AN ALGORITHMIC APPROACH TO RESOLUTIONS

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Abstract. We provide an algorithmic method for constructing projective resolutions of modules over quotients of path algebras. This algorithm is modified to construct minimal projective resolutions of linear modules over Koszul algebras.

1. Introduction

Projective resolutions play an important role in homological algebra. The existence of algorithmic methods has led to programs that construct projective resolutions of modules. For commutative rings we mention CoCoa System, Faugère’s GB, Macaulay, and Singular [8, 9, 17, 20]. On the other hand, for non-commutative rings there are fewer choices. For group algebras there are programs written for MAGMA by Jon Carlson [18]. The program Bergman [5] provides projective resolutions for quotients of free algebras by homogeneous ideals, and the program GRB [10] provides minimal projective resolutions for finite dimensional modules over finite dimensional quotients of path algebras.

The goal of this paper is to provide an algorithmic method that constructs projective resolutions of modules over quotients of path algebras. We also modify this algorithm to construct minimal linear projective resolutions of linear modules over a Koszul algebra in the last section. The construction uses both the theory of projective resolutions presented in [16] and the theory of Gröbner bases for path algebras [12].

Before giving a brief summary of needed background material in the next section, we provide an overview of the results of the paper. To this end we recall the definition of a path algebra, with a fuller explanation given in the next section. Let \( k \) be a field and let \( Q \) be a finite quiver; that is, a finite directed graph. The path algebra, \( kQ \), has \( k \)-basis consisting of all the finite directed paths in \( Q \), and multiplication is induced by concatenation of paths. For the remainder of this paper let \( R = kQ \) denote a path algebra. In Section 3, we present our main step in the construction of a projective resolution of a module over a quotient of a path algebra and then in Section 4, we show how to get a resolution using the main step. One of the more interesting theoretical results is that if \( I \) has a finite Gröbner basis, and \( M \) is a right \( R/I \)-module which is finitely presented as an \( R \)-module, then the construction yields a projective \( R/I \)-resolution of \( M \) such that each projective occurring in the resolution is finitely generated (even though \( R/I \) need not be noetherian). A discussion of the algorithmic aspects of the construction of the resolution follows in the next section. The final section applies a modified version

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of our construction to give an algorithmic method for constructing minimal linear projective resolutions of linear modules over Koszul algebras. Applications of this algorithm can be found in [6, 7].

2. Preliminaries

The main object of study in this paper is the construction of projective resolutions of modules over quotients of path algebras. In this section we recall notions and results on path algebras, Gröbner basis theory for path algebras [12] and projective resolutions of modules over quotients of these as presented in [16].

Let $k$ be a field and let $Q$ be a finite quiver. We denote the vertex set of $Q$ by $Q_0$, the arrow set by $Q_1$ and let $\mathcal{B}$ denote the set of finite directed paths in $Q$. The path algebra, $kQ$, as a $k$-vector space, has basis $\mathcal{B}$. Note that we view the vertices of $Q$ as paths of length 0. If $p, q \in \mathcal{B}$, then we define $p \cdot q = (pq)$ if the terminus vertex of $p$ is the origin vertex of $q$ and 0 otherwise. If $Q$ has a single vertex and $n$ arrows (loops), then $kQ$ is isomorphic to the free associative algebra on $n$ noncommuting variables. Hence the class of algebras we study include quotients of free algebras. We refer the reader to [3] for a fuller description of path algebras and their properties.

Beginning our background information, we summarize the theory of projective resolutions presented in [16]. Let $I$ be an ideal in $R = kQ$ and $\Lambda = R/I$. Let $X$ be an $R-R$-bimodule (respectively a left $R$-module, a right $R$-module) and $x$ in $X$. Then we say that a nonzero element $x$ is uniform (respectively left uniform, right uniform) if there exist $u$ and $v$ (respectively $u$ in $Q_0$, $v$ in $Q_0$) in $Q_0$ such that $x = u xv$ (respectively $x = u x$, $x = x v$). Note that if $Q$ has a single vertex then every nonzero $x$ in $X$ is uniform (resp., left uniform, right uniform).

If $X$ is a right $kQ$-module and $x$ is in $X$, then $\overline{x}$ denotes the natural residue class of $x$ in $X/XI$.

Suppose that $M$ is a finitely generated right $\Lambda$-module. Then, as shown in [16], there exist $t_n$ and $u_n$ in $\{0, 1, 2, \ldots \} \cup \infty$ with $u_0 = 0$, $\{f^n_i \}_{i \in T_n = \{1, \ldots , t_n\}}$, and $\{f'^n_i \}_{i \in U_n = \{1, \ldots , u_n\}}$ such that

(i) Each $f^n_i$ is a right uniform element of $R$ for all $i \in T_0$.

(ii) Each $f'^n_i$ is in $\Pi_{j \in T_{n-1}} f'^{n-1}_j R$ and is a right uniform element for all $i \in T_n$ and all $n \geq 1$.

(iii) Each $f'^n_i$ is in $\Pi_{j \in U_{n-1}} f^{n-1}_j I$ and is a right uniform element for all $i \in U_n$ and all $n \geq 1$.

(iv) For each $n \geq 2$,

\[
(\Pi_{i \in T_{n-1}} f'^{n-1}_i R) \cap (\Pi_{i \in T_{n-2}} f'^{n-2}_i I) = (\Pi_{i \in T_n} f^n_i R) \Pi (\Pi_{i \in U_n} f'^n_i R).
\]

The next result explains how the above elements give rise to a projective $\Lambda$-resolution of $M$. For this we need some notation. Let $f_1, \ldots , f_m$ be right uniform elements of $R$ and $v_1, \ldots , v_m$ vertices such that $f_i v_j = f_i$ for $i = 1, \ldots , m$. For $i = 1, \ldots , m$, let $\varepsilon_i = (\varepsilon_{i1}, \ldots , \varepsilon_{im})$ in $\mathbb{Z}^m$ be defined by $\varepsilon_{ij} = 0$ for $i \neq j$ and $\varepsilon_{ii} = f_i$. Let $\overline{\varepsilon}$ in $\Pi_{i=1}^m f_i R/ \Pi_{i=1}^m f_i I$ be defined in a similar fashion.

