is shown that \( \Delta \) given by
\[
\Delta(\varepsilon_i^n) = \sum_{r=0}^{n} \sum_{p=0}^{t_r} \sum_{q=0}^{t_{n-r}} c_{pq}(n, i, r) \varepsilon_p^r \otimes_k \varepsilon_q^{n-r}
\]
is such a lifting. A straightforward calculation then shows the following result.

**Theorem 9.9** ([BGSS, Theorem 2.3]). Let \( \Lambda = kQ/I \) be a Koszul algebra. Suppose that \( \eta : P^n \to \Lambda \) and \( \theta : P^m \to \Lambda \) represent elements in \( \text{HH}^*(\Lambda) \) and \( \eta(\varepsilon_i^n) = \lambda_i \) for \( i = 0, 1, \ldots, t_n \) and \( \theta(\varepsilon_i^m) = \lambda_i' \) for \( i = 0, 1, \ldots, t_m \). Then
\[
(\eta \ast \theta)(\varepsilon_i^{m+n}) = \sum_{p=0}^{t_m} \sum_{q=0}^{t_n} c_{pq}(n+m, i, n) \lambda_p \lambda_q
\]
for all \( i = 0, 1, \ldots, t_{m+n} \).

We apply these results to the quantized exterior algebra \( \Lambda = k\langle x, y \rangle / (x^2, xy + qyx, y^2) \) with \( q \) in \( k \setminus \{0\} \). It is shown in [BGMS] that a minimal projective resolution of \( \Lambda_0 = k \) as a right \( \Lambda \)-module is given inductively in terms of the elements
\[
f_i^n = f_{i-1}^{n-1} y + q^i f_i^{n-1} x
\]
for \( i = 0, 1, \ldots, n \), where \( f_0^0 = 1 \). Using Theorem 9.9 one can show the following.

**Theorem 9.10** ([BGMS]). Let \( \Lambda_q = k\langle x, y \rangle / (x^2, xy + qyx, y^2) \) with \( q \) in \( k \setminus \{0\} \). If \( q \) is not a root of unity, then
\[
\dim_k \text{HH}^*(\Lambda) = 5
\]
and
\[
\sum_{i \geq 0} \dim_k \text{HH}^i(\Lambda)t^i = 2 + 2t + t^2.
\]
This algebra gives a counterexample to a question of Dieter Happel as shown in [BGMS]: If the Hochschild cohomology groups $HH^n(\Gamma)$ of a finite dimensional algebra $\Gamma$ over a field $k$ vanish for all sufficiently large $n$, is the global dimension of $\Gamma$ finite?

Furthermore, since $E(\Lambda) = k(x,y)/(yx - qxy)$ is infinite dimensional and $\text{Im } \varphi_{\Lambda/\tau} = k$ when $q$ is not a root of unity, this gives an example of a “nice” selfinjective finite dimensional algebra where the finite generation properties $Fg1$ and $Fg2$ are not satisfied.

In general the image of the map $\varphi_{\Lambda/\tau}: HH^*(\Lambda) \to \text{Ext}^*_\Lambda(\Lambda/\tau, \Lambda/\tau)$ for a finite dimensional algebra $\Lambda$ is contained in the graded centre $Z_{gr}(E(\Lambda))$ of $E(\Lambda) = \text{Ext}^*_\Lambda(\Lambda/\tau, \Lambda/\tau)$. The behavior of $E(\Lambda)$ as a module over $Z_{gr}(E(\Lambda))$ and the image of $\varphi_{\Lambda/\tau}$ are important when considering the conditions $Fg1$ and $Fg2$. For Koszul algebras the corresponding map is $\varphi_{\Lambda_0}$ and its image is in fact the whole of $Z_{gr}(E(\Lambda))$.

**Theorem 9.11 ([BGSS, Theorem 4.1], [K]).** Let $\Lambda = kQ/I$ be a Koszul algebra. The image of the natural map $\varphi_{\Lambda_0}: HH^*(\Lambda) \to E(\Lambda)$ is the graded centre $Z_{gr}(E(\Lambda))$.

