Relative cotilting theory and almost complete cotilting modules

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1. Introduction

Relative cotilting modules and their connection to ordinary cotilting modules were first studied by Auslander and Solberg in [**6**] and [**7**]. Among other things, they show that relative cotilting modules induce ordinary partial cotilting modules. A special feature of such objects is that they can be relative cotilting modules in several relative theories. This will induce different complements of partial cotilting modules. Complements of partial cotilting and especially complements of almost complete tilting modules have been studied by Coelho, Happel and Unger in [**8**], [**9**] and [**10**]. We use this information to obtain information about how different relative theories for a relative cotilting module are related.

There is also an interesting connection to well known homological conjectures. In [**9**] it is shown that if the Finitistic Dimension Conjecture is true, then any almost complete cotilting module has a finite number of non-isomorphic indecomposable complements. So in fact they formulated the following conjecture: An almost complete cotilting module always has a finite number of non-isomorphic indecomposable complements. We show that given an artin algebra Λ, the Generalized Nakayama Conjecture is satisfied for Λ if and only if the injective almost complete cotilting modules over Λ only have a finite number of complements. This connection have been shown independently by Happel and Unger ([**11**]).

We first introduce the notion of faithful dimension of an arbitrary module M over an artin algebra Λ. Then we recall results on almost complete cotilting modules from [**8**], [**9**] and [**10**], and show that an almost complete cotilting is of faithful dimension n if and only if it has exactly $n+1$ non-isomorphic indecomposable complements. This will also give the connection to the Generalized Nakayama Conjecture. Next we turn to relative cotilting modules, and see how different complements of partial cotilting modules are related to different relative theories for relative cotilting modules. In the last section, we further investigate this connection, by looking at corresponding subcategories.

For an artin algebra Λ we denote by $\operatorname{mod}\Lambda$ the category of finitely generated left Λ-modules. All subcategories of $\operatorname{mod}\Lambda$ will be full additive subcategories. For a Λ-module M, we let $\delta(M)$ denote the number of non-isomorphic indecomposable summands. For a Λ-module M, let ${}^{\perp}M = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, M) =$

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(0) for all $i > 0\}$ and let $M^{\perp} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(M, X) = (0) \text{ for all } i > 0\}$. Now recall the following definitions and results from [**4**]. If $\mathcal{X}$ is a subcategory of $\operatorname{mod}\Lambda$ and C is in $\operatorname{mod}\Lambda$, then a map $X \to C$ is a right $\mathcal{X}$-approximation if X is in $\mathcal{X}$ and the induced map $\operatorname{Hom}_\Lambda(\ ,X)|_{\mathcal{X}} \to \operatorname{Hom}_\Lambda(\ ,C)|_{\mathcal{X}}$ is an epimorphism. The category $\mathcal{X}$ is called contravariantly finite if all C in $\operatorname{mod}\Lambda$ have a right $\mathcal{X}$-approximation. We also have the dual concepts of left $\mathcal{X}$-approximation and covariantly finite subcategories. For a module M in $\operatorname{mod}\Lambda$, the subcategory $\operatorname{add} M$ is both contravariantly finite and covariantly finite. For unexplained terminology we refer to [**3**].

2. The faithful dimension

In this section we introduce the notion of faithful dimension. In the next section we show that an almost complete cotilting module has exactly $n+1$ non-isomorphic indecomposable complements if and only if its faithful dimension is n.

Let M be a Λ-module. There is a complex

$$\eta\colon 0 \to \Lambda \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} M_n \to \cdots,$$

with $K_i = \operatorname{Coker} f_i$ for $i \geq 1$ and $K_0 = \Lambda$, such that each $K_i \to M_{i+1}$ is a minimal left $\operatorname{add} M$-approximation. Let η_n denote the truncated complex ending in M_n obtained from η. Then M is said to have *faithful dimension* n if η_n is exact, but η_{n+1} is not. If η is exact, then M has infinite faithful dimension. We denote this dimension by $\operatorname{fadim} M$. Assume M is a faithful module, that is there is a monomorphism $f\colon \Lambda \to M^r$. Then f factors through the minimal $\operatorname{add} M$-approximation g, but then g is also a monomorphism. Hence, if a module M is faithful, then it has at least faithful dimension 1. It is also easy to see, that a module M is faithful if and only if it is cofaithful (that is; there is an epimorphism $M' \to D\Lambda^{\mathrm{op}}$ with M' in $\operatorname{add} M$). We need the following notion from [**7**]. Let $\Gamma = \operatorname{End}_\Lambda(M)$. If the natural homomorphism $A \to \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(A, M), M)$ is an isomorphism, for a Λ-module A, then M is said to *dualize* A. If $M = M' \oplus M''$ and M' dualizes M, then M' is called a *dualizing summand*. The following result is a reformulation of a result of Auslander and Solberg ([**7**]).

Proposition 2.1. *Let M be in* $\operatorname{mod}\Lambda$.

(a) *M dualizes a module A if and only if there is an exact sequence $0 \to A \to M_1 \to M_2$ with M_i in $\operatorname{add} M$ for $i = 1, 2$ and $A \to M_1$ a left $\operatorname{add} M$-approximation.*
(b) *Let $\Gamma = \operatorname{End}_\Lambda(M)$. Then ${}_\Lambda M$ has faithful dimension at least 2 if and only if $\Lambda \simeq \operatorname{End}_\Gamma({}_\Gamma M)$.*

A module of faithful dimension at least 2 is then the same as a *faithfully balanced bimodule*, using the notion of [**1**].

For each Λ-module M there is a complex

$$\theta\colon \cdots \to M'_n \xrightarrow{f'_n} \cdots \xrightarrow{f'_2} M'_1 \xrightarrow{f'_1} D\Lambda^{\mathrm{op}} \to 0,$$

with $K'_i = \operatorname{Im} f'_i$ such that each $M'_i \to K'_i$ is a minimal right $\operatorname{add} M$-approximation. Let θ_n be the truncated complex starting in M'_n obtained from θ. The module M is then said to have *cofaithful dimension* n if θ_n is exact, but θ_{n+1} is not. We denote this dimension by $\operatorname{cofadim} M$.

Using this we obtain the following.

PROPOSITION 2.2. *Let M be a Λ-module, and let $\Gamma = \mathrm{End}(M)$. Then the following hold.*

(a) $\operatorname{fadim}{}_\Lambda M = \operatorname{cofadim}{}_\Lambda M$.
(b) *If* $\operatorname{fadim} M \geq 2$, *then the following are equivalent.*
  (i) $\operatorname{fadim} M = n$.
  (ii) $\mathrm{Ext}^i_\Gamma(M, M) = (0)$ *for* $i = 1, \ldots, n-2$ *and* $\mathrm{Ext}^{n-1}_\Gamma(M, M) \neq (0)$.

PROOF. Let α denote the natural homomorphism

$$\Lambda \to \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Lambda(\Lambda, M), {}_\Gamma M) = \mathrm{End}_\Gamma(M)$$

and α' the natural homomorphism

$$\Lambda^{\mathrm{op}} \to \mathrm{Hom}_{\Gamma^{\mathrm{op}}}(\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda^{\mathrm{op}}, D(M)), {}_{\Gamma^{\mathrm{op}}} D(M)) = \mathrm{End}_{\Gamma^{\mathrm{op}}}(D(M)).$$

Then it follows that we have the following commutative diagram.

$$\begin{array}{ccc} \Lambda & \longrightarrow & \mathrm{End}_\Gamma(M) \\ \| & & \downarrow \wr \\ \Lambda^{\mathrm{op}} & \longrightarrow & \mathrm{End}_{\Gamma^{\mathrm{op}}}(D(M)) \end{array},$$

where the rightmost vertical map is induced by the duality D. It follows from this diagram that $\operatorname{fadim} M = i$ if and only if $\operatorname{cofadim} M = i$ for $i = 0, 1$.

