Computational aspects of projective resolutions

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The main focus behind this talk was to understand the 12-dimensional Nichols algebra

\[ R = k\langle a, b, c \rangle / (a^2, b^2, c^2, ab + bc + ca, ac + ba + cb) \]

with respect to the cohomology ring \( H^*(R) = \bigoplus_{i \geq 0} \text{Ext}^n_R(k, k) \), where \( k \) is a field and \( k \) also denotes the trivial \( R \)-module. The cohomology ring is important, among other things, since it gives rise to support varieties of modules given certain finite generation conditions. See the talk of Sarah Witherspoon for further details and background.

One way of trying to understand the cohomology ring \( H^*(R) \) is to compute a projective resolution of the trivial module \( k \) and then compute the Yoneda algebra \( H^*(R) \). One projective resolution of an \( R \)-module is the so-called Bongartz-Butler-Gruenberg resolution: Let \( M \) be a finitely generated \( R \)-module, where \( R = kQ/I \) for a field \( k \), a quiver \( Q \) and an admissible ideal \( I \) in the path algebra \( kQ \). Given a projective presentation \( \eta: 0 \to P_1 \to P_0 \to M \to 0 \) of \( M \) as a \( kQ \)-module, it is known that the sequence of submodules of \( P_0 \),

\[ \cdots \subseteq P_1I^{n} \subseteq P_0I^{n} \subseteq P_1I^{n-1} \subseteq P_0I^{n-1} \subseteq P_1I^{n-2} \subseteq P_0I^{n-2} \subseteq P_1I \subseteq P_0I \subseteq P_1 \subseteq P_0, \]

modulo the ideal \( I \) induces a projective resolution of \( M \) as a \( R \)-module. This resolution can be analyzed using right Gröbner basis theory, and in [3] it is shown that knowing a finite set of equations this projective resolution and the Yoneda algebra \( \text{Ext}^*(M, M) \) can be computed for any finitely generated \( R \)-module. In general this resolution is far from minimal. For the above example one can show that the number of indecomposable projective summands in \( n \)-th projective in the resolution is \( 25^n \) and \( 3 \cdot 25^{n-1} \), when \( n \) is even and odd, respectively (and given a certain Gröbner basis). While the minimal projective resolution has a linear growth.

A more efficient projective resolution is described in [2, 1]. It takes the Bongartz-Butler-Gruenberg resolution as a starting point, but makes adjustments along the way. For instance, it can be shown that

\[ 0 \to P_0I \hookrightarrow P_1 \to \Omega^1_R(M) \to 0 \]

is a \( kQ \)-projective presentation of a first syzygy of \( M \) over \( R \). The projective module \( P_1 \) can be written as \( \prod_{i=1}^{t} w_i kQ \) for some \( w_i \) in \( P_0 \). If \( w_i \) is in \( P_0I \), then it will be mapped to zero in \( \Omega^1_R(M) \). Hence, consider only those \( w_i \) which is not in \( P_0I \). Denote this set by \( T_1 \subseteq \{ w_1, w_2, \ldots, w_t \} \). Then

\[ 0 \to P_0I \cap \prod_{w \in T_1} w kQ \hookrightarrow \prod_{w \in T_1} w kQ \to \Omega^1_R(R) \to 0 \]

is an exact sequence. We now use this sequence as \( \eta \) was used above. This gives rise to the projective resolutions described in [2, 1], where [1] explains in some detail the algorithm for constructing the resolution.
Using the software package QPA (see http://sourceforge.net/projects/quiverspathalg/), the structure of $H^*(R)$ has been computed up to degree 40. The dimension of $H^n(R)$ for $n = 0, 1, 2, \ldots, 40$ is the following:

$$1, 3, 5, 6, 7, 9, 11, 12, 13, 15, 17, 18, 19, 21, 23, 24, 25, \ldots$$

which is all positive integers congruent to $\{0, 1, 3, 5\}$ according to the On-Line Encyclopedia of Integer Sequences (oeis.org). If the dimension of $H^n(R)$ is given by this sequence, the Hilbert series for $H^*(R)$ is given by $$(1 + t)(1 + t + t^2)(1 - t)(1 - t^4).$$ This indicates that $H^*(R)$ has generators in degrees 1 and 4 as an algebra over $H^0(R)$. Again using the software package QPA, one can calculate that it has three generators in degree 1 and one generator in degree 4, as an algebra over the degree zero part. We hope to unravel the structure of the cohomology ring through an explicit knowledge of a projective resolution of $k$.

References