As an immediate consequence of the above result we explain how we now can prove the following result.

**Theorem 9.12 ([Ci]).** Let $\Lambda = kQ/J^2$ be a radical square zero algebra, where $Q$ is not an oriented cycle $(\tilde{h}_n)$. Then any product of two elements in $HH^{2*}(\Lambda)$ is zero.

**Proof.** Since $\Lambda = kQ/J^2$ is a quadratic monomial algebra, it follows from [GH] that $\Lambda$ is Koszul. By the basic properties of Koszul algebras $E(\Lambda) = kQ$. By the assumption on $Q$ it is known that $Z_{gr}(E(\Lambda)) = k$. Recall from Proposition 5.8 that $\ker \varphi_{\Lambda/\tau} = HH^{2*}(\Lambda)$ is a nilpotent ideal with nilpotency index at most the Loewy length of $\Lambda$. The claim follows from this. \qed

### 9.3. Recipe for support varieties for selfinjective Koszul algebras.

Theorem 9.11 in the previous subsection actually gives a recipe for checking if it is possible to find a non-trivial theory of support varieties over a finite dimensional selfinjective Koszul algebra. We apply this recipe for the exterior algebra to obtain a recent result of D. Eisenbud [Ei, Theorem 2.2] and more generally to Koszul duals of Artin-Schelter regular Koszul algebras.

Let $\Lambda$ be a finite dimensional selfinjective Koszul algebra over an algebraically closed field. In order for a graded subalgebra $H$ of $HH^*(\Lambda)$ to satisfy the condition $Fg2$, it is certainly necessary that $E(\Lambda)$ is a finitely generated module over $Z_{gr}(E(\Lambda))$. If this is the case and since the homomorphic image of any noetherian algebra is noetherian, we need to find a commutative noetherian graded subalgebra $H' \subseteq Z_{gr}(E(\Lambda))$ such that $Z_{gr}(E(\Lambda))$ is a finitely generated $H'$-module. Since $\varphi_{\Lambda_0}: HH^*(\Lambda) \to Z_{gr}(E(\Lambda))$ is onto, given that we have been successful so far, we obtain a graded subalgebra $H$ of $HH^*(\Lambda)$ satisfying $Fg1$ and $Fg2$ by choosing a graded preimage of $H'$ in $HH^*(\Lambda)$.

We shall apply the above recipe to reprove the following result. The result was first proved for graded modules in [ArAvH].

**Theorem 9.13 ([Ei, Theorem 2.2]).** Let $V$ be an $n$-dimensional vector space over a field $k$. Let $\Lambda = \wedge^nV$ be the exterior algebra of $V$ over $k$. 


Any non-projective indecomposable \( \Lambda \)-module \( M \) with bounded Betti numbers is periodic of period at most 2; that is, \( \Omega^t_\Lambda(M) \simeq M \) for \( t = 1 \) or \( t = 2 \).

**Proof.** Let \( \Lambda = \wedge^\infty V \), which is a finite dimensional selfinjective algebra. Moreover, we have seen that \( E(\Lambda) = k[x_0, x_1, \ldots, x_n] \). It is easy to see that

\[
Z_{gr}(k[x_0, x_1, \ldots, x_n]) = \begin{cases} k[\{x_i x_j\}_{i,j}], & \text{char } k \neq 2 \\ k[x_0, x_1, \ldots, x_n], & \text{char } k = 2 
\end{cases}
\]

In any case \( E(\Lambda) \) is a finitely generated module over \( H' = k[x_0^2, x_1^2, \ldots, x_n^2] \), and by choosing \( H \) as a graded inverse image of \( H' \), the conditions \( \text{Fg1} \) and \( \text{Fg2} \) are satisfied. Using Proposition 7.2 it follows that the period of an indecomposable periodic \( \Lambda \)-module \( M \) divides one of the degrees of \( x_i^2 \); that is, 2. The claim follows from this.