Theorem 2.1 ([16]). Let $M$ be a finitely generated right $\Lambda$-module and suppose that, for $n \geq 0$, $t_n$ and $u_n$ are in $\{0, 1, 2, \ldots \} \cup \infty$, and $\{f^n_i \}_{i \in T_0 = \{1, \ldots , t_0\}}$ and $\{f'^n_i \}_{i \in U_0 = \{1, \ldots , u_0\}}$ are chosen satisfying (i)-(iv) above. We have that $f^n_i = \sum_{j \in T_{n-1}} f'^{n-1}_j h^{-1}_{j,i}$, for some right uniform elements $h^{-1}_{j,i}$ in $R$. Let $L^n =$
(\Pi_{i \in T_n} f^n_i R/ \Pi_{i \in T_n} f^n_i I), and \epsilon^{n+1}: L^{n+1} \rightarrow L^n be given by \frac{f^n_i k_{j,i}}{f^n_j} \text{ in the } j\text{-th component of } \epsilon^{n+1}. \text{ Then}

\ldots \epsilon^{n+1} \xrightarrow{e} L^n \xrightarrow{e} L^{n-1} \xrightarrow{\epsilon^{n-1}} \ldots \epsilon^1 \xrightarrow{L^0} M \rightarrow 0

is a projective \Lambda-resolution of \mathcal{M}.

A more precise statement of the goal of this paper is to show how to algorithmically construct the \( t_n \)'s, the \( u_n \)'s, the \( f^n_i \)'s, and the \( f^n_{ij} \)'s. For this we need the theory of noncommutative Gröbner bases in path algebras and we review this theory. For more complete details we refer the reader to [12].

First we need to order the basis \( B \) of paths in \( Q \). We say > is an admissible order on \( B \) if the following properties hold.

1. The order > is a well-order.
2. If \( p, q \in B \) with \( p > q \) then for all \( r \in B \), \( pr > qr \) if both \( pr \) and \( qr \) are nonzero.
3. If \( p, q \in B \) with \( p > q \) then for all \( r \in B \), \( rp > rq \) if both \( rp \) and \( rq \) are nonzero.
4. If \( p = qr \) with \( p, q, \) and \( r \) paths in \( B \) then \( p \geq q \) and \( p \geq r \).

There are many admissible orders. For example, we arbitrarily order the set of vertices of \( Q \), and we arbitrarily order the set of arrows of \( Q \). Set each vertex smaller that any arrow. If \( p \) and \( q \) are paths of length at least one, then \( p > q \) if the length of \( p \) is greater than the length of \( q \), or, the lengths are equal and \( p = a_1 a_2 \cdots a_n \) and \( q = b_1 b_2 \cdots b_n \) with the \( a_i \)'s and \( b_i \)'s arrows, then there is some \( i \) such that \( a_j = b_j \) if \( j < i \) and \( a_i > b_i \). For the remainder of this section, let \( > \) be an admissible order on \( B \).

If \( x = \sum_{i=1}^n a_i p_i \in kQ \) with \( a_i \) nonzero elements of \( k \) and \( p_i \) distinct paths, then \( \text{tip}(x) = p_i \) if \( p_i \geq p_j \) for \( j = 1, \ldots, n \). If \( X \subseteq kQ \) then \( \text{tip}(X) = \{ \text{tip}(x) \mid x \in X \setminus \{0\} \} \). We say a subset \( \mathcal{G} \) of \( I \) is a Gröbner basis of \( I \) (with respect to >) if the ideal generated by \( \text{tip}(\mathcal{G}) \) equals the ideal generated by \( \text{tip}(I) \).

There is an extension of Buchberger's algorithm that allows one to construct Gröbner bases for ideals. A word of caution is needed here, in that, in general, even if \( I \) is finitely generated there may not be a finite Gröbner basis for \( I \). But the "algorithm" sequentially constructs sets \( \mathcal{G}_1, \mathcal{G}_2, \ldots \) so that \( \cup \mathcal{G}_i \) is a Gröbner basis and if there is a finite Gröbner basis then the "algorithm" terminates in a finite number of steps.

We provide a small example of a Gröbner basis in our setting, which we refer to later in the paper.

**Example 2.2.** Let \( Q \) be the quiver

\[ \begin{array}{c}
  v_1 \xrightarrow{a} v_2 \xrightarrow{b} v_4 \\
  v_1 \xrightarrow{c} v_3 \xrightarrow{d} v_4 \\
  v_3 \xrightarrow{e} v_5
\end{array} \]

Let \( I \) be the ideal generated by \( ab - cd \) and \( be \) in \( kQ \). Choose > to be the admissible order described earlier with \( v_5 < v_4 < v_3 < v_2 < v_1 < e < d < c < b < a \). The algorithm described in [12] yields the Gröbner basis \( \mathcal{G} = \{ab - cd, be, cde\} \).
If we change the order \( >_1 \) to \( >_2 \), where \( v_5 < \ldots < v_1 < a < b < c < d < e \), then one may check that the Gröbner basis now is \( G = \{ ab - cd, be \} \) (since \( \text{tip}(ab - cd) = cd \)).

