Relative homology and representation theory III.

Cotilting modules and Wedderburn correspondence.

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Introduction.

This paper, the last of a series of three papers studying the uses of relative homological algebra in the representation theory of artin algebras, is devoted to giving a rather explicit connection between the relative cotilting theory introduced in the previous paper and standard cotilting theory. The reader is referred to the previous papers in this series [2, 3] for basic definitions and results, as well as notations, concerning the relative homological algebra and theory of relative cotilting modules used in this paper. We now describe the main result of this paper.

Let $\operatorname{mod}\Lambda$ be the category of finitely generated left modules over an artin algebra Λ. Suppose F is an additive subfunctor of the additive bifunctor $\operatorname{Ext}^1_\Lambda(\ ,\ )\colon (\operatorname{mod}\Lambda)^{\mathrm{op}}\times\operatorname{mod}\Lambda\to \mathrm{Ab}$. Assume that F has enough projectives and injectives and that $\mathcal{P}(F)$, the subcategory of $\operatorname{mod}\Lambda$ consisting of the F-projectives, is $\operatorname{add}G$ for some G in $\operatorname{mod}\Lambda$. Then G is a generator for $\operatorname{mod}\Lambda$ since $\mathcal{P}(F)$ contains Λ. Let $\Gamma_G=\operatorname{End}_\Lambda(G)^{\mathrm{op}}$.

Let T in $\operatorname{mod}\Lambda$ be an F-cotilting module. We show that the Γ_G-module $\operatorname{Hom}_\Lambda(G,T)=(G,T)$ is a standard cotilting module and that the algebra $\operatorname{End}_{\Gamma_G}((G,T))$ is naturally isomorphic to $\Gamma=\operatorname{End}_\Lambda(T)$. In addition we show that the relative cotilting functor $\operatorname{Hom}_\Lambda(\ ,T)\colon\operatorname{mod}\Lambda\to\operatorname{mod}\Gamma$ is canonically isomorphic to the composition $\operatorname{Hom}_{\Gamma_G}(\ ,(G,T))\circ\operatorname{Hom}_\Lambda(G,\ )$ of the functors $\operatorname{Hom}_\Lambda(G,\ )\colon\operatorname{mod}\Lambda\to\operatorname{mod}\Gamma_G$ and the standard cotilting functor $\operatorname{Hom}_{\Gamma_G}(\ ,(G,T))\colon\operatorname{mod}\Gamma_G\to\operatorname{mod}\Gamma$. This shows how relative cotilting functors can be described in terms of standard cotilting functors. The proof of this result uses the notion of the Wedderburn correspondence introduced in [1] and also gives the following connection between the Wedderburn correspondence, and both standard and relative tilting and cotilting theory.

Suppose G is a generator for $\operatorname{mod}\Lambda$. Denote $F_{\operatorname{add}G}$ by F. Then F has enough projectives and injectives and $\mathcal{P}(F)=\operatorname{add}G$. Again letting $\Gamma_G=\operatorname{End}_\Lambda(G)^{\mathrm{op}}$, we get that G is an F-tilting module. Thus $\operatorname{Hom}_\Lambda(G,\ )\colon\operatorname{mod}\Lambda\to\operatorname{mod}\Gamma_G$ is a special case of a relative tilting functor. We then get that $\operatorname{Hom}_\Lambda(G,\ )\colon\operatorname{mod}\Lambda\to\operatorname{mod}\Gamma_G$ is canonically isomorphic to the composition $\operatorname{Hom}_\Gamma(\ ,(G,T))\circ\operatorname{Hom}_\Lambda(\ ,T)$ where $\operatorname{Hom}_\Lambda(\ ,T)\colon\operatorname{mod}\Lambda\to\operatorname{mod}\Gamma$ is a relative cotilting functor and $\operatorname{Hom}_\Gamma(\ ,(G,T))\colon\operatorname{mod}\Gamma\to\operatorname{mod}\Gamma_G$ is a standard cotilting functor with $\operatorname{id}_\Gamma(G,T)=2$. Further, we have that the Grothendieck groups F-$K_0(\operatorname{mod}\Lambda)$ and $K_0(\operatorname{mod}\Gamma_G)$ are isomorphic.

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1 Preliminaries.

For the basic definitions and results about relative homology and relative cotilting theory we refer the reader to the two papers [2] and [3]. However for the convenience of the reader we recall some of the most frequently used definitions and results from [2] and [3].

Let Λ be an artin algebra and let F be an additive subfunctor of $\mathrm{Ext}^1_\Lambda(\ ,\ )$ with enough projectives and injectives. Let $\mathcal{C}$ be any subcategory of $\mathrm{mod}\,\Lambda$. We denote by ${}^\perp\mathcal{C}$ the subcategory of $\mathrm{mod}\,\Lambda$ given by $\{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^i_F(X,\mathcal{C}) = 0, \text{for all } i > 0\}$ and by $\mathcal{C}^\perp$ the subcategory of $\mathrm{mod}\,\Lambda$ given by $\{Y \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^i_F(\mathcal{C},Y) = 0, \text{for all } i > 0\}$.

For an F-selforthogonal Λ-module T denote by $\mathcal{X}_T$ the subcategory of ${}^\perp T$ whose objects are the Λ-modules C such that there is an F-exact sequence

$$0 \to C \to T_0 \stackrel{f_0}{\to} T_1 \to \cdots \to T_n \stackrel{f_n}{\to} T_{n+1} \to \cdots$$

with T_i in $\mathrm{add}\,T$ and $\mathrm{Im}\,f_i$ in ${}^\perp T$ for all $i \geq 0$. In [3, Theorem 3.2] we showed that $\mathcal{X}_T = {}^\perp T$ for all F-cotilting modules T.

For a subcategory $\mathcal{C}$ in $\mathrm{mod}\,\Lambda$ we denote by $\widehat{\mathcal{C}}$ the subcategory of $\mathrm{mod}\,\Lambda$ whose objects are the Λ-modules M for which there is an F-exact sequence

$$0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to M \to 0$$

with C_i in $\mathcal{C}$.

Let $\mathcal{X}$ be a contravariantly finite F-generator in $\mathrm{mod}\,\Lambda$. Then the resolution dimension of a Λ-module M with respect to $\mathcal{X}$ is defined to be the minimum of n including infinity such that there exists an F-exact sequence

$$0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0,$$

where X_i is in $\mathcal{X}$. We denote this dimension by $\mathcal{X}\text{-resdim}_F\, M$. If $\mathcal{Y}$ is a subcategory of $\mathrm{mod}\,\Lambda$ then $\mathcal{X}\text{-resdim}_F\,\mathcal{Y}$ is defined to be $\sup\{\mathcal{X}\text{-}\,\mathrm{resdim}_F\, Y \mid Y \in \mathcal{Y}\}$.