We end this section by pointing out a class of algebras where the theory of the support varieties developed in the previous sections can be applied. In fact, the following is a generalization of the arguments we carried out for Theorem 9.13.

Let \( k \) be an algebraically closed field. Recall that a \( k \)-algebra \( R \) is called **connected graded** if \( R = \bigoplus_{i \geq 0} R_i \) is a finitely generated positively graded \( k \)-algebra with \( R_0 = k \). The **Gelfand-Kirillov dimension** of \( R \) is given by

\[
\text{GKdim } R = \inf \{ \alpha \in \mathbb{R} \mid \dim_k (\sum_{i=0}^n R_i) \leq n^\alpha \text{ for all } n \gg 0 \}.
\]

For \( R \) commutative this is nothing else than the Krull dimension of \( R \). Furthermore, recall that a connected graded \( k \)-algebra \( R \) is called **Artin-Schelter regular** of dimension \( d \) if the global dimension of \( R \) is \( d \), the Gelfand-Kirillov dimension of \( R \) is finite and \( R \) is Artin-Schelter Gorenstein; that is, \( \text{Ext}_R^i(k, R) = (0) \) for \( i \neq d \) while \( \text{Ext}_R^d(k, R) \simeq k \) up to shift.

**Theorem 9.14 ([Sm, Proposition 5.10]).** Let \( R \) be a connected graded Koszul algebra of finite global dimension over a field \( k \). Then \( R \) is Artin-Schelter Gorenstein if and only if \( E(R) \) is selfinjective.

In [M-V] this is generalized to the non-connected situation.

The behavior of some Artin-Schelter regular algebras over their centres is known. For example, the Skylanin algebras \( R \) in dimension 3 or 4 are given as \( R = R(E, \tau) \), where \( E \) is an elliptic curve and \( \tau \) is a point on \( E \). By [ATvdB, ST] \( R \) is a finitely generated module over its ordinary centre \( Z(R) \) if and only if \( \tau \) has finite order.

Let us assume that \( R = \bigoplus_{i \geq 0} R_i \) is a noetherian graded \( k \)-algebra, and assume that \( R \) is a finitely generated module over its centre \( Z(R) \) (essentially finite). By [AT] this implies that \( Z(R) \) is a noetherian ring. Since \( Z(R) \) is commutative and graded, we know that \( Z(R) = Z(R)_0[x_1, \ldots, x_n] \) for some homogeneous elements \( x_i \) in \( Z(R) \) where \( Z(R)_0 \) is noetherian. Now let us compare \( Z_{gr}(R) \) with \( Z(R) \). It is clear from the definitions that \( Z_{gr}(R)_{\text{even}} = Z(R)_{\text{even}} \). Suppose that the generators \( x_1, \ldots, x_t \) have odd degrees and that the rest have even degrees. Then it is easy to see that \( Z(R) \) is generated by \( \{x_1, \ldots, x_t\} \) as a module over \( Z_{gr}(R)_{\text{even}} \). We infer that \( R \) is a finitely generated module over \( Z_{gr}(R)_{\text{even}} \) (and \( Z_{gr}(R) \)). Hence we have essentially proved the following.
Proposition 9.15. Let $\Lambda$ be the Koszul dual of an Artin-Schelter regular noetherian Koszul algebra $R$ over a field $k$. Suppose that $R$ is a finitely generated module over $Z(R)$.

(a) $\Lambda$ is a finite dimensional selfinjective algebra.
(b) There exists a graded subalgebra $H$ of $\text{HH}^*(\Lambda)$ such that $\Lambda$ and $H$ satisfy $\text{Fg}1$ and $\text{Fg}2$.

Proof. The claim in (a) is one of the implications of Theorem 9.14. By the arguments preceding the proposition, $R = E(\Lambda)$ is a finitely generated module over $Z_{gr}(R)_{\text{even}}$, which is a commutative noetherian algebra. Since $\Lambda$ is Koszul, $\varphi_{\Lambda/k}: \text{HH}^*(\Lambda) \to R$ maps onto $Z_{gr}(R)$. Hence for $H$ we can choose a graded preimage of $Z_{gr}(R)_{\text{even}}$. 