So in the following we assume that $n \geq 2$. We first prove (i) and (ii) in (b) are equivalent. Let $\eta_n \colon 0 \to \Lambda \to M_1 \to \cdots \to M_n$ be the complex induced by taking minimal left $\operatorname{add} M$-approximations, $K_0 = \Lambda$ and $K_i = \mathrm{Coker}(K_i \to M_{i+1})$. We apply $\mathrm{Hom}_\Lambda(\ , {}_\Lambda M)$ to this complex and obtain an exact sequence

$$(M_n, M) \to \cdots \to (M_1, M) \to {}_\Gamma M \to 0$$

in $\operatorname{mod} \Gamma$. Since for each j we have $M_j \simeq ((M_j, M), M)$ and $K_0 = \Lambda \simeq ((\Lambda, M), M)$, by induction we obtain the following equivalence. For an integer $r \geq 1$ we have that $\mathrm{Ext}^i_\Gamma(M, M) = (0)$ for $i = 1, \ldots, r-1$ and $\mathrm{Ext}^r_\Gamma(M, M) \neq (0)$ if and only if there are exact commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{i-1} & \longrightarrow & M_{i-1} & \longrightarrow & K_i & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \longrightarrow & ((K_{i-1}, M), M) & \longrightarrow & ((M_i, M), M) & \longrightarrow & ((K_i, M), M) & \longrightarrow & 0 \end{array}$$

for each $i = 0, \ldots, r-1$ and an exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{r-1} & \longrightarrow & M_{r-1} & \longrightarrow & K_r & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow h_r & & \\ 0 & \longrightarrow & ((K_{r-1}, M), M) & \longrightarrow & ((M_r, M), M) & \xrightarrow{g_r} & ((K_r, M), M) & & \end{array}$$

where g_r is not an epimorphism and h_r is a proper monomorphism. But this is clearly equivalent to M having faithful dimension $r + 1$.

To prove that $\operatorname{cofadim} M = n$ is equivalent to (ii) in (b), choose a minimal projective resolution

$$\cdots \to P_1 \to P_0 \to D_\Gamma M \to 0$$

in Γ^{op}, and apply $\mathrm{Hom}_{\Gamma^{\mathrm{op}}}(\ ,{}_\Gamma M)$ to this exact sequence. We have $\Lambda^{\mathrm{op}} \simeq \mathrm{End}_\Gamma(M)^{\mathrm{op}} \simeq \mathrm{End}_{\Gamma^{\mathrm{op}}}(DM)$, and we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (DM, DM) & \longrightarrow & (P_0, DM) & \longrightarrow & \cdots & \longrightarrow & (P_n, DM) \\ & & \downarrow\wr & & \downarrow\wr & & & & \downarrow\wr \\ \eta'_n\colon 0 & \longrightarrow & \Lambda^{\mathrm{op}} & \longrightarrow & M''_1 & \longrightarrow & \cdots & \longrightarrow & M''_n \end{array}$$

with each M''_i in $\operatorname{add} DM$, and such that the inclusion maps are minimal left $\operatorname{add} DM$-approximations. The sequence η'_n is exact if and only if $\operatorname{fadim} DM = n$. Using that $\operatorname{fadim} DM = n$ if and only if $\operatorname{cofadim} M = n$, the same method as above gives the desired result. □

The Λ-modules with infinite faithful dimension seem to have special properties. Indeed, all tilting modules have infinite faithful dimension, and by the above also all cotilting modules have this property (see next section for the definition of cotilting modules). Let M be a Λ-module with $\operatorname{fadim} M = \infty$. Then the above proposition shows that M as a Γ-module is selforthogonal. Using our definitions it is easy to see that the selforthogoal modules M with $\operatorname{fadim} M = \infty$ coincides with the generalized tilting modules defined by Wakamatsu in [**13**]. These modules we call Wakamatsu tilting modules, and Wakamatsu cotilting modules are defined dually. Then Proposition 2.2 shows that a module is a Wakamatsu tilting module if and only if it is a Wakamatsu cotilting module. If M is such that $\delta(M) < \delta(\Lambda)$ and $\operatorname{fadim} M = \infty$, then it is not hard to show that $\delta({}_\Gamma M) > \delta(\Gamma)$. There is a conjecture that for a selforthogonal module this is impossible. So we formulate the following conjecture.

Conjecture. *For an artin algebra Λ, there is no module M in* $\operatorname{mod}\Lambda$ *with* $\delta(M) < \delta(\Lambda)$ *and* $\operatorname{fadim} M = \infty$.

This conjecture is also related to the Generalized Nakayama Conjecture. It says that in a minimal injective coresolution of ${}_\Lambda\Lambda$, all the indecomposable injective modules occur, when Λ is an artin algebra. In the language of faithful dimension this means that no injective module M, with $\delta(M) < \delta(\Lambda)$ has infinite faithful dimension.

3. Complements of almost complete cotilting modules

In this section we show that an almost complete cotilting module M has $n+1$ non-isomorphic indecomposable complements if and only if M has faithful dimension n. We also show a connection between almost complete cotilting modules and the Generalized Nakayama Conjecture.

We start by recalling definitions and results that we need later. A Λ-module T is a *cotilting* module if

(i) $\mathrm{Ext}^i_\Lambda(T, T) = (0)$ for all $i \geq 1$,
(ii) $\mathrm{id}_\Lambda T < \infty$,
(iii) there is an exact sequence $0 \to T_r \to \cdots \to T_0 \to D\Lambda^{\mathrm{op}} \to 0$ with T_i in $\operatorname{add} T$ for $i = 0, \ldots, r$.

It is known that if T is a cotilting module, then $\delta(T) = \delta(\Lambda)$ and ${}^\perp T \subseteq \operatorname{sub} T$, where $\operatorname{sub} T$ denotes the subcategory of modules cogenerated by T. A direct summand M in a cotilting module is called a *partial cotilting* module, if in addition $\delta(M) = \delta(\Lambda) - 1$ then M is called an *almost complete* cotilting module. If M is a partial

cotilting module, and X is a module such that $M \oplus X$ is a cotilting module, then X is called a *complement* of M. A Λ-module which satisfies (i) and (ii) is called an *exceptional* module. It is not known if an exceptional module M with $\delta(M) = \delta(\Lambda) - 1$ always is an almost complete module, although it is shown not to be true if $\delta(M) = \delta(\Lambda) - 2$ in [**12**]. Almost complete cotilting modules are studied in [**8**], [**9**] and [**10**]. The following theorem is contained is these papers.

THEOREM 3.1. *Let M be an almost complete cotilting module over an artin algebra Λ.*

(a) *There is a unique complement X_0, such that ${}^{\perp}M = {}^{\perp}(M \oplus X_0)$, this is called the Bongartz-complement of M.*
(b) *If M is non-faithful, this is the only complement of M.*
(c) *If M is faithful, there is an exact sequence*

$$\cdots \xrightarrow{f_3} M_1 \xrightarrow{f_2} M_0 \xrightarrow{f_1} X_0 \to 0$$

with $X_i = \operatorname{Ker} f_i$ such that M_i is in $\operatorname{add} M$ and $X_i \not\simeq X_j$ when $i \neq j$, and $\{X_i\}$ is a complete list of indecomposable complements of M. In addition each $0 \to X_{j+1} \to M_j$ is a minimal left $\operatorname{add} M$-approximation, and each $M_j \to X_j \to 0$ is a minimal right $\operatorname{add} M$-approximation, for $j \geq 0$.