We need the concept of a tip-reduced set of uniform elements of \( R \). If \( p \) and \( q \) are paths, we say \( p \) divides \( q \), denoted \( p | q \) (respectively, \( p \) left divides \( q \) denoted \( p \leftarrow q \), and \( p \) right divides \( q \) denoted \( p \rightarrow q \)) if \( q = rps \) for some paths \( r \) and \( s \) (resp., if \( q = ps \) for some path \( s \), and \( q = rp \) for some path \( r \)). We say a set of nonzero elements \( X \) in \( R \) is tip-reduced if, for \( x, y \in X \), \( \text{tip}(x) \mid \text{tip}(y) \) implies \( x = y \). Since \( > \) is a well-order on \( B \), we have the following result.

**Proposition 2.3** ([11, 12]). If \( X \) is a finite set of uniform elements of \( R \), there is a finite algorithm to produce a tip-reduced set of uniform elements \( Y \) of \( R \) such that the ideal generated by \( X \) equals the ideal generated by \( Y \).

To extend this concept to right projective \( R \)-modules, we must extend the notion of a tip. Let \( I \) be an index set and, for each \( i \), let \( v_i \in Q_0 \). Consider the right projective \( R \)-module \( P = \bigoplus_{i \in I} v_i R \). Let \( C \) be the set of all elements of \( P \) of the form \( x = (x_i)_{i \in I} \) such that, for all but one \( i \), \( x_i = 0 \), and, in that one coordinate, \( x_i \) is a path (with origin vertex \( v_i \)). Then \( C \) is a \( k \)-basis of \( P \). We now define a well-order \( >_P \) on \( C \). First let \( >_I \) be a well-order on \( I \). If \( x = (x_i) \) and \( y = (y_i) \) are elements of \( C \), then \( x >_P y \) if the nonzero path occurring in \( x \) is greater than the nonzero path occurring in \( y \) (using the admissible order \( >_B \)). If these paths are equal, then the coordinate that the nonzero entry occurs in \( x \) is greater than the coordinate that the nonzero entry occurs in \( y \) (using \( >_I \)). The reader may verify that \( >_P \) is a well-order on \( C \). The order \( >_P \) is dependent on the choice of \( >_I \) but in the remainder of this paper, for each set \( I \), we fix some well-order \( >_I \).

Keeping the notation of the previous paragraph, if \( w = (w_i) \) is a nonzero right uniform element of \( P \), we let \( \text{tip}(w) \) be the element of \( C \) such that the nonzero element of \( \text{tip}(w) \) is \( p \) in coordinate \( i^* \) where (i) \( \text{tip}(w_i) = p \), (ii) if \( w_j \neq 0 \) then \( p \geq \text{tip}(w_j) \), and (iii) \( i^* \geq_2 j \) for all \( j \) such that \( \text{tip}(w_j) = p \). We call \( p \) the tip path of \( w \) and denote it by \( \text{tippath}(w) \), and we call \( i^* \) the tip coordinate of \( w \) and denote it by \( \text{tipcoord}(w) \). Letting \( e_i = (e_{ij}) \) be in \( P \) with \( e_{ij} = \delta_{ij} v_i \), we see that \( \text{tip}(w = e_{i^*} p \), where \( p = \text{tippath}(w) \) and \( i^* = \text{tipcoord}(w) \). The proof of the following result is left to the reader after noting that if \( x \in P \) and \( p \in B \) such that \( \text{tip}(x)p = 0 \), then \( \text{tip}(xp) = \text{tip}(x)p \), \( \text{tippath}(xp) = \text{tippath}(x)p \), and \( \text{tipcoord}(xp) = \text{tipcoord}(x) \).

**Proposition 2.4.** Keeping the above notation, suppose \( x \) and \( y \) are right uniform elements of \( P \) and \( p, q \in B \). Then \( \text{tip}(x) >_P \text{tip}(y) \) implies \( \text{tip}(xp) >_P \text{tip}(yp) \), if both \( \text{tip}(x)p \) and \( \text{tip}(y)p \) are nonzero.

If \( x, y \in C \), then we say \( x \) left divides \( y \), written as \( x \mid_l y \), if there is some path \( p \) such that \( xp = y \). Note that \( x \mid_l y \) if and only if \( \text{tippath}(x) \mid_l \text{tippath}(y) \) and \( \text{tipcoord}(x) = \text{tipcoord}(y) \). We say a set of right uniform nonzero elements \( X \) of \( P \) is right tip-reduced if, for each \( x, y \in X \) with \( x \mid_l \text{tip}(y) \) implies \( x = y \).

**Proposition 2.5** ([11, 12]). Let \( P = \bigoplus_{i \in I} v_i R \) be as above. If \( X \) is a set of right uniform elements of \( P \), then there is a right tip-reduced subset \( Y \) of \( P \) of right uniform elements such that the submodule of \( P \) generated by \( X \) equals the submodule generated by \( Y \). Moreover, if \( X \) is a finite set, then there is a finite algorithm to produce such a \( Y \) with \( Y \) finite.
Example 2.7. Let \( Q \) be the quiver

\[
\begin{array}{c}
v_1 \\
\downarrow^a \\
v_2 \\
\downarrow^d
\end{array}
\]