A subcategory $\mathcal{W}$ of $\mathrm{mod}\,\Lambda$ is called an Ext_F*-injective cogenerator* for a subcategory $\mathcal{X}$ of $\mathrm{mod}\,\Lambda$ if (i) $\mathcal{W}$ is contained in $\mathcal{X} \cap \mathcal{X}^\perp$ and (ii) for each X in $\mathcal{X}$ there exists an F-exact sequence $0 \to X \to W \to X' \to 0$ in $\mathcal{X}$ with W in $\mathcal{W}$. The following two results are key results in our investigation of relative cotilting theory.

Theorem 1.1 ([3, Theorem 2.4]) *Let $\mathcal{X}$ be an F-resolving subcategory of $\mathrm{mod}\,\Lambda$ with an Ext_F-injective cogenerator $\mathcal{W}$. If $\widehat{\mathcal{X}} = \mathrm{mod}\,\Lambda$, we have the following.*

(a) *The subcategory $\mathcal{X}$ is contravariantly finite in $\mathrm{mod}\,\Lambda$.*

(b) $\mathcal{Y} = \mathcal{X}^\perp = \widehat{\mathcal{W}}$.

Theorem 1.2 ([3, Theorem 3.13 and Proposition 3.15]) *Let T be an F-cotilting module in $\mathrm{mod}\,\Lambda$ with $\mathrm{id}_F\, T = r$ and let $\Gamma = \mathrm{End}_\Lambda(T)$. Then we have the following.*

(a) *The subcategory $(\mathcal{P}(F),T) = \mathrm{add}\,T_0$ for a cotilting module T_0 over Γ with $\mathrm{id}_\Gamma\, T_0 \leq \max\{r,2\}$. The subcategory $(\mathcal{X}_T,T) = \mathcal{X}_{T_0} = {}^\perp T_0$.*

(b) *The module ${}_\Gamma T$ is a direct summand of a cotilting module T_0 over Γ with $\mathrm{add}\,T_0 = (\mathcal{P}(F),T)$, $\mathrm{id}_\Gamma\, T_0 \leq \max\{r,2\}$ and $\mathrm{id}_\Gamma\, T \leq r$. Moreover the natural homomorphism $X \to \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Gamma(X,T),T)$ is an isomorphism for all X in $\mathcal{X}_{T_0} = {}^\perp T_0$.*

The last notion we recall from [2, 3] is the notion of the relative Grothendieck group. Let $\mathbf{Z}(\mathrm{mod}\,\Lambda)$ denote the free abelian group with the isomorphism classes $[A]$ of modules A in $\mathrm{mod}\,\Lambda$ as basis. We define the *F-Grothendieck group* of $\mathrm{mod}\,\Lambda$, $F\text{-}K_0(\mathrm{mod}\,\Lambda)$, to be $\mathbf{Z}(\mathrm{mod}\,\Lambda)/F\text{-}R(\mathrm{mod}\,\Lambda)$, where $F\text{-}R(\mathrm{mod}\,\Lambda)$ is the subgroup of $\mathbf{Z}(\mathrm{mod}\,\Lambda)$ generated by the elements $[A]+[C]-[B]$ whenever there is an F-exact sequence of modules $0 \to A \to B \to C \to 0$.

2 Dualizing summands of cotilting modules.

Throughout this section let F be an additive subfunctor of $\mathrm{Ext}^1_\Lambda(\ ,\ )$ with enough projectives and injectives. Let T be an F-cotilting module and let $\Gamma = \mathrm{End}_\Lambda(T)$. Then by Theorem 1.2 (b) the module ${}_\Gamma T$ is a direct summand of a cotilting module T_0 over Γ with the property that the natural homomorphism $X \to \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Gamma(X,T),T)$ is an isomorphism for all X in $\mathcal{X}_{T_0}$. In particular the natural homomorphism $T_0 \to \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Gamma(T_0,T),T)$ is an isomorphism. We make the following definition for a module M over an artin algebra Γ. Let $M = M' \oplus M''$ and $\Lambda = \mathrm{End}_\Gamma(M')$. If the natural homomorphism $A \to \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Gamma(A,M'),M')$ is an isomorphism for some Λ-module A, then M' is said to *dualize* A. If this is true for $A = M$, then M' is said to be a *dualizing* summand of M.

Let $T = T_a \oplus T_b$ be a cotilting module over a artin algebra Γ with T_a a dualizing summand of T. Denote by Λ the artin algebra $\mathrm{End}_\Gamma(T_a)$ and $F = F_{\mathcal{P}(F)}$ where $\mathcal{P}(F) = \mathrm{Hom}_\Gamma(\mathrm{add}\,T, T_a)$. We show that T_a is an F-cotilting module in $\mathrm{mod}\,\Lambda$ and that $\mathrm{End}_\Lambda(T_a)$ is isomorphic to Γ. This show that every dualizing summand of a cotilting module is induced from itself as a relative cotilting module over its endomorphism ring.

We begin with the following general observation concerning dualizing summands of arbitrary Λ-modules.

Proposition 2.1 *Let M be an arbitrary module in* $\mathrm{mod}\,\Lambda$ *and let* $\Gamma = \mathrm{End}_\Lambda(M)$. *For a module A in* $\mathrm{mod}\,\Lambda$ *the natural homomorphism*

$$\alpha_A \colon A \to \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Lambda(A,M),M)$$

is an isomorphism if and only there exists an exact sequence

$$0 \to A \xrightarrow{f} M^n \to M^m,$$

where $f \colon A \to M^n$ is a left $\mathrm{add}\,M$*-approximation.*

Proof: Assume that $\alpha_A \colon A \to \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Lambda(A,M),M)$ is an isomorphism. The Γ-module $\mathrm{Hom}_\Lambda(A,M)$ is finitely generated, so we have a Γ-projective presentation

$$\mathrm{Hom}_\Lambda(M^m,M) \xrightarrow{(f_1,M)} \mathrm{Hom}_\Lambda(M^n,M) \xrightarrow{(f_0,M)} \mathrm{Hom}_\Lambda(A,M) \to 0,$$

which is induced from a complex $A \xrightarrow{f_0} M^n \xrightarrow{f_1} M^m$ in $\mathrm{mod}\,\Lambda$. Applying the functor $\mathrm{Hom}_\Gamma(\ ,M)$ to this sequence we obtain the following commutative diagram

$$\begin{array}{ccccccc}
0 & \to & ((A,M),M) & \to & ((M^n,M),M) & \to & ((M^m,M),M) \\
 & & \uparrow \wr\, \alpha_A & & \uparrow \wr\, \alpha_{M^n} & & \uparrow \wr\, \alpha_{M^m} \\
0 & \to & A & \xrightarrow{f_0} & M^n & \xrightarrow{f_1} & M^m,
\end{array}$$

where the upper row is exact. Hence the lower row is also exact. Moreover it has the property that every map $g \colon A \to M$ extends to M^n.