The proof is based on the following fact. If X is a indecomposable complement of M and X is generated by M, then the kernel Y of the minimal right $\operatorname{add} M$-approximation is again an indecomposable complement. Also if X is a indecomposable complement of M and Y is cogenerated by M, then the cokernel X of the minimal left $\operatorname{add} M$-approximation is an indecomposable complement. We generalize this to partial cotilting modules.

PROPOSITION 3.2. *Let M be a partial cotilting module with a complement X.*

(a) *If X is generated by M and $f\colon M' \to X$ is a minimal right $\operatorname{add} M$-approximation, then $\operatorname{Ker} f$ is also a complement of M.*
(b) *If X is cogenerated by M and $g\colon X \to M''$ is a minimal left $\operatorname{add} M$-approximation, then $\operatorname{Coker} f$ is also a complement of M.*

PROOF. (a) Let $\eta\colon 0 \to Y \to M' \xrightarrow{f} X \to 0$ be exact, with f a minimal right $\operatorname{add} M$-approximation. By [**8**, Corollary 1.2], it is enough to show that $M \oplus Y$ is an exceptional module. Wakamatsus Lemma gives that $\operatorname{Ext}^1_\Lambda(M, Y) = (0)$, and then it is easily seen by applying $\operatorname{Hom}_\Lambda(M, \ )$ to η that $\operatorname{Ext}^i_\Lambda(M, Y) = (0)$ for $i > 1$ as well. By applying $\operatorname{Hom}_\Lambda(\ , M)$ to η we get that $\operatorname{Ext}^i_\Lambda(Y, M) = (0)$. By dimension shift we obtain $\operatorname{Ext}^i_\Lambda(Y, Y) \simeq \operatorname{Ext}^{i+1}_\Lambda(X, Y) \simeq \operatorname{Ext}^i_\Lambda(X, X) = (0)$. It is easy to see that $\operatorname{id} Y < \infty$ since $\operatorname{id}(M \oplus X) < \infty$.

(b) is similar. □

The following direct observation from Theorem 3.1 suggests that homological properties of the endomorphism ring of an almost complete cotilting module give information about the number of complements.

PROPOSITION 3.3. *Let M be an almost complete cotilting module over Λ, and let $\Gamma = \operatorname{End}_\Lambda(M)$. If $\operatorname{gldim} \Gamma = r < \infty$, then M has at most $r + 2$ complements.*

PROOF. Assume M has at least $r+3$ complements $X_0, X_1, \ldots, X_{r+2}$, such that there is an exact sequence

$$0 \to X_{r+2} \to M_{r+2} \to \cdots \to M_1 \to X_0 \to 0,$$

as in Theorem 3.1. Apply $\mathrm{Hom}_\Gamma(\ ,M)$ to this sequence, and obtain the exact sequence

$$0 \to (X_2, M) \to (M_3, M) \to \cdots \to (M_{r+2}, M) \to (X_{r+2}, M) \to 0.$$

Then (X_2, M) is projective, since $\mathrm{gldim}\,\Gamma = r$. But since X_2 is dualized by M, by Proposition 2.1, this means that X_2 is in $\mathrm{add}\,M$, a contradiction. □

After having decribed the number of complements of M in terms of $\mathrm{fadim}\,M$, we give the precise relationship between the number of complements and the homological properties over its endomorphism ring. First we point out a connection to the Generalized Nakayama Conjecture.

Since the injective dimension of the complement X_r in Theorem 3.1 is at least r, the number of complements of an almost complete cotilting module M over Λ is less than or equal to one plus the finitistic dimension of Λ. If we reformulate the Generalized Nakayama Conjecture in terms of almost complete cotilting modules, it says: no almost complete injective cotilting module has infinite faithful dimension. With this formulation the following is a direct consequence of Theorem 3.1 and Proposition 2.2.

THEOREM 3.4. *Let Λ be an artin algebra. Then Λ satisfies the Generalized Nakayama Conjecture if and only if the almost complete injective cotilting modules over Λ only have a finite number of complements.*

PROOF. An artin algebra Λ satisfies the Generalized Nakayama Conjecture if and only if no injective almost complete cotilting module has infinite faithful dimension. Let M be an injective almost complete cotilting module with injective Bongartz-complement X_0, such that $\mathrm{add}(X_0 \oplus M) = \mathrm{add}\,D\Lambda^{\mathrm{op}}$. Then it is easy to see that M has infinite cofaithful dimension if and only if M has an infinite number of complements, using Theorem 3.1. □

This is also a consequence of the following more general fact below, which also gives a procedure for computing the number of indecomposable non-isomorphic complements of an almost complete cotilting module, without actually computing any complement. In proving this we need the following lemma.

LEMMA 3.5. *Let X and M be Λ-modules such that X is generated by M, and $0 \to Y \to M' \xrightarrow{g} X \to 0$ is an exact sequence with M' in $\mathrm{add}\,M$. Assume $\mathrm{Ext}^1_\Lambda(A, Y) = (0)$ for a Λ-module A. Let $f\colon A \to M''$ be a minimal left $\mathrm{add}\,M$-approximation. Then f is also a minimal left $\mathrm{add}(M \oplus X)$-approximation.*

PROOF. Let $A \to M_1 \oplus X_1$ with M_1 in $\mathrm{add}\,M$ and X_1 in $\mathrm{add}\,X$ be a minimal $\mathrm{add}(M \oplus X)$-approximation, and consider the pullback-diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y_1 & \longrightarrow & Z & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y_1 & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_1 \oplus X_1 & \longrightarrow & 0
\end{array}$$

with Y_1 in $\mathrm{add}\,Y$ and $M_2 \to X_1$ a minimal right $\mathrm{add}\,M$-approximation. Then the upper row splits, and the induced map $A \to Z \to M_1 \oplus M_2$ is an $\mathrm{add}(M \oplus X)$-approximation. □

We now describe the number of indecomposable non-isomorphic complements of an almost complete cotilting module M in terms of $\mathrm{fadim}\,M$.

THEOREM 3.6. *Let M be an almost complete cotilting module in* $\operatorname{mod}\Lambda$. *Then M has exactly $n+1$ indecomposable non-isomorphic complements if and only if* $\operatorname{fadim} M = n$

PROOF. For $n = 0$ the statement follows from Theorem 3.1.

Assume M has $n+1$ indecomposable non-isomorphic complements $X_0, \dots, X_n$. We want to show that $\operatorname{fadim} M \geq n$.