with admissible order \( > \) on \( B \) defined earlier with \( v_1 < v_2 < a < b < c < d \). Let \( P = v_1 R \Pi v_1 R \Pi v_2 R \Pi v_2 R \) with order \( >_P \) induced by \( > \) and \( (v_1, 0, 0, 0) > (0, v_1, 0, 0) > (0, 0, v_2, 0) > (0, 0, 0, v_2) \). Let \( X = \{ f_1, f_2, f_3 \} \) in \( P \), where \( f_1 = (ac, ba + ac, 0, cac) \), \( f_2 = (v_1, bc + ac, d, 0) \) and \( f_3 = (a, b, da + cb, da) \). We right tip-reduce \( X \). We see that \( \text{tip}(f_1) = (0, bac, 0, 0) \), \( \text{tip}(f_2) = (0, bc, 0, 0) \) and \( \text{tip}(f_3) = (0, 0, da, 0) \). Since \( \text{tip}(f_2) \mid \text{tip}(f_1) \), we replace \( f_1 \) by \( f'_1 = f_1 - f_2 ac = (0, 0, -dac, cac) \). Then \( \text{tip}(f'_1) = (0, 0, dac, 0) \) and we see that \( \text{tip}(f_3) \mid \text{tip}(f'_1) \). Hence we replace \( f'_1 \) by \( f''_1 = f'_1 + f_3 c = (ac, bc + ac, d + cac) \). Then \( \text{tip}(f''_1) = (0, 0, 0, dac) \). Hence the set \( X^* = \{ f''_1, f_2, f_3 \} \) is right tip-reduced and we have that (i) \( X \) and \( X^* \) generated the same submodule, say \( P' \) of \( P \) and (ii) \( P' = f''_1 R \Pi f_2 R \Pi f_3 R \).

We note that, given a set subset \( X \) of \( \Pi_{i \in I} v_i R \), there is no unique right tip-reduced \( X' \) generating the same submodule as \( X \).
We need one final definition. Suppose $p$ and $q$ are paths. We say $q$ and $p$ overlap or $q$ overlaps $p$ if there exist paths $r$ and $s$ such that $pr = sq$. We allow $s$ to be a vertex, but $r$ is a path of length at least length 1. An overlap relation may be illustrated in the following way.

![Overlap Diagram]

3. The main step

In this section we present the main step of our construction of a projective resolution of a module over a quotient, $\Lambda$, of a path algebra. Beginning with a presentation of a $\Lambda$-module over the path algebra, we show how to find a presentation, over the path algebra, of the first syzygy of the module over $\Lambda$. This gives rise to an inductive algorithm for finding a projective resolution of a module over $\Lambda$ described in the next section.

Let $I$ be an ideal in a path algebra $R = kQ$, let $\Lambda = R/I$, and let $G = \{g_i^2\}_{i \in I}$ be a uniform, tip-reduced Gröbner basis for the ideal $I$ with respect to some admissible order $>$. Let $M$ be a right $\Lambda$-module. By [11] there exists an $R$-presentation of $M$ of the form

$$0 \rightarrow (\Pi_{i \in T_1} f_i^1 R) \Pi (\Pi_{j \in U_1} f_j^{1'} R) \xrightarrow{H}\Pi_{i \in T_0} f_i^0 R \xrightarrow{\varepsilon} M \rightarrow 0,$$

where

(i) $H^1$ is an inclusion,

(ii) $f_i^0$’s, $f_i^1$’s and $f_j^{1'}$ are right uniform,

(iii) $f_i^1$ is in $\Pi_{j \in U_1} f_j^0 I$ for all $j$ in $U_1$.

(iv) the set $\{f_i^1\}_{i \in T_1} \cup \{f_j^{1'}\}_{j \in U_1}$ is right tip-reduced.

Our goal is to construct sets $\{f_i^2\}_{i \in T_2}$ and $\{f_j^{2'}\}_{j \in U_2}$, such that $\{f_i^2\}_{i \in T_2} \cup \{f_j^{2'}\}_{j \in U_2}$ is a right uniform and right tip-reduced set in $\Pi_{i \in T_1} f_i^0 R$, the set $\{f_i^2\}_{i \in U_2}$ is in $\Pi_{i \in T_1} f_i^1 I$, and

$$0 \rightarrow (\Pi_{i \in T_2} f_i^2 R) \Pi (\Pi_{j \in U_2} f_j^{2'} R) \xrightarrow{H^2}\Pi_{i \in T_1} f_i^1 R \rightarrow \Omega^1_{\Lambda}(M) \rightarrow 0,$$

is an exact sequence of right $R$-modules, where $H^2$ is an inclusion map and $\Omega^1_{\Lambda}(M)$ is the kernel of $\Pi_{i \in T_2} f_i^2 R/ \Pi_{i \in T_0} f_i^0 I \rightarrow M$.

Recall from [16] that we want to construct the $f_i^2$’s and the $f_j^{2'}$’s so that

$$(\Pi_{i \in T_1} f_i^1 R) \cap (\Pi_{i \in T_0} f_i^0 I) = (\Pi_{i \in T_2} f_i^2 R) \Pi (\Pi_{i \in U_2} f_i^{2'} R).$$

This equality can be seen from the following short exact sequence of right $R$-modules

$$0 \rightarrow (\Pi_{i \in T_1} f_i^1 R) \cap (\Pi_{i \in T_0} f_i^0 I) \rightarrow \Pi_{i \in T_1} f_i^1 R \rightarrow \Omega^1_{\Lambda}(M) \rightarrow 0$$

The existence of this exact sequence is obtained by considering the exact sequence of right $R$-modules given by the left hand column of the following commutative diagram:

![Diagram]
To construct the \( f_i^2 \)'s, we need some preliminary definitions. Let \( p \) be a path in \( Q \) of length at least one. We define \( X(p) \) to be the set of paths \( q \) that satisfy the following conditions:

1. \( p \parallel q \).
2. There is some \( g_i^2 \in \mathcal{G} \) such that \( \text{tip}(g_i^2) \parallel_r q \).
3. If there are paths \( r \) and \( s \) and \( g_i^2 \in \mathcal{G} \) such that \( q = r \text{tip}(g_i^2)s \) then \( s \) is a vertex (and hence \( i = j \) since \( \{g_i^2\}_{i \in I} \) is tip-reduced).

The following figures illustrate (1) and (2) in the definition of \( X(p) \):

\[
\text{tip}(g_i^2) \quad \text{or} \quad \text{tip}(g_i^2) \quad \text{or} \quad \text{tip}(g_i^2)
\]

where \( q \) is the path indicated by the dashed lines.