10. Support varieties for complexes

This section is devoted to defining support varieties of a bounded complex of finitely generated modules over a finite dimensional algebra $\Lambda$. With the same finiteness conditions $\text{Fg}1$ and $\text{Fg}2$, there seem to be the same rich theory as for modules. Even though we show that, if we are interested in complexes with non-trivial variety, we naturally are led back to the module situation. However considering complexes still has one advantage. The seemingly different behavior of selfinjective and Gorenstein algebras seems to disappear. We end this section with a brief discussion on a general framework within to define support varieties. Further details on the results discussed here can be found in [BKSS].

10.1. The definition for complexes. We first revisit the definition for modules. Let $\Lambda$ be a finite dimensional algebra over an algebraically closed field $k$. Let $P: \cdots \to P^n \to P^{n-1} \to \cdots \to P^0 \to 0$ be a minimal projective resolution of $\Lambda$ as a $\Lambda'$-module, which is quasi-isomorphic to the stalk complex of $\Lambda$ concentrated in degree 0 via the natural map $P \xrightarrow{\mu} \Lambda$. The definition of the support variety for a module uses the homomorphism $\varphi_{\Lambda}: \text{HH}^*(\Lambda) \to \text{Ext}_\Lambda^*(M, M)$. We have that $\text{HH}^*(\Lambda) \simeq \Pi_{i \in \mathbb{Z}} \text{Hom}_{D(\text{mod} \Lambda)}(\Lambda, i)$ and $\text{Ext}_\Lambda^*(M, M) \simeq \Pi_{i \in \mathbb{Z}} \text{Hom}_{D(\Lambda)}(M, M[i])$, where an element $\eta: P \to \Lambda[n]$ in $\text{HH}^*(\Lambda)$ corresponds to the map

$$\Lambda \xrightarrow{\mu} P \xrightarrow{\eta} \Lambda[n]$$

in $D(\text{mod} \Lambda^c)$. Then we obtain the natural map $\text{HH}^*(\Lambda) \to \Pi_{i \in \mathbb{Z}} \text{Hom}_{D(\Lambda)}(M, M[i])$ given by

$$\eta \mapsto \Lambda \otimes \Lambda M \simeq M \xrightarrow{\mu \otimes 1_M} P \otimes \Lambda M \xrightarrow{\eta \otimes 1_M} \Lambda[n] \otimes \Lambda M \simeq M[n]$$

Since the map $\mu \otimes \Lambda X^*$ is a quasi-isomorphism for all stalk complexes $X^*$, it follows that $\mu \otimes \Lambda X^*$ is a quasi-isomorphism for all complexes in $D(\text{mod} \Lambda^c)$. Then one can show that there is a natural map

$$\varphi_{\Lambda^c}: \text{HH}^*(\Lambda) \to \Pi_{i \in \mathbb{Z}} \text{Hom}_{D(\text{mod} \Lambda)}(X^*, X^*[i]) = \text{Hom}_{D(\text{mod} \Lambda)}^{*}(X^*, X^*)$$

given by

$$\Lambda \otimes \Lambda X^* \simeq X^* \xrightarrow{\mu \otimes 1_X} P \otimes \Lambda X^* \xrightarrow{\eta \otimes 1_X} \Lambda[n] \otimes \Lambda X^* \simeq X^*[n]$$
Similarly as for modules we obtain that $\text{Hom}_{D(A)}(X^*, Y^*)$ becomes a left and a right $HH^\ast(A)$-module via $\varphi_Y$- and $\varphi_X$, respectively. Moreover, for $\eta \in HH^m(A)$ and $\theta \in \text{Hom}_{D(A)}(X^*, Y^*[n])$ we have that

$$
\eta \cdot \theta = \varphi_Y \cdot (\eta \theta) = (-1)^{mn} \theta \varphi_X \cdot = (-1)^{mn} \theta \cdot \eta
$$

Let $H$ be a graded subalgebra such that $A$ and $H$ satisfy $\text{Fg1}$. Then we define the support variety of a pair of complexes in $D^b(\text{mod } \Lambda)$ as a subvariety of $\text{MaxSpec } H$ similarly as for a pair of modules.