If $n = 1$, then by Theorem 3.1 we have that $\operatorname{fadim} M \geq 1$. Assume now $n \geq 2$, such that we have an exact sequence

$$\theta_n \colon 0 \to X_n \to M^n \to \cdots \to M^1 \to X_0 \to 0.$$

By induction, $\operatorname{fadim} M \geq n-1$ and there is an exact sequence

$$\eta_{n-1} \colon 0 \to \Lambda \to M_1 \to \cdots \to M_{n-1} \to K_{n-1} \to 0.$$

We need only to show that K_{n-1} is cogenerated by M. Since X_{n-1} is generated by M, the map $\Lambda \to M_1$ is an $\operatorname{add}(M \oplus X_{n-1})$-approximation, by Lemma 3.5. Thus, by Wakamatsus Lemma ([**13**]) we have $\operatorname{Ext}^1_\Lambda(K_1, X_{n-1}) = (0)$. Also by Wakamatsus Lemma we have that K_i is in ${}^\perp M = {}^\perp(M \oplus X_0)$ for $i = 1, \dots, n-1$. By dimension shift, first along η_{n-1} and then along θ_n we then obtain $(0) = \operatorname{Ext}^1_\Lambda(K_1, X_{n-1}) \simeq \operatorname{Ext}^{n-1}_\Lambda(K_{n-1}, X_{n-1}) \simeq \operatorname{Ext}^1_\Lambda(K_{n-1}, X_1)$. Since $M \oplus X_0$ is a cotilting module, we have that ${}^\perp(M \oplus X_0) \subseteq \operatorname{sub}(M \oplus X_0)$. But K_{n-1} is in ${}^\perp(M \oplus X_0)$ since ${}^\perp(M \oplus X_0) = {}^\perp M$. Thus, there is a monomorphism $0 \to K_{n-1} \to M' \oplus X_0^r$ with M' in $\operatorname{add} M$ and $r \geq 0$. In the pullback-diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X_1^r \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & K_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_1^r \arrow[r] & M' \oplus M_1^r \arrow[r] & M' \oplus X_0^r \arrow[r] & 0
\end{tikzcd}$$

the upper sequence splits. This shows that K_{n-1} is cogenerated by M. Thus, $\operatorname{fadim} M \geq n$.

Assume now that $\operatorname{fadim} M = n$. We want to show that M has at least $n+1$ complements.

If $n = 1$, then M has at least two indecomposable non-isomorphic complements by Theorem 3.1. Assume $n \geq 2$. By induction M has n indecomposable non-isomorphic complements $X_0, \dots, X_{n-1}$, and we have exact sequences

$$\eta_n \colon 0 \to \Lambda \to M_1 \to \cdots \to M_n \to K_n \to 0$$

and

$$\theta_{n-1} \colon 0 \to X_{n-1} \to M^{n-1} \to \cdots \to M^1 \to X_0 \to 0.$$

It is enough to show that X_{n-1} is generated by M. By dimension shift along η_n and θ_{n-1} we have that $\operatorname{Ext}^1_\Lambda(K_1, X_{n-1}) \simeq \operatorname{Ext}^n_\Lambda(K_n, X_{n-1}) \simeq \operatorname{Ext}^1_\Lambda(K_1, X_0)$. But $\operatorname{Ext}^1_\Lambda(K_1, X_0) = (0)$ since by Wakamatsus Lemma K_1 is in ${}^\perp M = {}^\perp(M \oplus X_0)$.

There is an integer s and an epimorphism $\Lambda^s \to X_{n-1} \to 0$. Then in the pushout-diagram

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^s \arrow[r] \arrow[d] & M_1^s \arrow[r] \arrow[d] & K_1^s \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_{n-1} \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & K_1^s \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

the lower row will be an split exact sequence. This shows that X_{n-1} is generated by M. The kernel of the minimal right add M-approximation of X_{n-1} is then also a complement, and M has at least $n+1$ indecomposable non-isomorphic complements. The claim of the theorem follows from this. □

Using the description of the faithful dimension of a module in terms of its endomorphism ring we have the following immediate corollary.

Corollary 3.7. *Let M be an almost complete cotilting module over Λ and let $\Gamma = \operatorname{End}_\Lambda(M)$. Denote by α the natural homomorphism*

$$\Lambda \to \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(\Lambda, M), M) = \operatorname{End}_\Gamma(M).$$

(a) *M has a unique indecomposable complement if and only if α is not a monomorphism.*
(b) *M has exactly two indecomposable non-isomorphic complements if and only if α is a monomorphism but not an epimorphism.*
(c) *If M has at least three indecomposable non-isomorphic complements, then M has exactly $n+1$ indecomposable non-isomorphic complements if and only if $\operatorname{Ext}^i_\Gamma(M, M) = (0)$ for $i = 1, \dots, n-2$ and $\operatorname{Ext}^{n-1}_\Gamma(M, M) \neq (0)$.*

4. Relative cotilting modules

We start by recalling some definitions and results from [**5**] and [**6**]. Let Λ be an artin algebra. Denote by $\mathcal{P}(\Lambda)$ and $\mathcal{I}(\Lambda)$ the full subcategories of projective and injective modules in $\operatorname{mod}\Lambda$, respectively.

Let F be an additive subbifunctor of $\operatorname{Ext}^1_\Lambda(\ ,\ )\colon (\operatorname{mod}\Lambda)^{\mathrm{op}} \times \operatorname{mod}\Lambda \to \mathrm{Ab}$. Then an exact sequence $\eta\colon 0 \to A \to B \to C \to 0$ is said to be *F-exact* if η is in $F(C, A)$. A module P is called *F-projective* (or relative projective) if $F(P,\ ) = 0$ and the full subcategory of F-projective modules is denoted by $\mathcal{P}(F)$. Dually we define the full subcategory of F-injective modules $\mathcal{I}(F)$. The relative projective and the relative injective modules are related by the equality $\mathcal{P}(F) = \mathcal{P}(\Lambda) \cup \operatorname{Tr} D\,\mathcal{I}(F)$.

For a subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$ define

$$F_{\mathcal{X}}(C, A) = \{0 \to A \to B \to C \to 0 \mid (\ , B)|_{\mathcal{X}} \to (\ , C)|_{\mathcal{X}} \to 0 \text{ exact}\}$$

and

$$F^{\mathcal{X}}(C, A) = \{0 \to A \to B \to C \to 0 \mid (B,\ )|_{\mathcal{X}} \to (A,\ )|_{\mathcal{X}} \to 0 \text{ exact}\}$$

for all pairs of modules A and C in $\operatorname{mod}\Lambda$. It is proved in [**5**] that $F_{\mathcal{X}}$ and $F^{\mathcal{X}}$ are additive subbifunctors of $\operatorname{Ext}^1_\Lambda(\ ,\ )$. The constructions are related by $F_{\mathcal{X}} = F^{D\operatorname{Tr}\mathcal{X}}$, and we have $\mathcal{P}(F_{\mathcal{X}}) = \mathcal{P}(\Lambda) \cup \operatorname{add}\mathcal{X}$. Moreover, an additive subbifunctor F has enough projectives and injectives if and only if $F = F_{\mathcal{P}(F)}$ and $\mathcal{P}(F)$ is functorially finite in $\operatorname{mod}\Lambda$. In this case we let $\operatorname{Ext}^i_F(\ ,\ )$ denote the i-th derived functor of

F, where we have that $F = \mathrm{Ext}^1_F(\ ,\ )$. By $\mathrm{id}_F\, M$ we denote the relative injective dimension of a Λ-module M.

The use of this subject in the representation theory of artin algebras have been studied by Auslander and Solberg in the series [**5**], [**6**] and [**7**]. For further results on relative homology and relative cotilting modules we refer to these papers.