If \( q \in X(p) \) and \( q = q' \text{tip}(g_i^2) \), call \( g_i^2 \) the end relation of \( q \). We break \( X(p) \) into two disjoint sets. Let

\[
O(p) = \{ q \in X(p) \mid \text{the tip of the end relation of } q \text{ and } p \text{ overlap} \}
\]

and

\[
N(p) = X(p) \setminus O(p).
\]

Elements \( q \) in \( O(p) \) can be describe by the following diagram

\[
\text{tip}(g_i^2)
\]

where \( z \) is a path of length at least one (in particular we allow \( z = p \)). Again, \( q \) is the path indicated by the dashed line. Elements \( q \) in \( N(p) \) are illustrated by the following diagram

\[
\text{tip}(g_i^2)
\]

where \( z \) is a path of length at least zero.
We can now define \( T_2 \), the index set for the \( f^2_i \)'s, and \( U_2 \), the index set for the \( f^2_i \)'s. Let \( T_2 = \{(i, q) \mid i \in T_1 \text{ and } q \in O(\text{tippath}(f^1_i)) \} \) and \( U_2 = \{(i, q) \mid i \in T_1 \text{ and } q \in N(\text{tippath}(f^1_i)) \} \). We remark here that \( T_2 \) and \( U_2 \) are countable sets, since \( T_1 \) and \( \mathcal{B} \) are countable sets. To define the \( f^2_i \), suppose that \( s = (i, q) \in T_2 \) and that \( \text{tipcoord}(f^1_i) = i^* \). From the definition of \( T_2 \), we see that \( q = \text{tippath}(f^1_i)p = q' \text{ tip}(g^2_j) \) for some paths \( p \) and \( q' \) and \( g^2_j \in \mathcal{G} \) is the end relation of \( q \). Consider \( f^1_i)p - \varepsilon_i \text{ c}(q'g^2_j) \), where \( \varepsilon_i \) is defined as in Section 2 and \( c = \frac{\text{coefficient of tip}(f^1_i)}{\text{coefficient of tip}(g^2_j)} \) in \( k \). We see that \( f^1_i)p - \varepsilon_i \text{ c}(q'g^2_j) \) is right uniform. Note that \( \varepsilon_i \text{ c}(q'g^2_j) \) is in \( \Pi \in T_0 f^0_R \) and has only one non-zero component, namely \( q'g^2_j \) in the same component as \( \text{tipcoord}(f^1_i) \).

Clearly \( \pi(f^1_i)p - \varepsilon_i \text{ c}(q'g^2_j) = 0 \), so that \( f^1_i)p - \varepsilon_i \text{ c}(q'g^2_j) \) is in \( (\Pi \in T_1 f^1_i) \Pi (\Pi \in U_1 f^1_i) \). Hence,

\[
f^1_i)p - \varepsilon_i \text{ c}(q'g^2_j) = \sum_{j \in T_1} f^1_i r_j + \sum_{j \in U_1} f^1_i s_j
\]

for some \( r_j \) and \( s_j \) in \( R \). By the unicity of the sums, there is a vertex \( v \) such that \( f^1_i)v = f^1_i r_j \) and \( \varepsilon_i \text{ c}(q'g^2_j) = \varepsilon_i \text{ c}(q'g^2_j) \), \( f^1_i r_j = f^1_i s_j \) for all \( j \) in \( T_1 \) and \( f^1_i r_j = f^1_i s_j \) for all \( j \) in \( U_1 \). Since \( \{f^1_i\} \cup \{f^1_i\} \in U_1 \) is a right tip-reduced right Gröbner basis for \( \Pi \in T_0 f^0_R \) \( \Pi (\Pi \in U_1 f^1_i) \) and since \( \text{tip}(f^1_i)p = \varepsilon_i \text{ c}(q'g^2_j) \), we see that \( \text{tip}(f^1_i)p > \text{tip}(f^1_i)r_j \) for all \( j \) in \( T_1 \). Let \( f^2 = f^1_i r_j - \sum_{j \in T_1} f^1_i r_j \). Then, since \( \varepsilon_i \text{ c}(q'g^2_j) \) and each \( f^1_i \) is in \( \Pi \in T_0 f^0_R \), the element \( f^2_i \) is in \( \Pi \in T_1 f^1_i \Pi (\Pi \in U_1 f^1_i) \). Moreover, we see that \( f^2_i \) is right uniform. Thus, for each \( s \in T_2 \), we have constructed an \( f^2_i \). Note that \( \text{tip}(f^2_i) = \text{tip}(f^1_i)p \).

We now construct the \( f^2_i \)'s. Let \( s = (i, q) \in U_2 \). From the definition of \( U_2 \), there is a path \( z \) and a \( g^2_j \) in \( \mathcal{G} \) such that \( q = \text{tippath}(f^1_i)z \text{ tip}(g^2_j) \). Define \( f^2_i = f^1_i z g^2_j \). We have that each \( f^2_i \) is in \( \Pi \in T_1 f^1_i \). It is clear that \( f^2_i \in (\Pi \in T_1 f^1_i) \Pi (\Pi \in U_1 f^1_i) \) and that \( \text{tip}(f^2_i) = \text{tip}(f^1_i)z \text{ tip}(g^2_j) \).

The next result proves the main properties of the \( f^1_i \)'s and the \( f^2_i \)'s.

**Theorem 3.1.**

\[
(\Pi \in T_1 f^1_i) \cap (\Pi \in T_0 f^0) = (\Pi \in T_0 f^1_i) \Pi (\Pi \in U_1 f^1_i)
\]

and

\[
\{f^2_i\} \cup \{f^2_i\} \in U_2
\]

is right uniform and right tip-reduced and hence a right uniform and right tip-reduced right Gröbner basis for \( (\Pi \in T_1 f^1_i) \cap (\Pi \in T_0 f^0) \). Furthermore, each \( f^2_i \) is in \( \Pi \in T_1 f^1_i \).