Assume that there is an exact sequence $0 \to A \xrightarrow{f_0} M^n \xrightarrow{f_1} M^m$ such that $f_0 \colon A \to M^n$ is a left $\mathrm{add}\,M$-approximation. Let $K = \mathrm{coker}\,f_0$, then choose or modify f_1 if necessary such that $K \to M^m$ is a left $\mathrm{add}\,M$-approximation. Then the induced sequence

$$\mathrm{Hom}_\Lambda(M^m,M) \to \mathrm{Hom}_\Lambda(M^n,M) \to \mathrm{Hom}_\Lambda(A,M) \to 0$$

is exact. Therefore we have the following commutative exact diagram

$$\begin{array}{ccccccc}
0 & \to & A & \xrightarrow{f_0} & M^n & \xrightarrow{f_1} & M^m \\
 & & \downarrow \alpha_A & & \downarrow \wr\, \alpha_{M^n} & & \downarrow \wr\, \alpha_{M^m} \\
0 & \to & ((A,M),M) & \to & ((M^n,M),M) & \to & ((M^m,M),M).
\end{array}$$

Hence, α_A is an isomorphism.

From this result the following corollary follows immediately.

Corollary 2.2 *Let M be an arbitrary module in* $\operatorname{mod}\Lambda$ *and let* $\Gamma = \operatorname{End}_\Lambda(M)$. *Assume that the natural homomorphism*

$$\alpha_A \colon A \to \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(A, M), M)$$

is an isomorphism for a module A in $\operatorname{mod}\Lambda$. *Then the natural homomorphism*

$$\alpha'_A \colon A \to \operatorname{Hom}_{\Gamma'}(\operatorname{Hom}_\Lambda(A, M'), M')$$

is an isomorphism for any module M' in $\operatorname{mod}\Lambda$ *with* $\operatorname{add} M' = \operatorname{add} M$ *and* $\Gamma' = \operatorname{End}_\Lambda(M')$.

In rest of this section let $T = T_a \oplus T_b$ be a cotilting module over an artin algebra Γ with $\operatorname{id}_\Gamma T = r$, where T_a is a dualizing summand of T. Let T' be a module such that $\operatorname{add} T' = \operatorname{add} T_a$. Denote by $\Lambda = \operatorname{End}_\Gamma(T')$. Then T' is a Γ-Λ-bimodule. Therefore $\operatorname{Hom}_\Gamma(\operatorname{add} T, T')$ is a subcategory of $\operatorname{mod}\Lambda$ and $F = F_{\operatorname{Hom}_\Gamma(\operatorname{add} T, T')}$ is a subfunctor of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ with enough projectives and injectives. The projectives of F are $\mathcal{P}(F) = \operatorname{Hom}_\Gamma(\operatorname{add} T, T')$ by [2, Proposition 1.10]. Our first aim is to show that T' is an F-cotilting module over Λ. Thus showing that every dualizing summand of a cotilting module is induced from some relative cotilting module. Using this notation we have the following result.

Proposition 2.3
(a) *The functors* $\operatorname{Hom}_\Lambda(\ , {}_\Lambda T'_{\Gamma^{\mathrm{op}}}) \colon \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ *and* $\operatorname{Hom}_\Gamma(\ , {}_\Gamma T'_{\Lambda^{\mathrm{op}}}) \colon \operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$ *are an adjoint pair of contravariant functors, that is, we have the isomorphism*

$$\phi \colon \operatorname{Hom}_\Gamma(A, \operatorname{Hom}_\Lambda(C, {}_\Lambda T'_{\Gamma^{\mathrm{op}}})) \to \operatorname{Hom}_\Lambda(C, \operatorname{Hom}_\Gamma(A, {}_\Gamma T'_{\Lambda^{\mathrm{op}}}))$$

functorial in both variables for all A in $\operatorname{mod}\Gamma$ *and C in* $\operatorname{mod}\Lambda$ *given by $\phi(f)(c)(a) = f(a)(c)$ for f in* $\operatorname{Hom}_\Gamma(A, \operatorname{Hom}_\Lambda(C, {}_\Lambda T'_{\Gamma^{\mathrm{op}}}))$.

(b) *For all X in $\mathcal{X}_T$ the natural homomorphism*

$$X \to \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Gamma(X, T'), T')$$

is an isomorphism.

Proof: (a) Define $\psi \colon \operatorname{Hom}_\Lambda(C, \operatorname{Hom}_\Gamma(A, {}_\Gamma T'_{\Lambda^{\mathrm{op}}})) \to \operatorname{Hom}_\Gamma(A, \operatorname{Hom}_\Lambda(C, {}_\Lambda T'_{\Gamma^{\mathrm{op}}}))$ by $\psi(g)(a)(c) = g(c)(a)$ for g in $\operatorname{Hom}_\Lambda(C, \operatorname{Hom}_\Gamma(A, {}_\Gamma T'_{\Lambda^{\mathrm{op}}}))$. It is easy to see that ϕ and ψ are inverse isomorphisms functorial in both variables.

(b) Let X be in $\mathcal{X}_T$. Then there is an exact sequence $0 \to X \to T_0 \xrightarrow{d} T_1$, where $\operatorname{Im} d$ and $\operatorname{Coker} d$ are in ${}^\perp T$. Since T' dualizes T by Corollary 2.2, the claim follows easily from this.

An immediate consequence of this is the following result.

Corollary 2.4 (a) *For all X in $\mathcal{X}_T$ and all A in* $\operatorname{mod}\Gamma$

$$\operatorname{Hom}_\Gamma(\ , T') \colon \operatorname{Hom}_\Gamma(A, X) \to \operatorname{Hom}_\Lambda((X, T'), (A, T'))$$

is an isomorphism functorial in both variables.

(b) *The map* $\Gamma \to \operatorname{End}_\Lambda({}_\Lambda T')$ *given by $\gamma \mapsto f_\gamma$ where $f_\gamma(t) = \gamma \cdot t$ for all t in T' is an isomorphism.*

The isomorphism of the homomorphism groups in Corollary 2.4 extends to an isomorphism of the Ext-groups.

Proposition 2.5 *For all X and Y in $\mathcal{X}_T$*

$$\operatorname{Hom}_\Gamma(\ ,T')\colon \operatorname{Ext}^i_\Gamma(X,Y) \to \operatorname{Ext}^i_F((Y,T'),(X,T'))$$

is an isomorphism functorial in both variables for all $i > 0$.

Proof: Let X and Y be in $\mathcal{X}_T$ and let $0 \to X \to T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} \cdots$ be a coresolution of X in $\operatorname{add} T$ with $\operatorname{Im} d_i$ in ${}^\perp T$. Then it is easy to see that the exact sequence

$$\cdots \to (T_1,T') \to (T_0,T') \to (X,T') \to 0$$

is an F-exact projective resolution of (X,T'). When applying $\operatorname{Hom}_\Lambda(\ ,(Y,T'))$ to this sequence we obtain the following commutative diagram by Corollary 2.4

$$\begin{array}{ccccccccc} 0 & \to & ((X,T'),(Y,T')) & \to & ((T_0,T'),(Y,T')) & \to & ((T_1,T'),(Y,T')) & \to \cdots \\ & & \uparrow\wr & & \uparrow\wr & & \uparrow\wr & \\ 0 & \to & (Y,X) & \to & (Y,T_0) & \to & (Y,T_1) & \to \cdots. \end{array}$$

The cohomology of the upper sequence is $\operatorname{Ext}^i_F((X,T'),(Y,T'))$ and the cohomology of the lower sequence is $\operatorname{Ext}^i_\Gamma(Y,X)$, hence our desired result. The isomorphism between the Ext-groups is induced by the functorial isomorphism of the homomorphism groups, therefore it follows that the isomorphism between the Ext-groups is functorial in both variables.