Let F be an additive subbifunctor of $\mathrm{Ext}^1_\Lambda(\ ,\ )$ with enough projectives and injectives. Recall from [**6**] that a Λ-module T is an *F-cotilting module* if

(i) $\mathrm{Ext}^i_F(T,T) = (0)$ for all $i \geq 1$,
(ii) $\mathrm{id}_F\, T < \infty$,
(iii) for each F-injective module I there is an F-exact sequence

$$0 \to T_n \to \cdots \to T_1 \to I \to 0$$

with the T_i in $\mathrm{add}\, T$.

It is known that the number of indecomposable non-isomorphic summands in an F-cotilting module equals the number of indecomposable non-isomorphic F-projective modules. Therefore $F = F_{\mathrm{add}(\Lambda\oplus X)}$ for some module X. We assume that all subbifunctors in the rest of the paper are of this form, so it is enough to specify $\mathcal{P}(F)$ to define the subbifunctors F we consider.

The following theorem is proven in [**6**].

THEOREM 4.1. *Let T be a F-cotilting module in* $\mathrm{mod}\,\Lambda$, *with* $\mathcal{P}(F) = \mathrm{add}(\Lambda \oplus X)$ *for a Λ-module X. Let* $\Gamma = \mathrm{End}_\Lambda(T)$.

(a) $C = \mathrm{Hom}_\Lambda(\Lambda \oplus X, T) = {}_\Gamma T \oplus (X,T)$ *is a Γ-cotilting module.*
(b) *The functors* $\mathrm{Hom}_\Lambda(\ ,T)\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ *and* $\mathrm{Hom}_\Gamma(\ ,T)\colon \mathrm{mod}\,\Gamma \to \mathrm{mod}\,\Lambda$ *induce dualities between the categories* ${}^\perp{}_\Lambda T = \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^i_F(X,T) = (0)$ *for all* $i > 0\}$ *and* ${}^\perp({}_\Gamma T \oplus (X,T))$.
(c) ${}_\Gamma T$ *is a dualizing summand of C.*

Some interesting consequences of this are the following observations. If T is an $F_{\mathrm{add}(\Lambda\oplus Y)}$-cotilting module where some indecomposable summand of Y do not occur in X, then this induces another complement $\mathrm{Hom}_\Lambda(Y,T)$ of the partial complete cotilting module ${}_\Gamma T$. Moreover, if the module X in the above theorem is indecomposable, the relative cotilting module T in $\mathrm{mod}\,\Lambda$ induces an ordinary almost complete cotilting module T in $\mathrm{mod}\,\Gamma$, with an indecomposable complement $\mathrm{Hom}_\Lambda(X,T)$. Furthermore, different relative theories give rise to different complements.

Not all almost complete cotilting modules are induced from relative cotilting modules as we now point out. By Theorem 4.1 and Theorem 3.1 it follows that an almost complete cotilting module M is a dualizing summand in a cotilting module if and only if M has at least three complements $X_0, X_1, X_2, \ldots$, and in this case M is a dualizing summand in $M \oplus X_i$ for $i \geq 2$. The following general result is proved in [**7**].

THEOREM 4.2. *Let $M \oplus X$ be an cotilting module in* $\mathrm{mod}\,\Gamma$ *and let* $\Lambda = \mathrm{End}_\Gamma(M)$. *Define F to be the subbifunctor with* $\mathcal{P}(F) = \mathrm{add}(\Lambda \oplus (X,M))$. *Then M is a dualizing summand of the Γ-module $M \oplus X$ if and only if ${}_\Lambda M$ is an F-cotilting module in* $\mathrm{mod}\,\Lambda$.

We are especially interested in the case when M is an almost complete cotilting module. Then we have the following immediate consequence using Theorem 3.1.

Corollary 4.3. *Let M be an almost complete cotilting module in* $\operatorname{mod}\Gamma$ *with at least three complements $X_0, X_1, X_2, \dots,$ and let $\Lambda = \operatorname{End}_\Gamma(M)$. For each $i \geq 2$ let F_i be the subbifunctor of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ such that $\mathcal{P}(F_i) = \operatorname{add}(\Lambda \oplus (X_i, M))$. Then ${}_\Lambda M$ is an F_i-cotilting module for all $i \geq 2$.*

5. Changing relative theories for a relative cotilting module

In the previous section we observed that in some situations it is possible that a module is a relative cotilting module for several relative theories. This section is devoted to giving necessary and sufficient criteria for this. The first theorem gives sufficient criteria.

Theorem 5.1. *Let $F = F_{\operatorname{add}(G\oplus A)}$ and $F' = F_{\operatorname{add}(G\oplus B)}$ for two Λ-modules A and B, where G is a generator, and such that A and B has no summands in* $\operatorname{add} G$. *Assume that there exists an exact sequence $\eta\colon 0 \to A \to P \to B \to 0$ such that η is $F^{\operatorname{add} T}$-exact and P is in $\mathcal{P}(F) \cap \mathcal{P}(F')$.*

(a) *Assume η is $F^{\operatorname{add} G}$-exact. Then if T is an F-cotilting module, T is also an F'-cotilting module.*
(b) *Assume η is $F_{\operatorname{add} G}$-exact. Then if T is an F'-cotilting module, T is also an F-cotilting module if there is an $F^{\operatorname{add} T}$-exact sequence $G'' \to G' \to A \to 0$ with G' and G'' in* $\operatorname{add} G$ *such that $G' \to A$ is a right* $\operatorname{add} G$*-approximation.*

Proof. (a) Assume T is a F-cotilting module and let $\Gamma = \operatorname{End}(T)$. Then (G, T) is a partial cotilting module with a complement (A, T), and such that (G, T) is a dualizing summand in $(G, T) \oplus (A, T)$, by Theorem 4.2. Since η is $F^{\operatorname{add} T}$-exact, the sequence

$$0 \to (B, T) \to (P, T) \to (A, T) \to 0$$

is exact. Since η is $F^{\operatorname{add} G}$-exact, $(P, T) \to (A, T)$ is a minimal $\operatorname{add}(G, T)$-approximation. But then, by Proposition 3.2 (B, T) is another complement of (G, T) and (G, T) is a dualizing summand in $(G, T) \oplus (B, T)$. But then by Theorem 4.2, T is an $F_{\operatorname{add}(G\oplus((B,T),T)}$-cotilting module. Since (A, T) is a complement of (G, T) and ${}_\Gamma T$ is a direct summand in (G, T), we have $\operatorname{Ext}^1_\Gamma((A, T), T) = (0)$. But then we obtain the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & ((A,T),T) \arrow[r] \arrow[d, "\wr"] & ((P,T),T) \arrow[r] \arrow[d, "\wr"] & ((B,T),T) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & P \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

such that $B \simeq ((B, T), T)$. This implies that T is a F'-cotilting module.

(b) By using $\operatorname{Hom}_\Lambda(\ , T)$ on the $F^{\operatorname{add} T}$-exact sequence $G'' \to G' \to A \to 0$ we obtain that (A, T) is dualized by (G, T). Then the rest is dual to (a). □

We now show that this induces a necessary condition for a Λ-module T to be a relative cotilting module for two different relative theories $F_{\operatorname{add}(G\oplus A)}$ and $F_{\operatorname{add}(G\oplus B)}$, where A are B are indecomposable modules and G is a generator.