**Definition 10.1.** For two complexes $X^*$ and $Y^*$ in $D^b(\text{mod } \Lambda)$ the support variety $V_H(X^*, Y^*)$ is the subvariety of $\text{MaxSpec } H$ given by

$$
V_H(X^*, Y^*) = \text{MaxSpec}(H/ \text{Ann}_H \text{Hom}_{D(A)}(X^*, Y^*))
$$

Similarly as for short exact sequences of modules, given a triangle $X_1 \to X_2 \to X_3 \to X_1[-1]$ in $D^b(\text{mod } \Lambda)$, then $V_H(X_r, Y) \subseteq V_H(X_s, Y) \cup V_H(X_t, Y)$ for $\{r, s, t\} = \{1, 2, 3\}$. Using this and that $D^b(\text{mod } \Lambda)$ as a triangulated category is generated by $\Lambda/\tau$ we have the following.

**Theorem 10.2 ([BKSS]).** For any complex $X^*$ in $D^b(\text{mod } \Lambda)$

$$
V_H(X^*, \Lambda/\tau) = V_H(X^*, X^*) = V_H(\Lambda/\tau, X^*)
$$

Similarly as in the module case define the support variety $V_H(X^*)$ of a complex $X^*$ in $D^b(\text{mod } \Lambda)$ to be $V_H(X^*) = V_H(X^*, \Lambda/\tau)$. Also as before $V_H(X^*)$ always contains $\mathfrak{m}_{gr}$ for a non-zero complex $X^*$ in $D(\text{mod } \Lambda)$. The variety of $X^*$ is trivial if $V_H(X^*) = \{\mathfrak{m}_{gr}\}$. Moreover, for a module $M$ it is easy to see that the support variety of $M$ when we view $M$ as a module and $M$ as a stalk complex concentrated in one degree coincide.

For modules the assumptions we needed to get a good theory of support varieties were the two conditions $\text{Fg1}$ and $\text{Fg2}$. A similar condition for complexes would be that all the $H$-modules $\text{Hom}_{D(A)}(X^*, Y^*)$ for $X^*$ and $Y^*$ in $D^b(\text{mod } \Lambda)$ are finitely generated. Next we see that this condition for complexes is in fact equivalent to the condition $\text{Fg2}$ for modules.

**Proposition 10.3 ([BKSS]).** Suppose $\Lambda$ and $H$ satisfy $\text{Fg1}$. Then the following are equivalent.

(i) $\text{Ext}_A^\ast(\Lambda/\tau, \Lambda/\tau)$ is a finitely generated $H$-module.

(ii) $\text{Ext}_A^\ast(M, N)$ is a finitely generated $H$-module for all $M$ and $N$ in mod $\Lambda$.

(iii) $\text{Hom}_{D(A)}^\ast(X^*, Y^*)$ is a finitely generated $H$-module for all $X^*$ and $Y^*$ in $D^b(\text{mod } \Lambda)$.

Given that the support variety of a finitely generated projective module is trivial, the support variety of a stalk complex of a finitely generated projective module is also trivial. Since any bounded complex of finitely generated projective modules (a perfect complex) can be obtained by a sequence of mapping cones involving stalk complexes of finitely generated projective modules, it follows that the variety of a perfect complex is trivial. Under the usual finiteness conditions $\text{Fg1}$ and $\text{Fg2}$ the converse is also true.