Since Γ is in $\mathcal{X}_T = {}^\perp T$, the following corollary follows immediately from Proposition 2.5.

Corollary 2.6 (a) $\operatorname{Ext}^i_F((\mathcal{X}_T,T'),T') = 0$ *for all $i > 0$.*
(b) $\operatorname{Ext}^i_F(T',T') = 0$ *for all $i > 0$.*

With these preliminary results we can show that the Λ-module T' is an F-cotilting module.

Proposition 2.7 *With the notation as above we have the following.*
(a) *The subcategory $(\mathcal{X}_T,T')$ is F-resolving contravariantly finite in $\operatorname{mod}\Lambda$ with $(\mathcal{X}_T,T')$-$\operatorname{resdim}_F(\operatorname{mod}\Lambda) \leq \max\{\operatorname{id}_\Gamma T, 2\}$ and $(\mathcal{X}_T,T')^\perp = \widehat{\operatorname{add} T'}$.*
(b) *The subcategory $\widehat{\operatorname{add} T'}$ is F-coresolving covariantly finite in $\operatorname{mod}\Lambda$ with $\operatorname{id}_F(\widehat{\operatorname{add} T'}) \leq \max\{\operatorname{id}_\Gamma T, 2\}$.*
(c) *The module T' is an F-cotilting module in $\operatorname{mod}\Lambda$ with $\operatorname{id}_F T' \leq \max\{r, 2\}$.*

Proof: (a) First we want to show that $(\mathcal{X}_T,T')$ is an F-resolving subcategory of $\operatorname{mod}\Lambda$. Since $\operatorname{Ext}^i_\Gamma(X,Y) \simeq \operatorname{Ext}^i_F((Y,T'),(X,T'))$ for all X and Y in $\mathcal{X}_T$ and all $i > 0$ by Proposition 2.5, the subcategory $(\mathcal{X}_T,T')$ is closed under F-extensions and $\mathcal{P}(F)$ is contained in $(\mathcal{X}_T,T')$. So, it remains to prove that $(\mathcal{X}_T,T')$ is closed under kernels of epimorphisms in F-exact sequences.

Let $\eta\colon 0 \to A \to (X_2,T') \to (X_3,T') \to 0$ be F-exact with X_2 and X_3 in $\mathcal{X}_T$. This sequence is induced from an exact sequence $X_3 \xrightarrow{f} X_2 \xrightarrow{g} X_1 \to 0$ where $A \simeq (X_1,T')$. Since X_3 is in $\mathcal{X}_T$, the following diagram is exact and commutative

$$\begin{array}{ccccccccc} 0 & \to & ((X_3,T'),T') & \to & ((X_2,T'),T') & \to & ((X_1,T'),T') & \to & 0 \\ & & \uparrow\wr & & \uparrow\wr & & \uparrow\alpha & & \\ & & X_3 & \xrightarrow{f} & X_2 & \xrightarrow{g} & X_1 & \to & 0. \end{array}$$

Hence, f is a monomorphism and α is an isomorphism. Then it follows that $\operatorname{Ext}^i_\Gamma(X_1,T) = 0$ for $i > 1$, since X_2 and X_3 is in $\mathcal{X}_T$. Since η is F-exact, the following diagram is exact and commutative

$$\begin{array}{ccccc} ((T,T'),(X_2,T')) & \to & ((T,T'),(X_1,T')) & \to & 0 \\ \uparrow\wr & & \uparrow\wr & & \\ (X_2,T) & \to & (X_3,T) & \to & \operatorname{Ext}^1_\Gamma(X_1,T) \to 0. \end{array}$$

Therefore, $\operatorname{Ext}^1_\Gamma(X_1, T) = 0$ and we have shown that X_1 is in $\mathcal{X}_T$. Since $A \simeq (X_1, T')$, it follows that $(\mathcal{X}_T, T')$ is closed under kernels of epimorphisms in F-exact sequences. Thus we have shown that $(\mathcal{X}_T, T')$ is F-resolving.

Next we want to show that $(\mathcal{X}_T, T')$-$\operatorname{resdim}_F(\operatorname{mod}\Lambda)$ is finite. Since $\mathcal{P}(F)$ is contained in $(\mathcal{X}_T, T')$, it is enough to show that the i-th syzygy in a relative projective resolution of any module C in $\operatorname{mod}\Lambda$ is in $(\mathcal{X}_T, T')$ for large enough i. Let $(T_1, T') \to (T_0, T') \to C \to 0$ be the start of a relative projective resolution of C in $\operatorname{mod}\Lambda$. This sequence is induced from an exact sequence $T_0 \to T_1 \to B \to 0$ in $\operatorname{mod}\Gamma$, so that $\Omega^2_F(C) \simeq (B, T')$. A relative projective resolution of (B, T') corresponds to a succession of minimal left $\operatorname{add} T$-approximations, $B \to T_2 \to B_1 \to 0$, $B_1 \to T_3 \to B_2 \to 0$ and so on. By [3, Lemma 3.12] the module B is in $\mathcal{X}_T$ if $\operatorname{id}_\Gamma T \leq 2$ and $B_{\operatorname{id}_\Gamma T - 2}$ is in $\mathcal{X}_T$ if $\operatorname{id}_\Gamma T > 2$. Therefore we have that $(\mathcal{X}_T, T')$-$\operatorname{resdim}_F(\operatorname{mod}\Lambda) \leq \max\{\operatorname{id}_\Gamma T, 2\}$.

Since $(\mathcal{X}_T, T')$ is an F-resolving subcategory of $\operatorname{mod}\Lambda$ with $(\mathcal{X}_T, T')$-$\operatorname{resdim}_F(\operatorname{mod}\Lambda)$ finite, it suffices to find an Ext_F-injective cogenerator for $(\mathcal{X}_T, T')$ in order to show that $(\mathcal{X}_T, T')$ is contravariantly finite in $\operatorname{mod}\Lambda$ by Theorem 1.1. The subcategory $\operatorname{add} T'$ is contained in $(\mathcal{X}_T, T') \cap (\mathcal{X}_T, T')^\perp$ by Corollary 2.6 (a). Let X be in $\mathcal{X}_T$ and let $0 \to X' \to P \to X \to 0$ be the projective cover of X. Then the sequence $0 \to (X, T') \to (P, T') \to (X', T') \to 0$ is F-exact by Proposition 2.5 with (P, T') in $\operatorname{add} T'$ and (X', T') in $(\mathcal{X}_T, T')$. Hence $\operatorname{add} T'$ is an Ext_F-injective cogenerator for $(\mathcal{X}_T, T')$. Then by Theorem 1.1 the subcategory $(\mathcal{X}_T, T')$ is contravariantly finite in $\operatorname{mod}\Lambda$ and $(\mathcal{X}_T, T')^\perp = \widehat{\operatorname{add} T'}$.