Proposition 5.2. *Let A and B be two indecomposable Λ-modules such that T is an $F = F_{\operatorname{add}(G\oplus A)}$-cotilting and an $F' = F_{\operatorname{add}(G\oplus B)}$-cotilting module. Then there exist n exact sequences*

$$\eta_i\colon 0 \to X_i \to P_{i+1} \to X_{i+1} \to 0$$

for $i = 0, \dots, n-1$, where P_i is in $\operatorname{add} G$, *η_i is $F_{\operatorname{add}(G\oplus X_i)}$-exact and $F^{\operatorname{add}(G\oplus T\oplus X_{i+1})}$-exact, and either $X_0 = A$ and $X_n = B$ or $X_0 = B$ and $X_n = A$.*

Proof. Let $\Gamma = \operatorname{End}_\Lambda(T)$ and $U = \operatorname{Hom}_\Lambda(G, T)$. Then U is an almost complete cotilting module over Γ with complements $\operatorname{Hom}_\Lambda(A, T)$ and $\operatorname{Hom}_\Lambda(B, T)$. Then, by Proposition 3.1 we can assume that there is an exact sequence in $\operatorname{mod}\Gamma$

$$0 \to Y_n \to U_n \to U_{n-1} \to \cdots \to U_1 \to Y_0 \to 0$$

consisting of the short exact sequences $\eta'_{i-1}\colon 0 \to Y_i \to U_i \to Y_{i-1} \to 0$, where the inclusions $Y_i \to U_i$ are minimal left $\operatorname{add} U$-approximations and $Y_0 = (A, T)$ and $Y_n = (B, T)$. Applying $\operatorname{Hom}_\Gamma(\ , T)$ to the exact sequences η'_i we obtain exact sequences

$$\eta_i\colon 0 \to (Y_i, T) \to (U_{i+1}, T) \to (Y_{i+1}, T) \to 0$$

for $i = 0, \dots, n-1$. Since T is an F-cotilting module and an F'-cotilting module, we have that $(Y_0, T) = ((A, T), T) \simeq A$ and $(Y_n, T) = ((B, T), T) \simeq B$. Similarly, $(U_i, T) = ((G_i, T), T) \simeq G_i$ for some G_i in $\operatorname{add} G$.

Consider $\eta_0\colon 0 \to (Y_0, T) \to (U_1, T) \to (Y_1, T) \to 0$. Since Y_0 and U_1 are in ${}^\perp(U \oplus Y_0)$, it follows that Y_1 is in ${}^\perp(U \oplus Y_0)$. Since $\operatorname{Ext}^1_\Gamma(Y_0, Y_1) \simeq \operatorname{Ext}^1_F((Y_1, T), (Y_0, T))$ via the functor $\operatorname{Hom}_\Gamma(\ , T)$, the sequence η_0 is F-exact.

Applying $\operatorname{Hom}_\Lambda(\ , G \oplus T \oplus (Y_1, T))$ on the F-exact sequence η_0 gives rise to the long exact sequence

$$((U_1, T), G \oplus T \oplus (Y_1, T)) \to ((Y_0, T), G \oplus T \oplus (Y_1, T)) \to \operatorname{Ext}^1_F((Y_1, T), G \oplus T \oplus (Y_1, T)) \to 0.$$

Similarly as above we have that

$$\begin{aligned}\operatorname{Ext}^1_F((Y_1, T), G \oplus T \oplus (Y_1, T)) &\simeq \operatorname{Ext}^1_\Gamma((G \oplus T \oplus (Y_1, T), T), ((Y_1, T), T)) \\ &\simeq \operatorname{Ext}^1_\Gamma(U \oplus \Gamma \oplus Y_1, Y_1) = (0),\end{aligned}$$

therefore the sequence η_0 is $F^{\operatorname{add}(G\oplus T\oplus (Y_1,T))}$-exact.

Now it follows from Theorem 5.1 that T is an $F_{\operatorname{add}(G\oplus(Y_1,T))}$-cotilting module, and we can repeat the above procedure for η_1, η_2 and so on. We let $X_i = (Y_i, T)$ and $P_{i+1} = (U_{i+1}, T)$ for $i = 0, \dots, n-1$, and this completes the proof of the proposition. □

The following is an obvious consequence.

Corollary 5.3. *If Λ has finite global dimension and* $\operatorname{gldim}\Lambda = n$, *then a given Λ-module T can be a relative cotilting module for relative theories $F = F_{\operatorname{add}(\Lambda\oplus A)}$ with A indecomposable for at most n non-isomorphic indecomposable modules A.*

We also consider the situation where the generator G is equal to Λ. Then the hypothesis in Theorem 5.1 simplifies considerably.

Lemma 5.4. *If $G = \Lambda$ and the modules A and B are indecomposable, we have the following equivalence. The sequence η is $F^{\operatorname{add} T}$-exact and $F^{\operatorname{add} G}$-exact with P in $\mathcal{P}(F) \cap \mathcal{P}(F')$ if and only if and $\operatorname{Ext}^1_\Lambda(B, \Lambda \oplus T) = (0)$ with P in $\mathcal{P}(\Lambda)$.*

Proof. Since $\mathcal{P}(F) = \operatorname{add}(\Lambda \oplus A)$ and $\mathcal{P}(F') = \operatorname{add}(\Lambda \oplus B)$, a Λ-module is projective in both relative theories if and only if it is an ordinary projective module. But then η is $F^{\operatorname{add} T}$-exact if and only if $\operatorname{Ext}^1_\Lambda(B, T) = (0)$, and η is $F^{\operatorname{add} G}$-exact if and only if $\operatorname{Ext}^1_\Lambda(B, \Lambda) = (0)$. □

Next we show that a corresponding relation between the relative injective modules gives the same result.

PROPOSITION 5.5. *Let A and B be two non-projective modules, with $\mathcal{P}(F) = \operatorname{add}(G \oplus A)$ and $\mathcal{P}(F') = \operatorname{add}(G \oplus B)$. Then the following are equivalent.*

(a) *There is a sequence $\eta\colon 0 \to A \to P \to B \to 0$, with P in $\mathcal{P}(F) \cap \mathcal{P}(F')$ such that η is $F^{\operatorname{add}(G \oplus T)}$-exact.*

(b) *There is a sequence $\eta'\colon 0 \to D\operatorname{Tr} A \to I \to D\operatorname{Tr} B \to 0$, with I in $\mathcal{I}(F) \cap \mathcal{I}(F')$ such that η' is $F_{\operatorname{add} T}$-exact and $F^{\operatorname{add} D\operatorname{Tr} G}$-exact.*

PROOF. We prove only that (a) implies (b), since the opposite is dual. Assume $\eta\colon 0 \to A \to P \to B \to 0$ is an exact sequence with P in $\mathcal{P}(F) \cap \mathcal{P}(F')$ and that η is $F^{\operatorname{add}(G \oplus T)}$-exact. We first show that we then have an exact sequence $\eta'\colon 0 \to D\operatorname{Tr} A \to I \to D\operatorname{Tr} B \to 0$ with I in $\mathcal{I}(F) \cap \mathcal{I}(F')$. We choose minimal projective resolutions for A and B, and with the Horse Shoe Lemma we obtain the following exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Q_1 \arrow[r] \arrow[d] & Q_1 \oplus F_1 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q_0 \arrow[r] \arrow[d] & Q_0 \oplus F_0 \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

From this we can construct a commutative diagram with exact columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B^* \arrow[r] \arrow[d] & P^* \arrow[r] \arrow[d] & A^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_0^* \arrow[r] \arrow[d] & F_0^* \oplus Q_0^* \arrow[r] \arrow[d] & Q_0^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1^* \arrow[r] \arrow[d] & F_1^* \oplus Q_1^* \arrow[r] \arrow[d] & Q_1^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Tr} B \arrow[r] \arrow[d] & \operatorname{Tr} P \oplus Q \arrow[r] \arrow[d] & \operatorname{Tr} A \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with Q in $\mathcal{P}(\Lambda^{\mathrm{op}})$. Since η is $F^{\operatorname{add} G}$-exact, the upper row is exact. The middle rows are obviously exact. Therefore the lower row is exact, and we have the exact sequence

$$\eta'\colon 0 \to D\operatorname{Tr} A \xrightarrow{f} I \to D\operatorname{Tr} B \to 0$$

with I in $\mathcal{I}(F) \cap \mathcal{I}(F')$.