**Theorem 10.4 ([BKSS]).** Suppose that $\Lambda$ and $H$ satisfy $\text{Fg1}$ and $\text{Fg2}$. Let $X^*$ be in $D^b(\text{mod } \Lambda)$. Then $V_H(X^*)$ is trivial if and only if $X^*$ is a perfect complex.
Suppose that $\Lambda$ and $H$ satisfy $Fg1$ and $Fg2$. Since the subcategory $D^\text{perf}(\mod \Lambda)$ of perfect complexes of $D^b(\mod \Lambda)$ is a Serre subcategory, we can form the Verdier quotient $D^b(\mod \Lambda)/D^\text{perf}(\mod \Lambda)$. This quotient is interesting since it is the natural habitat for the complexes with non-trivial variety. Recall that if $\Lambda$ and $H$ satisfy $Fg1$ and $Fg2$, then $\Lambda$ is Gorenstein. It is known that $D^b(\mod \Lambda)/D^\text{perf}(\mod \Lambda)$ is equivalent to $\mod \Lambda$ when $\Lambda$ is selfinjective and equivalent to $\text{CM}(\Lambda)$ when $\Lambda$ is Gorenstein (see [Buc, Ha, R]). Here, $\text{CM}(\Lambda)$ denotes the full subcategory of $\mod \Lambda$ consisting of all the modules $M$ with $\text{Ext}^i_M(M, \Lambda) = (0)$ for all $i \geq 1$. Hence, we are back in the module situation. However, whether $\Lambda$ is selfinjective or Gorenstein, the complexes with trivial variety have the same description.

10.2. General setup for support varieties. The construction of support varieties of complexes carried out above can be generalized to a more abstract framework. When analyzing the underlying structure one finds that we have a triangulated monoidal category $(\mathcal{C}, \otimes, T)$ acting on a triangulated category $(\mathcal{A}, \Sigma)$. By a triangulated monoidal category we basically mean a triangulated category with a tensor product having a monoidal identity, hence having a bifunctor $\mathcal{C} \times \mathcal{C} \overset{\otimes}{\longrightarrow} \mathcal{C}$. An action of the category $\mathcal{C}$ on $\mathcal{A}$ is the existence of a bifunctor $\mathcal{C} \times \mathcal{A} \overset{\alpha}{\longrightarrow} \mathcal{A}$ having certain conditions. For instance, for the monoidal identity object $e$, there should exist isomorphisms $e \otimes c \simeq c \simeq c \otimes e$ for all objects $c \in \mathcal{C}$ and $e \cdot a \simeq a$ for all objects $a \in \mathcal{A}$. The assumptions on the categories and the action involved implies that the graded ring $\text{End}^*_\mathcal{C}(e) = \bigoplus_{q \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(e, T^q(e))$ is graded commutative. This ring plays the role of the Hochschild cohomology ring. Moreover the ring $\text{End}^*_\mathcal{C}(e)$ acts on $\text{Hom}_\mathcal{A}(a, b) = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{A}}(a, \Sigma^p(b))$ on the left and the right with the same “graded commutativity” as in the module situation. To define the support variety of a pair of objects $(a, b)$ in $\mathcal{A}$, we choose a commutative noetherian graded subalgebra $H$ of $\text{End}^*_\mathcal{C}(e)$, and we define the support variety $V_H(a, b)$ as the subvariety of $\text{MaxSpec} H$ given by

$$V_H(a, b) = \text{MaxSpec}(H/ \text{Ann}_H \text{Hom}_\mathcal{A}(a, b)).$$

To recover the support varieties for a group ring $kG$ of a finite group $G$ we can choose $\mathcal{C} = \mathcal{A} = D(\mod kG)$, and the monoidal structure and the action are both given by the tensor product $- \otimes_k -$. For a finite dimensional algebra $\Lambda$ we can choose $\mathcal{C} = D(\mod \Lambda^e)$ and $\mathcal{A} = D(\mod \Lambda)$, with the monoidal structure given by $- \otimes^L_{\Lambda^e}$ and the action given by $- \otimes^L_{\Lambda}$ to recover the support varieties given by a graded subalgebra of the Hochschild cohomology ring.

11. Epilogue

Here we discuss topics which were not mentioned in the lectures. Also we dwell on differences with the theory for group rings and possible analogues of results which are true for group rings.