(b) By (a) $\widehat{\operatorname{add} T'} = (\mathcal{X}_T, T')^\perp$, so by [3, Proposition 2.2 (a)] the subcategory $\widehat{\operatorname{add} T'}$ is F-coresolving covariantly finite in $\operatorname{mod}\Lambda$. Since $(\mathcal{X}_T, T')$-$\operatorname{resdim}_F(\operatorname{mod}\Lambda) \leq n$ if and only if $\operatorname{id}_F (\mathcal{X}_T, T')^\perp \leq n$ by [3, Theorem 2.5], it follows from (a) that $\operatorname{id}_F \widehat{\operatorname{add} T'} \leq \max\{\operatorname{id}_\Gamma T, 2\}$.

(c) Since (X_T, T') is F-resolving contravariantly finite with (X_T, T')-$\operatorname{resdim}_F(\operatorname{mod}\Lambda)$ finite, the subcategory $(\mathcal{X}_T, T') \cap (\mathcal{X}_T, T')^\perp$ is $\operatorname{add} T''$ for some F-cotilting module T'' over Λ by [3, Proposition 3.22]. It is not hard to see that $(\mathcal{X}_T, T') \cap (\mathcal{X}_T, T')^\perp = \operatorname{add} T'$. Therefore T' is an F-cotilting module with $\operatorname{id}_F T' \leq \max\{\operatorname{id}_\Gamma T, 2\}$, since $\operatorname{id}_F \widehat{\operatorname{add} T'} \leq \max\{\operatorname{id}_\Gamma T, 2\}$.

Then the following characterization of dualizing summands of cotilting modules is an immediate consequence from the above results and the remark in the introduction to this section.

Theorem 2.8 *Let $T = T_1 \oplus T_2$ be a cotilting module in* $\operatorname{mod}\Gamma$ *and let $\Lambda = \operatorname{End}_\Gamma(T_1)$. Define F to be the subfunctor of* $\operatorname{Ext}^1_\Lambda(\ ,\ )$ *given by $F_{\mathcal{P}(F)}$, where $\mathcal{P}(F) = (\operatorname{add} T, T_1)$. Then T_1 is a dualizing summand of the Γ-module T if and only if T_1 is an F-cotilting module in* $\operatorname{mod}\Lambda$.

One natural question is whether or not the dualizing summand T_a of T determines the cotilting module T uniquely. We observed in [3] that a module T' could be a relative cotilting with respect to two different relative theories. The endomorphism ring of T' is independent of the relative theories. It is easy to see that this implies that T' as a module over its endomorphism ring is a dualizing summand of two different cotilting modules.

3 Wedderburn correspondence.

Let $T = T_a \oplus T_b$ be a cotilting module over an artin algebra Γ, where T_a is a dualizing summand of T. Denote by $\Sigma = \operatorname{End}_\Gamma(T)$. Then $\operatorname{Hom}_\Gamma(T_a, T)$ is a projective Σ-module, moreover we show that it is a Wedderburn projective. The existence of a relative cotilting module for $\operatorname{mod}\Lambda$ implies that $\mathcal{P}(F) = \operatorname{add} G$ for some generator G of $\operatorname{mod}\Lambda$ by [3, Corollary 3.14]. This section is devoted to investigating the connections between all these concepts, cotilting modules with dualizing summands, relative cotilting modules, generators and Wedderburn projectives.

First we recall the notion of a Wedderburn projective and Wedderburn the correspondence from [1].

Let Γ be an artin algebra and P a projective module in $\operatorname{mod}\Gamma$. Let $\Lambda_P = \operatorname{End}_\Gamma(P)^{\mathrm{op}}$ and $G_P = \operatorname{Hom}_\Gamma(P, \Gamma)$. Then we have the following functors

$$\operatorname{Hom}_\Gamma(P,\ )\colon \operatorname{mod}\Gamma \to \operatorname{mod}\Lambda_P$$

and

$$\mathrm{Hom}_{\Lambda_P}(G_P,\ )\colon \mathrm{mod}\,\Lambda_P \to \mathrm{mod}\,\Gamma.$$

The Λ_P-module G_P is a generator for $\mathrm{mod}\,\Lambda_P$. The module P is called a *Wedderburn projective* Γ-module if the canonical homomorphism $\Gamma \to \mathrm{Hom}_{\Lambda_P}(G_P, G_P)$ given by $\gamma \mapsto f_\gamma$ where $f_\gamma(g) = g\cdot\gamma$ for g in G_P, is an isomorphism. Moreover, if P is a Wedderburn projective, then $\mathrm{Hom}_{\Lambda_P}(G_P, \Lambda_P)$ is isomorphic to P as Λ-modules by [1, Proposition 8.2].

Let Λ be an artin algebra and G a generator for $\mathrm{mod}\,\Lambda$. Let $\Gamma_G = \mathrm{End}_\Lambda(G)^{\mathrm{op}}$ and $P_G = \mathrm{Hom}_\Lambda(G, \Lambda)$. Then we have the following functors

$$\mathrm{Hom}_\Lambda(G,\ )\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma_G$$

and

$$\mathrm{Hom}_{\Gamma_G}(P_G,\ )\colon \mathrm{mod}\,\Gamma_G \to \mathrm{mod}\,\Lambda.$$

With the notation above we recall the following result from [1].

Proposition 3.1 *Let Λ be an artin algebra and G a generator for $\mathrm{mod}\,\Lambda$. Then we have the following.*

(a) *The functor $\mathrm{Hom}_\Lambda(G,\ )\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma_G$ is a fully faithful functor which is a right adjoint of $\mathrm{Hom}_{\Gamma_G}(P_G,\ )\colon \mathrm{mod}\,\Gamma_G \to \mathrm{mod}\,\Lambda$.*

(b) *The composition $\mathrm{Hom}_{\Gamma_G}(P_G,\ )\circ \mathrm{Hom}_\Lambda(G,\ )$ is functorial isomorphic to the identity functor on $\mathrm{mod}\,\Lambda$.*

(c) *The functor $\mathrm{Hom}_\Lambda(G,\ )|_{\mathrm{add}\,G}\colon \mathrm{add}\,G \to \mathcal{P}(\Gamma_G)$ is an equivalence of categories.*

(d) *The homomorphism $\mathrm{Hom}_\Lambda(G,\ )\colon \mathrm{Hom}_\Lambda(\Lambda,\Lambda) \to \mathrm{Hom}_{\Gamma_G}(P_G, P_G)$ is an isomorphism inducing an isomorphism $\Lambda \to \mathrm{End}_{\Gamma_G}(P_G)^{\mathrm{op}}$ of rings.*

It follows from Proposition 3.1 that the Γ_G-module P_G is a Wedderburn projective. A module M in $\mathrm{mod}\,\Lambda$ is said to be a *Wedderburn module* if it is either a generator for $\mathrm{mod}\,\Lambda$ or it is a Wedderburn projective Λ-module. A pair (Λ, M) is called a *Wedderburn pair* if M is a Wedderburn module. The operation End on (Rings, Modules) given by $\mathrm{End}(\Lambda, M) = (\mathrm{End}_\Lambda(M)^{\mathrm{op}}, \mathrm{Hom}_\Lambda(M, \Lambda))$ is an involution on the collection of Wedderburn pairs and it is called the *Wedderburn correspondence.*