Since η is $F^{\operatorname{add} T}$-exact all morphisms $A \to T$ factors through $0 \to A \to P$, such that $\underline{\operatorname{Hom}}_\Lambda(A,T) = (0)$. We use the Auslander-Reiten formula and obtain $\operatorname{Ext}^1_\Lambda(T, D\operatorname{Tr} A) = (0)$. But this means that η' is $F_{\operatorname{add} T}$-exact. Since η is $F^{\operatorname{add} G}$-exact, then $(P,G) \to (A,G) \to 0$ is exact. From general theory, $\underline{\operatorname{Hom}}_\Lambda(A,G) \simeq \overline{\operatorname{Hom}}_\Lambda(D\operatorname{Tr} A, D\operatorname{Tr} G)$. Then, if h in $\operatorname{Hom}_\Lambda(D\operatorname{Tr} A, D\operatorname{Tr} G)$, there is a g in $\operatorname{Hom}_\Lambda(I, D\operatorname{Tr} G)$ such that $h - g \circ f$ factors through an injective module. But since a map from $D\operatorname{Tr} A$ to an injective module clearly factors through f, we see that h factors through f. Therefore the sequence η' is $F^{\operatorname{add} D\operatorname{Tr} G}$-exact. □

Another formulation of the Generalized Nakayama Conjecture states that if $\operatorname{Ext}^i_\Lambda(D\Lambda^{\mathrm{op}} \oplus X, X) = (0)$ for all $i > 0$, then X is injective. This conjecture is obviously a consequence of the more general conjecture that if $\operatorname{Ext}^i_\Lambda(N,N) = (0)$ for all $i \geq 0$ for a Λ-module N, then $\delta(N) \leq \delta(\Lambda)$.

For a arbitrary Λ-module M, by $\overline{M}$ we mean the maximal direct summand of M with no injective summands. We need the following lemma, where we leave the proof to the reader.

Lemma 5.6. *Let X be a non-injective Λ-module such that $\operatorname{Ext}^i_\Lambda(D\Lambda^{\mathrm{op}} \oplus X, X) = (0)$ for all $i = 1, \dots, n$.*

(a)
$$\operatorname{Ext}^i_\Lambda(D\Lambda^{\mathrm{op}}, \Omega^{-j}_\Lambda(X)) = (0)$$
for $j = 0, \dots, n-1$ and $0 < i < n - j$ and
$$\operatorname{Ext}^i_\Lambda(\Omega^{-j}_\Lambda(X), \Omega^{-k}_\Lambda(X)) = (0)$$
for $0 \leq j < i + k \leq n$.

(b) *The number of non-injective indecomposable summands of X and $\Omega^{-i}_\Lambda(X)$ are the same for $0 \leq i \leq n$.*

(c) *If $0 \leq i < j \leq n$, then $\overline{\Omega^{-i}_\Lambda(X)} \not\simeq \overline{\Omega^{-j}_\Lambda(X)}$.*

By this we obtain an alternative way to prove that if the number of complements of an almost complete cotilting module is always finite, the Generalized Nakayama Conjecture is true for all artin algebras.

Proposition 5.7. *Let X be a non-injective indecomposable module in $\operatorname{mod}\Lambda$ with $\operatorname{Ext}^i_\Lambda(D\Lambda^{\mathrm{op}} \oplus X, X) = (0)$ for $i = 1, \dots, n$. Let $T = D\Lambda^{\mathrm{op}} \oplus X$ and $\Gamma = \operatorname{End}_\Lambda(T)$. Then T is an injective almost complete Γ-cotilting module, and T has at least $n+2$ complements.*

Proof. Let X be an indecomposable non-injective Λ-module, such that $\operatorname{Ext}^i_\Lambda(D\Lambda^{\mathrm{op}} \oplus X, X) = (0)$ for all $i \geq 1$. Let $T = D\Lambda^{\mathrm{op}} \oplus X$ and let $\Gamma = \operatorname{End}_\Lambda(T)$. Since $\operatorname{id}_\Lambda X \geq n$, for each $i = 0, \dots, n-2$ we have an exact sequence $\eta_i \colon 0 \to \Omega^{-i}_\Lambda(X) \to I_i \to \Omega^{-(i+1)}_\Lambda(X) \to 0$. Let $F_i = F^{\operatorname{add}(D\Lambda^{\mathrm{op}} \oplus \Omega^{-i}_\Lambda(X))}$, then by part (a) in Lemma 5.6 the sequence η_i is $F^{\mathcal{I}(F_{i+1})}$-exact and $F_{\mathcal{I}(F_i) \oplus \operatorname{add} T}$-exact. Therefore, by repeated use of Theorem 5.1 the module T is an $F^{\operatorname{add}(D\Lambda^{\mathrm{op}} \oplus \Omega^{-i}_\Lambda(M))} = F_{\operatorname{add}(\Lambda \oplus \operatorname{Tr} D(\Omega^{-i}_\Lambda(M)))}$-cotilting module for all $i = 0, \dots, n-1$. But Lemma 5.6 gives that all these n relative theories are different. Then we know that the injective almost complete cotilting module T has at least n complements which are dualized by T, and therefore at least $n+2$ complements. □

6. Subcategory correspondence

It is shown in [**2**] that for a given artin algebra Λ, there is a 1-1 correspondence between the basic cotilting modules and contravariantly finite resolving subcategories of $\operatorname{mod}\Lambda$ with finite resolution dimension. Using this correspondence one can reformulate questions about complements of almost complete cotilting modules to questions about subcategories of $\operatorname{mod}\Lambda$. If M is an almost complete cotilting module with at least three complements, we have seen that it is induced by a relative cotilting module. So, questions about these types of almost complete cotilting modules can be translated to questions about relative cotilting modules. In this section we give the precise relationship.

First we recall the definitions and results we need later. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod}\Lambda$. Then $\mathcal{X}$ is called resolving if it is closed under extensions and kernels of epimorphisms, and contains the projective modules. Dually, we define coresolving subcategories. The resolution-dimension $\mathcal{X}$-$\operatorname{resdim} C$ of a module C is the minimal number t including infinity such that there exists an exact sequence

$$\eta\colon 0 \to X_t \to X_{t-1} \to \cdots \to X_0 \to C \to 0$$

with the X_i in $\mathcal{X}$. Then $\mathcal{X}$-$\operatorname{resdim}(\operatorname{mod}\Lambda)$ is defined to be the $\sup\{\mathcal{X}\text{-}\operatorname{resdim} C \mid C \in \operatorname{mod}\Lambda\}$. We define dually $\mathcal{X}$-$\operatorname{coresdim}(\operatorname{mod}\Lambda)$. We let $\widehat{\mathcal{X}}$ be the full subcategory of modules with a finite resolution like η. For a relative theory F, a module X and a category $\mathcal{X}$ the notions ${}^{\perp_F}X$, $X^{\perp_F}$ and $(\widehat{\mathcal{X}})_F$ should now be self-explanatory.