**Proof.** We have seen that the \( f^2_i \)'s and the \( f^2_i \)'s are in \( (\Pi \in T_1 f^1_i) \cap (\Pi \in T_0 f^0) \), that they are right uniform elements, and that each \( f^2_i \) is in \( \Pi \in T_1 f^1_i \).

We note that if \( s \in T_2 \) with \( s = (i, q) \) and \( g^2_j \) is the end relation of \( q \), then there are paths \( p \) and \( q' \) such that \( \text{tippath}(f^1_i)p = q' \text{ tip}(g^2_j) \). We have seen that \( \text{tip}(f^1_i)p = \text{tip}(f^1_i)p \). Let \( i^* = \text{tipcoord}(f^1_i) \). Then \( \text{tip}(f^1_i)p \) occurs in the \( i^* \)-th component of \( f^2 \) viewed as an element of \( \Pi \in T_0 f^0 \). On the other hand, if \( s = (i, q) \in U_2 \) with \( q \) having end relation \( g^2_j \), then there is path \( z \) such that \( q = \text{tip}(f^1_i)z \text{ tip}(g^2_j) \). We see that \( \text{tip}(f^2_i) = \text{tip}(f^1_i)z \text{ tip}(g^2_j) \) in the \( \text{tip}(f^1_i) \) coordinate of \( \Pi \in T_0 f^0 \).
Next we show that $\{f^2_1\}_{s \in T_2} \cup \{f^2_2\}_{s \in U_2}$ is right tip-reduced. Suppose not. Since $\{f^2_1\}_{s \in T_1}$ is right tip-reduced, it is clear that $\{f^2_2\}_{s \in T_2}$ and $\{f^2_2\}_{s \in U_2}$ are both right tip-reduced sets. Suppose that, for some $s \in T_2$ and $s' \in U_2$, $\text{tip}(f^2_2) = \text{tip}(f^2_2)$ and $\text{tip}(f^2_2)$ left divides $\text{tip}(f^2_2)$. Let $s = (i, q)$ and $s' = (i', q')$. We see that either $\text{tip}(f^2_2)$ left divides $\text{tip}(f^2_2)$ or vise versa. In either case, since $\{f^2_2\}$ is right tip-reduced, we conclude that $i = i'$. But then, since $\text{tip}(f^2_2)$ left divides $\text{tip}(f^2_2)$, we conclude that the end relation of $q$ appears before the end relation of $q'$ which contradicts property (3) of the definition of $X(q')$. Hence $\text{tip}(f^2_2)$ does not left divides $\text{tip}(f^2_2)$. A similar argument shows that $\text{tip}(f^2_2)$ does not left divides $\text{tip}(f^2_2)$. We conclude that $\{f^2_2\}_{s \in T_2} \cup \{f^2_2\}_{s \in U_2}$ is right tip-reduced.

Since $\{f^2_2\}_{s \in T_2} \cup \{f^2_2\}_{s \in U_2}$ is right uniform right tip-reduced, the submodule generated by this set can be written as $(\Pi_{s \in T_2} f^2_2 R) \cap (\Pi_{s \in U_2} f^2_2 R)$ by Proposition 2.6. It remains to show $\{f^2_2\}_{s \in T_2} \cup \{f^2_2\}_{s \in U_2}$ generates $(\Pi_{s \in T_2} f^1_1 R) \cap (\Pi_{s \in T_2} f^0_1 I)$. We have already proven that

$$(\Pi_{s \in T_2} f^2_2 R) \cap (\Pi_{s \in U_2} f^2_2 R) \subseteq (\Pi_{s \in T_2} f^1_1 R) \cap (\Pi_{s \in T_2} f^0_1 I).$$

Suppose that $\{f^2_2\}_{s \in T_2} \cup \{f^2_2\}_{s \in U_2}$ does not generate $(\Pi_{s \in T_2} f^1_1 R) \cap (\Pi_{s \in T_2} f^0_1 I)$. Let $x \in (\Pi_{s \in T_2} f^1_1 R) \cap (\Pi_{s \in T_2} f^0_1 I)$ such that $\text{tip}(x)$ is minimal with respect to the property that $x \notin (\Pi_{s \in T_2} f^2_2 R) \cap (\Pi_{s \in U_2} f^2_2 R)$. Since $x$ is in $\Pi_{s \in T_2} f^1_1 R$ and since $f^1_1$'s are tip-reduced, it follows that $\text{tip}(x) = \text{tip}(f^1_1)p$ for some $i$ in $T_1$ and some path $p$. On the other hand, $x$ is in $\Pi_{s \in T_2} f^0_1 I$, hence $\text{tip}(x) = \epsilon_i q \text{tip}(g^2_2)z$ for some $j$ in $T_0$ and some paths $q$ and $z$. Thus $\text{tip}(x) = \text{tip}(f^1_1)p = q \text{tip}(g^2_2)z$. For all possible $g^2_2$'s such that $\text{tip}(x) = \epsilon_i q \text{tip}(g^2_2)z$, choose $j$ such that $q$ has minimal length. Either $\text{tip}(g^2_2)$ overlaps $\text{tip}(f^1_1)$ or not.

If they do overlap, then there exists an $t$ in $T_2$ such that $\text{tip}(f^2_2)z = \text{tip}(x)$. Since the tip of $x - cf^2_2z$ is smaller than $\text{tip}(x)$ for some $c$ in $k$, the difference $x - cf^2_2z$ is in $(\Pi_{s \in T_2} f^2_2 R) \cap (\Pi_{s \in U_2} f^2_2 R)$. This is a contradiction.