Let P be a Wedderburn projective in $\mathrm{mod}\,\Lambda$. Denote by $\Delta = \mathrm{End}_\Lambda(P)^{\mathrm{op}}$ and $G = \mathrm{Hom}_\Lambda(P, \Lambda)$. Then by the above results the canonical homomorphisms $\Lambda \to \mathrm{Hom}_\Delta(G, G)^{\mathrm{op}}$ and $P \to \mathrm{Hom}_\Delta(G, \Delta)$ are isomorphisms as algebras and as Λ-modules respectively. Let $F = F_{\mathrm{add}\,G}$, then $\mathcal{P}(F) = \mathrm{add}\,G$ and $\mathcal{I}(F) = \mathcal{I}(\Delta) \cup \mathrm{add}\,D\,\mathrm{Tr}\,G$. Assume that T is a Δ-module such that $\mathrm{add}\,T = \mathcal{I}(F)$. Then T is an F-cotilting module in $\mathrm{mod}\,\Delta$ with $\mathrm{id}_F\,T = 0$. Denote by $\Gamma = \mathrm{End}_\Delta(T)$. By Theorem 1.2 (a) the module $T_0 = \mathrm{Hom}_\Delta(G, T)$ is a cotilting module over Γ with $\mathrm{id}_\Gamma\,T_0 \leq 2$ and $\mathrm{id}_\Gamma\,T = 0$. Moreover, $\mathrm{id}_\Gamma\,T_0 = 2$ by [3, Proposition 3.26]. If $\Lambda' = \mathrm{End}_\Gamma(T_0)$, then ${}_{\Lambda'}T_0$ is a cotilting module over Λ'. Since T is an F-cotilting module in $\mathrm{mod}\,\Delta$, we have that

$$\mathrm{End}_\Gamma(T_0) = \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Delta(G,T), \mathrm{Hom}_\Delta(G,T)) \simeq \mathrm{Hom}_\Delta(G,G)^{\mathrm{op}}.$$

Because P is a Wedderburn projective in $\mathrm{mod}\,\Lambda$, it follows that $\mathrm{End}_\Gamma(T_0) \simeq \Lambda$. This shows that to every Wedderburn projective P there is a cotilting module naturally associated with it. Moreover, we have the following result.

Proposition 3.2 *Let P be a Wedderburn projective in $\mathrm{mod}\,\Lambda$. Then there exists a cotilting module T_0 in $\mathrm{mod}\,\Lambda$, such that if $\Gamma = \mathrm{End}_\Lambda(T_0)$, the module ${}_\Gamma T_0$ has the properties* (i) *${}_\Gamma T_0$ is a cotilting module in $\mathrm{mod}\,\Gamma$ with $\mathrm{id}_\Gamma\,T_0 = 2$,* (ii) *if T_a the maximal injective summand of T_0, then it is a dualizing summand of T_0 and* (iii) *$P \simeq \mathrm{Hom}_\Gamma(T', T_0)$ for some module T' with $\mathrm{add}\,T' = \mathrm{add}\,T_a$.*

Proof: By the above considerations we have shown that there exists a cotilting module T_0 over Λ such that $\mathrm{id}_\Gamma\,T_0 = 2$, since $\Gamma \simeq \mathrm{End}_\Lambda(T_0)$. This proves (i).

(ii) Let $G = Q \oplus M$, where Q is projective and M has no nonzero projective summands. This gives the decomposition $T_0 = \mathrm{Hom}_\Delta(Q, T) \oplus \mathrm{Hom}_\Delta(M, T)$ of T_0. Denote by $T_a = \mathrm{Hom}_\Delta(Q, T)$

and $T_b = \mathrm{Hom}_\Delta(M,T)$. Since $\mathrm{add}\, T_a = \mathrm{add}\, {}_\Gamma T$ and ${}_\Gamma T$ dualizes T_0, the module T_a is dualizing summand of T_0 and $\mathrm{id}_\Gamma\, T_a = 0$. It remains to show that T_a is a maximal injective summand of T_0. Assume that I is an injective summand of $T_b = \mathrm{Hom}_\Delta(M,T)$. The functor $\mathrm{Hom}_\Delta(\ ,T)\colon \mathrm{mod}\,\Delta \to \mathrm{mod}\,\Gamma$ induces a duality between $\mathcal{X}_T = \mathrm{mod}\,\Lambda$ and ${}^\perp T_0$ by [3, Corollary 3.6 (a)]. Then $I \simeq \mathrm{Hom}_\Delta(X,T)$ for some summand X of M. Using the duality again it follows that X must be projective. By assumption M has no nonzero projective summands, so $X = 0$. Therefore T_b does not have any nonzero injective summands and T_a is the maximal injective summand of T_0.

(iii) We have that $\mathrm{add}\, {}_\Gamma T = \mathrm{add}\, T_a$. We want to prove that $P \simeq \mathrm{Hom}_\Gamma(T,T_0)$. Since T is an F-cotilting module, we have that

$$\mathrm{Hom}_\Gamma(T,T_0) = \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Delta(\Delta,T),\mathrm{Hom}_\Delta(G,T)) \simeq \mathrm{Hom}_\Delta(G,\Delta).$$

Because P is a Wedderburn projective in $\mathrm{mod}\,\Lambda$, we have that $\mathrm{Hom}_\Delta(G,\Delta) \simeq P$. Hence $\mathrm{Hom}_\Gamma(T,T_0) \simeq P$.

Let $T = T_a \oplus T_b$ be a cotilting module over an artin algebra Γ with T_a a dualizing summand of T. Denote by $\Lambda = \mathrm{End}_\Gamma(T)$. Then we observed above that at least in one special case $\mathrm{Hom}_\Gamma(T',T)$ is a Wedderburn projective projective in $\mathrm{mod}\,\Lambda$ for all modules such that $\mathrm{add}\, T' = \mathrm{add}\, T_a$. Next we prove that this is true in general.