By [**2**], the map $T \mapsto {}^{\perp}T$ is a 1-1 correspondence between the basic cotilting modules and the contravariantly finite resolving subcategories $\mathcal{X}$ of $\operatorname{mod}\Lambda$ with $\mathcal{X}$-$\operatorname{resdim}(\operatorname{mod}\Lambda) < \infty$. There is also a 1-1 correspondence between the basic cotilting modules and the covariantly finite coresolving subcategories $\mathcal{X}$ with $\mathcal{X}$-$\operatorname{coresdim}(\operatorname{mod}\Lambda) < \infty$. This is given by the map $T \mapsto ({}^{\perp}T)^{\perp}$. It is also known that $({}^{\perp}T)^{\perp} = \widehat{\operatorname{add} T} = \mathcal{I}^{\infty}(\Lambda) \cap T^{\perp}$.

Assume M is an almost complete cotilting module in $\operatorname{mod}\Gamma$ with complements $X_0, X_1, X_2, \ldots$. Then we have an exact sequence

$$\eta\colon \cdots \to M_1 \to M_0 \to X_0 \to 0$$

as in Theorem 3.1. This exact sequence induces a properly descending chain of subcategories of $\operatorname{mod}\Gamma$

$$\cdots \subseteq {}^{\perp}(M \oplus X_2) \subseteq {}^{\perp}(M \oplus X_1) \subseteq {}^{\perp}(M \oplus X_0) = {}^{\perp}M.$$

We also have a properly ascending chain of subcategories

$$\cdots \supseteq \mathcal{I}^{\infty}(\Gamma) \cap (M \oplus X_2)^{\perp} \supseteq \mathcal{I}^{\infty}(\Gamma) \cap (M \oplus X_1)^{\perp} \supseteq \mathcal{I}^{\infty}(\Gamma) \cap (M \oplus X_0)^{\perp}.$$

The Generalized Nakayama Conjecture states that these sequences always are finite. We have in the earlier sections seen how questions about the complements of an almost complete cotilting module M can be translated to questions about relative theories for a relative cotilting module, by looking at the endomorphism ring $\Lambda = \operatorname{End}_{\Gamma}(M)$. Motivated by this, we show that the subcategories ${}^{\perp}(M \oplus X_i)$ and $\mathcal{I}^{\infty}(\Gamma) \cap (M \oplus X_i)^{\perp}$ have dual versions in $\operatorname{mod}\Lambda$.

Proposition 6.1. *Let M be an almost complete cotilting module in* $\operatorname{mod}\Gamma$, *and assume M has at least three complements $X_0, X_1, X_2, \ldots$. Let $\Lambda = \operatorname{End}_{\Gamma}(M)$ and let F_i be the relative theories in* $\operatorname{mod}\Lambda$ *given by $\mathcal{P}(F_i) = \operatorname{add}(\Lambda \oplus (X_i, M))$ for $i \geq 2$.*

(a) *The categories* ${}^{\perp_{F_i}}M$ *and* ${}^{\perp}(M \oplus X_i)$ *are dual by the functors* $\operatorname{Hom}_\Lambda(\ ,M)$ *and* $\operatorname{Hom}_\Gamma(\ ,M)$.

(b) *The categories* $\mathcal{I}^\infty(F_i) \cap M^{\perp_{F_i}}$ *and* $\mathcal{I}^\infty(\Gamma) \cap (M \oplus X_{i-2})^\perp$ *are dual by the functors* $D\operatorname{Hom}_\Lambda(M,\ )$ *and* $D\operatorname{Hom}_\Gamma(M,\ )$.

(c) *The categories* $(\widehat{\operatorname{add} M})_{F_i}$ *and* $\operatorname{add}(\widehat{M \oplus X_{i-2}})$ *are dual by the same functors as in* (b).

Proof. (a) This follows directly from results in [**6**].

(b) The categories $\mathcal{I}^\infty(F_i) \cap M^{\perp_{F_i}}$ and $\mathcal{I}^\infty(\Gamma) \cap (D(M, \mathcal{I}(F_i)))^\perp$ are dual by the given functors ([**6**]), so we need only to show that $D(M, \mathcal{I}(F_i)) = \operatorname{add}(M \oplus X_{i-2})$. Since $\mathcal{I}(F_i) = \operatorname{add}(D\Lambda^{\mathrm{op}} \oplus D\operatorname{Tr}(X_i, M))$, this follows from the equalities (i) $D\operatorname{Hom}_\Lambda(M, D\Lambda^{\mathrm{op}}) \simeq {}_\Gamma M$ and (ii) $D\operatorname{Hom}_\Lambda(M, D\operatorname{Tr}(X_j, M)) \simeq X_{j-2}$, By adjoitness, we have

$$D\operatorname{Hom}_\Lambda(M, D\Lambda^{\mathrm{op}}) \simeq DD(\Lambda^{\mathrm{op}} \otimes_\Lambda M)$$

This proves (i), since $\Lambda^{\mathrm{op}} \otimes_\Lambda M \simeq {}_\Gamma M$. Using adjointness again we obtain that

$$D\operatorname{Hom}(M, D\operatorname{Tr}(X_j, M)) \simeq \operatorname{Tr}(X_j, M) \otimes_\Lambda M.$$

Thus, we need to show that $\operatorname{Tr}(X_j, M) \otimes_\Lambda M \simeq X_{j-2}$. Consider the exact sequence

$$0 \to X_j \to M'' \to M' \to X_{j-2} \to 0$$

induced by taking minimal $\operatorname{add} M$-approximations in $\operatorname{mod}\Gamma$. By applying the functor $\operatorname{Hom}_\Lambda(\ ,M)$ to this we obtain an exact sequence in $\operatorname{mod}\Lambda$

$$0 \to (X_{j-2}, M) \to (M', M) \to (M'', M) \to (X_j, M) \to 0.$$

This is a projective presentation of (X_j, M) and we obtain the exact sequence

$$((M'', M), \Lambda) \to ((M', M), \Lambda) \to \operatorname{Tr}(X_j, M) \to 0.$$

We have that $\operatorname{Hom}_\Lambda((M, M), \Lambda) \otimes_\Lambda {}_\Lambda M_{\Gamma^{\mathrm{op}}} \simeq \Lambda \otimes_\Lambda {}_\Lambda M_{\Gamma^{\mathrm{op}}} \simeq {}_\Gamma M$, and if $\widetilde{M}$ is in $\operatorname{add} M$, then $\operatorname{Hom}_\Lambda((\widetilde{M}, M), \Lambda) \otimes M \simeq \widetilde{M}$. By this we obtain the exact commutative diagram

$$\begin{array}{ccccccc}
((M'', M), \Lambda) \otimes M & \longrightarrow & ((M', M), \Lambda) \otimes M & \longrightarrow & \operatorname{Tr}(X_j, M) \otimes M & \longrightarrow & 0 \\
\downarrow \wr & & \downarrow \wr & & \downarrow & & \\
M'' & \longrightarrow & M' & \longrightarrow & X_{j-2} & \longrightarrow & 0
\end{array}$$

such that $\operatorname{Tr}(X_j, M) \otimes M \simeq X_{j-2}$. This concludes the proof of (b).

(c) From [**6**] we have that for any (relative) F-cotilting module T we have $(\widehat{\operatorname{add} M})_F = \mathcal{I}^\infty(F) \cap M^{\perp_F}$. □

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