One starting point for the support varieties for group rings of finite groups over a field of characteristic $p$ was given by Quillen in [Q], where the group cohomology ring $H^*G(k)$ modulo nilpotent elements is described as an inverse limit of cohomology rings of the elementary abelian $p$-subgroups of $G$. For elementary abelian groups $E$, Carlson defined a rank variety $V^\#_E(M)$ for any finitely generated module $M$ in [Carl2]. In addition showing that $V^\#_E(M)$ is contained in $V_E(M)$ and that their dimensions are equal. It was shown in [AvrS] that $V^\#_E(M)$ and $V_E(M)$ indeed are equal as varieties. Similarly as for the group cohomology ring,
the support variety of any finitely generated module $M$ is stratified by restrictions of elementary abelian subgroups of $G$ as shown in [AvrS]. There is no analogue of the rank varieties for elementary abelian groups known for support varieties over a finite dimensional algebra defined in terms of the Hochschild cohomology ring of the algebra. Also, a stratification of the support varieties is not known in general. However, there are some results of Erdmann and Holloway indicating that there might be an analogue of the rank varieties also in the more general setting of finite dimensional algebras (see [EH, EH2]).

For group rings one has that $V_G(M \otimes_k N) = V_G(M) \cap V_G(N)$ is true for any finitely generated $kG$-modules $M$ and $N$. There is no analogue of this known for support varieties over finite dimensional algebras in general. It is not even clear what the analogue should be. On suggestion could be the following: For any module $B$ in $\text{mod} \Lambda$ and $M$ in $\text{mod} \Lambda$, is it true that

$$V_H(B \otimes_{\Lambda} M) = V_H(B \otimes_{\Lambda} \Lambda/\tau) \cap V_H(M)?$$

We have one indication for this through Theorem 6.1, which answers the question affirmatively for $B = M$ when $\text{Fg1}$ and $\text{Fg2}$ are satisfied. Initial investigations into a possible classification of certain thick “tensor” ideals in the stable module category of a selfinjective algebra, lead us to believe that this is a desirable property to have.

For a finite group $G$ the stable category $\text{mod} kG$ of $kG$-modules is a triangulated category, and thick tensor ideals have been classified in terms of the group cohomology ring $[\text{BenCR}]$. This has been extended to finite dimensional cocommutative Hopf algebras in $[\text{FPe}]$. Here connections with infinite dimensional modules are made through the idempotent modules in $\text{mod} kG$ introduced in $[\text{R}]$. For a finite dimensional selfinjective algebra $\Lambda$ satisfying $\text{Fg1}$ and $\text{Fg2}$ a similar classification of thick subcategories of $\text{mod} \Lambda$ seems to be a very hard problem. However, the attempts done so far show that there might be idempotent bimodules playing the same role as the idempotent modules a la Rickard (see $[\text{R2}]$).

Let $\Lambda$ be a finite dimensional algebra over a field $k$. We have seen that the conditions $\text{Fg1}$ and $\text{Fg2}$ have been the essential assumption in all of our results. By Proposition 5.7 the properties $\text{Fg1}$ and $\text{Fg2}$ are equivalent to that $\text{HH}^*(\Lambda)$ is a noetherian ring and $\text{Ext}_\Lambda^*(\Lambda/\tau, \Lambda/\tau)$ is a finitely generated module over $\text{HH}^*(\Lambda)$. So it would be very interesting to have a characterization of the finite dimensional algebras satisfying these properties. Since the Hochschild cohomology ring $\text{HH}^*(\Lambda)$ modulo the ideal $\mathcal{N}$ generated by all the homogeneous nilpotent elements is always commutative, $\text{HH}^*(\Lambda)/\mathcal{N}$ is noetherian if and only if it is a finitely generated $k$-algebra. So this connects the conjecture mentioned in Section 5 with the above question.

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References


Farnsteiner, R., Complexity of tame blocks of infinitesimal group schemes, preprint.


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