Proposition 3.3 *Let $T = T_a \oplus T_b$ be a cotilting module over an artin algebra Γ with T_a a dualizing summand of T. Denote by $\Lambda = \mathrm{End}_\Gamma(T)$. Then the module $\mathrm{Hom}_\Gamma(T',T)$ is a Wedderburn projective in $\mathrm{mod}\,\Lambda$ for all modules T' such that $\mathrm{add}\, T' = \mathrm{add}\, T_a$.*

Proof: Let T' be a module such that $\mathrm{add}\, T' = \mathrm{add}\, T_a$. Denote by $\Sigma = \mathrm{End}_\Gamma(T')$, the subfunctor $F = F_{\mathrm{Hom}_\Gamma(\mathrm{add}\, T,T')}$ and $G = \mathrm{Hom}_\Gamma(T,T')$. By Proposition 2.7 the module ${}_\Sigma T'$ is an F-cotilting module. The module G is a generator for $\mathrm{mod}\,\Sigma$. Therefore, if $\Delta = \mathrm{End}_\Sigma(G)^{\mathrm{op}}$, the module $P = \mathrm{Hom}_\Sigma(G,\Sigma)$ is a Wedderburn projective over Δ. Since T' is an F-cotilting module, we have the following isomorphisms

$$\Delta = \mathrm{Hom}_\Sigma(G,G)^{\mathrm{op}} = \mathrm{Hom}_\Sigma(\mathrm{Hom}_\Gamma(T,T'),\mathrm{Hom}_\Gamma(T,T'))^{\mathrm{op}} \simeq \mathrm{Hom}_\Gamma(T,T) = \Lambda$$

and

$$P = \mathrm{Hom}_\Sigma(G,\Sigma) = \mathrm{Hom}_\Sigma(\mathrm{Hom}_\Gamma(T,T'),\mathrm{Hom}_\Gamma(T',T')) \simeq \mathrm{Hom}_\Gamma(T',T).$$

Hence $\mathrm{Hom}_\Gamma(T',T)$ is a Wedderburn projective in $\mathrm{mod}\,\Lambda$.

This result also shows that the converse of Proposition 3.2 is true. Hence we have proved that the statements (a) and (b) in the following proposition are equivalent.

Proposition 3.4 *Let P be a module in $\mathcal{P}(\Lambda)$. Then the following are equivalent.*

(a) *The module P is a Wedderburn projective in* $\mathrm{mod}\,\Lambda$.

(b) *There exists a cotilting module T in* $\mathrm{mod}\,\Lambda$*, such that if $\Gamma = \mathrm{End}_\Lambda(T)$, the module ${}_\Gamma T$ has the properties* (i) *${}_\Gamma T$ is a cotilting module in* $\mathrm{mod}\,\Gamma$ *with* $\mathrm{id}_\Gamma\, T = 2$, (ii) *if T_a the maximal injective summand of T, then it is a dualizing summand of T and* (iii) *$P \simeq \mathrm{Hom}_\Gamma(T',T)$ for some module T' with* $\mathrm{add}\, T' = \mathrm{add}\, T_a$.

(c) *There exists an exact sequence*

$$0 \to \Lambda^{\mathrm{op}} \xrightarrow{f} \mathrm{Hom}_\Lambda(P,\Lambda)^n \to \mathrm{Hom}_\Lambda(P,\Lambda)^m,$$

where $f\colon \Lambda^{\mathrm{op}} \to \mathrm{Hom}_\Lambda(P,\Lambda)^n$ is a left $\mathrm{Hom}_\Lambda(P,\Lambda)$-approximation.

Proof: (b) *implies* (c). Let $T = T_a \oplus T_b$ be a cotilting module over an artin algebra Γ with T_a a dualizing summand of T. Denote by $\Lambda = \mathrm{End}_\Gamma(T)$. Let T' be a module such that $\mathrm{add}\, T' = \mathrm{add}\, T_a$. Then T' dualizes T and therefore there exists an exact sequence

$$0 \to T \xrightarrow{f} (T')^n \to (T')^m,$$

where $f\colon T \to (T')^n$ is a left $\operatorname{add} T'$-approximation by Proposition 2.1. Applying the functor $\operatorname{Hom}_\Gamma(T, \ )$ to this sequence gives rise to the exact sequence

$$0 \to \operatorname{Hom}_\Gamma(T,T) \overset{\operatorname{Hom}_\Gamma(T,f)}{\to} \operatorname{Hom}_\Gamma(T,T')^n \to \operatorname{Hom}_\Gamma(T,T')^m,$$

where $\operatorname{Hom}_\Gamma(T,f)\colon \operatorname{Hom}_\Gamma(T,T) \to \operatorname{Hom}_\Gamma(T,T')^n$ is a left $\operatorname{add}\operatorname{Hom}_\Gamma(T,T')$-approximation. Let $P = \operatorname{Hom}_\Gamma(T',T)$. Since ${}_\Gamma T$ is a cotilting module, we have the following isomorphism

$$\operatorname{Hom}_\Lambda(P,\Lambda) = \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Gamma(T',T), \operatorname{Hom}_\Gamma(T,T)) \simeq \operatorname{Hom}_\Gamma(T,T').$$

Hence, the above exact sequence has the following form

$$0 \to \Lambda^{\mathrm{op}} \xrightarrow{f'} \operatorname{Hom}_\Lambda(P,\Lambda)^n \to \operatorname{Hom}_\Lambda(P,\Lambda)^m,$$

where $f'\colon \Lambda^{\mathrm{op}} \to \operatorname{Hom}_\Lambda(P,\Lambda)^n$ is a left $\operatorname{add}\operatorname{Hom}_\Lambda(P,\Lambda)$-approximation. This completes the proof of (b) *implies* (c).

(c) *implies* (a). Assume that there exists an exact sequence

$$0 \to \Lambda^{\mathrm{op}} \xrightarrow{f} \operatorname{Hom}_\Lambda(P,\Lambda)^n \to \operatorname{Hom}_\Lambda(P,\Lambda)^m,$$

where $f'\colon \Lambda^{\mathrm{op}} \to \operatorname{Hom}_\Lambda(P,\Lambda)^n$ is a left $\operatorname{add}\operatorname{Hom}_\Lambda(P,\Lambda)$-approximation. By Proposition 2.1 this is equivalent to the natural homomorphism

$$\Lambda^{\mathrm{op}} \to \operatorname{Hom}_\Delta(\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda^{\mathrm{op}}, \operatorname{Hom}_\Lambda(P,\Lambda)), \operatorname{Hom}_\Lambda(P,\Lambda))$$

being an isomorphism, where $\Delta = \operatorname{End}_{\Lambda^{\mathrm{op}}}(\operatorname{Hom}_\Lambda(P,\Lambda))$. Since $\operatorname{Hom}_\Lambda(\ ,\Lambda)\colon \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda^{\mathrm{op}})$ is a duality, we have that

$$\Delta = \operatorname{End}_{\Lambda^{\mathrm{op}}}(\operatorname{Hom}_\Lambda(P,\Lambda)) \simeq \operatorname{End}_\Lambda(P)^{\mathrm{op}} = \Lambda_P.$$

Therefore the above natural homomorphism

$$\Lambda^{\mathrm{op}} \to \operatorname{Hom}_{\Lambda_P}(\operatorname{Hom}_\Lambda(P,\Lambda), \operatorname{Hom}_\Lambda(P,\Lambda))$$

is given by $\lambda \mapsto f_\lambda$, where $f_\lambda(g) = g \cdot \lambda$ for g in $G = \operatorname{Hom}_\Lambda(P,\Lambda)$. This implies that the canonical homomorphism $\Lambda \to \operatorname{Hom}_{\Lambda_P}(G,G)$ is an isomorphism, so that P is a Wedderburn projective in $\operatorname{mod}\Lambda$.

Remark: (1) If P is a basic module, it is easy to see that T' in the above proposition is T_a.