If $\text{tip}(g^2_2)$ does not overlap $\text{tip}(f^1_1)$, then there is some $t$ in $U_2$ such that $\text{tip}(f^2_2)z = \text{tip}(x)$ for some path $z$ in $Q$. A similar argument as above leads to a contradiction. This completes the proof. 

**Example 3.2.** We continue Example 2.2. First we use the order $>_1$. Let $\Lambda = kQ/I$ where $I$ is generated by $ab - cd$ and $be$. Let $M = v_1 \Lambda/\tau$, where $\tau$ is the Jacobson radical of $\Lambda$. It is immediate that, as a right $R$-module, has a projective presentation

$$0 \rightarrow aR \oplus cR \xrightarrow{H} v_1 R \rightarrow M \rightarrow 0,$$

where $H^1(a) = a$ and $H^1(c) = c$. Recall that $G = \{ab - cd, be, cde\}$. Let $g^2_2 = ab - cd$, $g^2_2 = be$, and $g^2_2 = cde$. Then $\text{tip}(g^2_2) = ab$, $\text{tip}(g^2_2) = be$, and $\text{tip}(g^2_2) = cde$. Let $f^0_1 = v_1$, $f^1_1 = a$, and $f^2_1 = c$. We see that $T_1 = \{1, 2\}$. We find $T_2 = \{(i, q) \mid i \in T_1 \text{ and } q \in Q(\text{tip}(f^1_1))\}$ and $U_2 = \{(i, q) \mid i \in T_1 \text{ and } q \in N(\text{tip}(f^1_1))\}$. First note that $X(\text{tip}(f^1_1)) = X(a) = \{ab\} = O(a)$ and $X(\text{tip}(f^1_1)) = X(c) = \{cde\} = O(c)$. Hence $T_2 = \{(1, ab), (2, cde)\}$ and $U_2 = \emptyset$. For $(1, ab)$ we calculate $f^1_{1b} - v_1 g^2_2 = ab - v_1(ab - cd) = cd = f^2_{1d}$. Therefore $f^2_{1(ab)} = f^1_{1b} - f^2_{1d}$ =
\[ f_{(2, cde)}^2 = f_{12}^2 \text{cd} = cde. \] Thus we have
\[ f_{(1, ab)}^2 R II f_{(2, cde)}^2 R \xrightarrow{H^2} f_{1}^1 R II f_{2}^2 R \]
with \( H^2(f_{(1, ab)}^2) = f_{12}^1 b - f_{3}^2 d \) and \( H^2(f_{(2, cde)}^2) = f_{12}^3 d \).

If we change the order to >2, we still have \( f_{12}^1 = v_1, f_{2}^2 = a \) and \( f_{3}^3 = c \). But now \( G = \{ab - cd, be\} \) with \( \text{tip}(ab - cd) = cd \) and \( \text{tip}(be) = be \). The reader may check that \( T_2 = \{(2, cd)\}, U_2 = 0, f_{(2, cd)}^2 = ab - cd \) and
\[ f_{(2, cd)}^2 R \xrightarrow{H^2} f_{1}^1 R II f_{2}^2 R, \]
where \( H^2(f_{(2, cd)}^2) = f_{12}^1 b - f_{3}^2 d. \)

4. Constructing resolutions

This section is devoted to constructing a projective resolution of a module over a quotient of a path algebra using the main step of the previous section.

Let \( M \) be a right \( \Lambda \)-module and suppose that we have an \( R \)-presentation of \( M \) of the form
\[ (1) \quad 0 \rightarrow (\Pi_{i \in T_1} f_{1}^i R) II (\Pi_{j \in U_1} f_{1}^j R) \xrightarrow{H^1} (\Pi_{i \in T_2} f_{1}^i R) \rightarrow M \rightarrow 0, \]
where the \( f_{1}^i \)'s, \( f_{1}^j \)'s and \( f_{1}^{1'} \)'s are right uniform elements, each \( f_{1}^i \) is in \( \Pi_{i \in T_2} f_{1}^i R \), and \( \{f_{1}^i\}_{i \in T_1} \cup \{f_{1}^{1'}\}_{i \in U_1} \) is right tip-reduced.

In the previous section we showed how to construct \( f_{1}^{2'} \)'s and \( f_{1}^{3'} \)'s such that
\[ (\Pi_{i \in T_1} f_{1}^i R) \cap (\Pi_{i \in T_2} f_{1}^i R) = (\Pi_{i \in T_1} f_{1}^{2'} R) II (\Pi_{i \in U_1} f_{1}^{3'} R) \]
and
\[ 0 \rightarrow (\Pi_{i \in T_1} f_{1}^{2'} R) II (\Pi_{j \in U_2} f_{1}^{3'} R) \xrightarrow{H^2} (\Pi_{i \in T_2} f_{1}^i R) \rightarrow \Omega_{\Lambda}^1(M) \rightarrow 0 \]
is an exact sequence of right \( R \)-modules, where \( \Omega_{\Lambda}^1(M) \) is \( \text{Ker}(\Pi_{i=1}^n f_{1}^0 R/\Pi_{i=1}^n f_{1}^0 I \rightarrow M) \).