(2) It follows directly from the above proposition that if P is a Wedderburn projective in $\operatorname{mod}\Lambda$, then all modules Q such that $\operatorname{add} Q = \operatorname{add} P$ are also Wedderburn projective. In particular, a module P is a Wedderburn projective if and only if the corresponding basic module is a Wedderburn projective.

Let F be an additive subfunctor of $\operatorname{Ext}^1_\Lambda(\ , \ )$ with enough projectives and injectives. Let T be an F-cotilting module and $\Gamma = \operatorname{End}_\Lambda(T)$. Then $\mathcal{P}(F) = \operatorname{add} G$ for some generator G in $\operatorname{mod}\Lambda$ by [3, Corollary 3.14]. The module $T_0 = (G,T)$ is a cotilting module in $\operatorname{mod}\Gamma$ with ${}^\perp T_0 = (X_T, T)$ by Theorem 1.2 (a). Let $\Sigma = \operatorname{End}_\Gamma(T_0)$ and $\Gamma_G = \operatorname{End}_\Lambda(G)^{\mathrm{op}}$. Similarly as in Proposition 2.1 and Corollary 2.4 one can show that $\operatorname{Hom}_\Lambda(\ ,T)\colon \operatorname{Hom}_\Lambda(A,C) \to \operatorname{Hom}_\Gamma((C,T),(A,T))$ is an isomorphism for all modules A in $\operatorname{mod}\Lambda$ and all modules C in $\mathcal{X}_T$. Then it follows that $\phi = \operatorname{Hom}_\Lambda(\ ,T)\colon \Gamma_G \to \Sigma$ is an isomorphism. Since G is a generator for $\operatorname{mod}\Lambda$, the pair $(\Gamma_G, P_G = \operatorname{Hom}_\Lambda(G,\Lambda))$ is the Wedderburn pair corresponding to (Λ, G) by the Wedderburn correspondence. Then we have the following diagram of functors

$$\begin{array}{ccc} \operatorname{mod}\Lambda & \overset{\operatorname{Hom}_\Lambda(\ ,T)}{\longrightarrow} & \operatorname{mod}\Gamma \\ \Big\downarrow {\scriptstyle \operatorname{Hom}_\Lambda(G, \ )} & & \Big\downarrow {\scriptstyle \operatorname{Hom}_\Gamma(\ ,(G,T))} \\ \operatorname{mod}\Gamma_G & \overset{\phi^{-1}}{\longrightarrow} & \operatorname{mod}\Sigma, \end{array}$$

where we also have a diagonal functor $\mathrm{Hom}_{\Gamma_G}(\ ,(G,T))\colon \mathrm{mod}\,\Gamma_G \to \mathrm{mod}\,\Gamma$. By the above remark the composition $\mathrm{Hom}_{\Gamma}(\ ,(G,T))\circ \mathrm{Hom}_{\Lambda}(\ ,T)$ is functorial isomorphic to $\phi^{-1}\circ \mathrm{Hom}_{\Lambda}(G,\ )$ when restricted to $\mathcal{X}_T$. It follows by applying Theorem 1.2 (a) twice that the Γ_G-module (G,T) is a cotilting module with $\mathrm{id}_{\Gamma_G}(G,T) \leq \max\{\mathrm{id}_F T, 2\}$ and where $\Gamma \simeq \mathrm{End}_{\Gamma_G}((G,T))$. Since $\mathrm{Hom}_{\Lambda}(G,\ )\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma_G$ is a fully faithful functor by Proposition 3.1 (a), we have that $\mathrm{Hom}_{\Gamma_G}(\ ,(G,T))\circ \mathrm{Hom}_{\Lambda}(G,\ ) \simeq \mathrm{Hom}_{\Lambda}(\ ,T)\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$. Hence we have shown the following result.

Proposition 3.5 *Every relative cotilting functor* $\mathrm{Hom}_{\Lambda}(\ ,T)\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ *is isomorphic to the composition of functors* $\mathrm{Hom}_{\Gamma_G}(\ ,(G,T))\circ \mathrm{Hom}_{\Lambda}(G,\ )$*, where* $\mathrm{Hom}_{\Lambda}(G,\ )\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma_G$ *is the usual functor and* $\mathrm{Hom}_{\Gamma_G}(\ ,(G,T))\colon \mathrm{mod}\,\Gamma_G \to \mathrm{mod}\,\Gamma$ *is a standard cotilting functor.*

Let G be a generator for $\mathrm{mod}\,\Lambda$. Denote $F_{\mathrm{add}\,G}$ by F. Then F has enough projectives and injectives. The projectives of the subfunctor are $\mathcal{P}(F) = \mathrm{add}\,G$ by [2, Proposition 1.10]. Let $\Gamma_G = \mathrm{End}_{\Lambda}(G)^{\mathrm{op}}$. Then G is an F-tilting module, so that the functor $\mathrm{Hom}_{\Lambda}(G,\ )\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma_G$ is a special case of a relative tilting functor.

Let T be a module in $\mathrm{mod}\,\Lambda$ such that $\mathrm{add}\,T = \mathcal{I}(F)$, then T is a F-cotilting module with $\mathcal{X}_T = \mathrm{mod}\,\Lambda$. Denote by $\Gamma = \mathrm{End}_{\Lambda}(T)$ and let G be a Λ-module such that $\mathrm{add}\,G = \mathcal{P}(F)$. From the above discussion it follows that the functor $\mathrm{Hom}_{\Lambda}(G,\ )\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma_G$ is isomorphic to $\mathrm{Hom}_{\Gamma}(\ ,(G,T))\circ \mathrm{Hom}_{\Lambda}(\ ,T)$, where $\mathrm{Hom}_{\Lambda}(\ ,T)\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ is the relative cotilting functor and $\mathrm{Hom}_{\Gamma}(\ ,(G,T))\colon \mathrm{mod}\,\Gamma \to \mathrm{mod}\,\Gamma_G$ is a standard cotilting functor. By [3, Proposition 3.18] the abelian groups $F - K_0(\mathrm{mod}\,\Lambda)$, $K_0(\mathrm{mod}\,\Gamma)$ and $K_0(\mathrm{mod}\,\Gamma_G)$ are all isomorphic. Using these remarks and Theorem 1.2, it is easy to see that we have the following result.

Proposition 3.6 *Let* Λ*,* G*,* T *and* F *be as above. Then the following is true.*

(a) *The functor* $\mathrm{Hom}_{\Lambda}(G,\ )\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma_G$ *is the composition of the relative cotilting functor* $\mathrm{Hom}_{\Lambda}(\ ,T)\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ *and the standard cotilting functor* $\mathrm{Hom}_{\Gamma}(\ ,(G,T))\colon \mathrm{mod}\,\Gamma \to \mathrm{mod}\,\Gamma_G$.

(b) *The Grothendieck groups* F-$K_0(\mathrm{mod}\,\Lambda)$ *and* $K_0(\mathrm{mod}\,\Gamma_G)$ *are isomorphic.*

(c) *The* Γ_G*-module* (G,T) *is a cotilting module with* $\mathrm{id}_{\Gamma_G}(G,T) = 2$.

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