From our construction, (i) \( H^2 \) is an inclusion map, (ii) the elements \( f_{1}^{2'} \)'s and \( f_{1}^{3'} \)'s are right uniform, (iii) each \( f_{1}^{2'} \) is in \( \Pi_{i \in T_2} f_{1}^i R \), and (iv) the set \( \{f_{1}^{2'}\}_{i \in T_2} \cup \{f_{1}^{3'}\}_{i \in U_2} \) is right tip-reduced. Replacing \( M \) by \( \Omega_{\Lambda}^1(M) \), we may view \( f_{1}^{2'} \) as \( f_{1}^0 \)'s and the \( f_{1}^{3'} \) as \( f_{1}^1 \)'s and, applying our main step, we may construct elements \( f_{1}^{4'} \)'s and \( f_{1}^{5'} \)'s in \( \Pi_{i=1}^{n-1} f_{1}^{2'} R \) so that
\[ (\Pi_{i \in T_3} f_{1}^{2'} R) \cap (\Pi_{i \in T_1} f_{1}^i R) = (\Pi_{i \in T_3} f_{1}^{4'} R) II (\Pi_{i \in U_3} f_{1}^{5'} R) \]
and
\[ 0 \rightarrow (\Pi_{i \in T_3} f_{1}^{4'} R) II (\Pi_{j \in U_3} f_{1}^{5'} R) \xrightarrow{H^3} (\Pi_{i \in T_2} f_{1}^i R) \rightarrow \Omega_{\Lambda}^2(M) \rightarrow 0 \]
is an exact sequence of right \( R \)-modules with \( \Omega_{\Lambda}^2(M) \) being \( \text{Ker}(\Pi_{i=1}^{n-1} f_{1}^0 R/\Pi_{i=1}^{n-1} f_{1}^0 I \rightarrow \Omega_{\Lambda}^1(M)) \), (i) \( H^3 \) is an inclusion map, (ii) the elements \( f_{1}^{4'} \)'s and \( f_{1}^{5'} \)'s are right uniform, (iii) each \( f_{1}^{4'} \) is in \( \Pi_{i \in T_2} f_{1}^i R \), and (iv) the set \( \{f_{1}^{4'}\}_{i \in T_3} \cup \{f_{1}^{5'}\}_{i \in U_3} \) is right tip-reduced.

Repeating the above procedure, we obtain, for \( n \geq 2 \) elements \( f_{1}^{n} \)'s and \( f_{1}^{n'} \)'s in \( \Pi_{i=1}^{n-1} f_{1}^{n-1} R \) so that
\[ (\Pi_{i \in T_{n-1}} f_{1}^{n-1} R) \cap (\Pi_{i \in T_{n-2}} f_{1}^{n-2} R) = (\Pi_{i \in T_{n}} f_{1}^{n} R) II (\Pi_{i \in U_{n}} f_{1}^{n'} R) \]
and
\[ 0 \rightarrow (\Pi_{i \in T_{n-1}} f_{1}^{n} R) II (\Pi_{j \in U_{n}} f_{1}^{n'} R) \xrightarrow{H^n} (\Pi_{i \in T_{n-1}} f_{1}^{n-1} R) \rightarrow \Omega_{\Lambda}^n(M) \rightarrow 0 \]
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is an exact sequence of right \( R \)-modules with \( \Omega_\Lambda^{n-1}(M) \) being \( \text{Ker}(\cup_{i=1}^{n-2} f_i f_{i+1}^{-1} : f_i^{-1} I \to \Omega_\Lambda^{n-2}(M)) \). (i) \( H^n \) is an inclusion map, (ii) the elements \( f_i^n \)'s and \( f_i^{n+1} \)'s are right uniform, (iii) each \( f_i^n \)' is in \( \Pi_{i \in T_n-1} f_{i+1}^{n-1} I \), and (iv) the set \( \{ f_i^n \}_{i \in T_n} \cup \{ f_i^{n+1} \}_{i \in T_n} \) is right tip-reduced.

Since each \( f_i^n \) is in \( \Pi_{i \in T_n-1} f_{i+1}^{n-1} R \), we may write

\[
\begin{align*}
  f_i^n &= \sum_{t \in T_n-1} f_{i+1}^{n-1} h_i^{n-1,n} \\
\end{align*}
\]

for some elements \( h_i^{n-1,n} \) in \( R \). Let

\[
L^n = \Pi_{i \in T_n} f_i^n R / \Pi_{i \in T_n} f_i I.
\]

Let \( v_i^n \) be the vertex in \( Q \) such that \( f_i^n v_i^n = f_i^n \). We see that \( L^n \) is isomorphic to \( \Pi_{i \in T_n} v_i^n \Lambda \) for all \( n \geq 0 \), hence it is a projective \( \Lambda \)-module. Define \( e_{n+1} : L^{n+1} \to L^n \) by \( e_{n+1}(f_i^{n+1}) = f_i^n h_i^{n,n+1} \) in the component of \( L^n \) corresponding to \( \overline{f_i} \). Now applying Theorem 2.1 we conclude that the resolution \((L, e)\)

\[
\ldots \xrightarrow{e_n} L^n \xrightarrow{e_{n-1}} \ldots \xrightarrow{e_1} L^0 \to M \to 0
\]

is a projective \( \Lambda \)-resolution of \( M \). We call \((L, e)\) the resolution associated to \((1)\).

**Example 4.1.** We now continue Example 3.2. Under the ordering \( \succ \), we have \( f_1^3 = v_1, f_1^1 = a, f_1^2 = c, f_2^3 = f_1^3 b - f_2^2 d, \) and \( f_2^2 = f_2^2 d e \). We find the \( f_i^3 \)'s.

Write \( T_2 \) as \( \{1, 2\} \). Then \( \text{tip}(f_1^2) = (0, b) \) and \( \text{tip}(f_2^2) = (0, de) \) in \( f_1^1 R \cup f_2^2 R \).

Hence \( X(\text{tippath}(f_1^2)) = X(b) = \{be\} \) and \( X(\text{tippath}(f_2^2)) = X(de) = \emptyset \). Thus \( T_3 = \{(1, be)\} \) and \( U_3 = \emptyset \). For \( (1, be) \), we calculate

\[
\begin{align*}
  f_1^2 b e - f_1^4 g_2^2 &= f_1^4 b e - f_2^2 d e - f_1^4 b e = -f_1^4 d e = -f_2^2.
\end{align*}
\]

Hence \( f_1^3 = f_1^2 b + f_2^2 \) and we get

\[
\begin{align*}
  f_1^3 R &\xrightarrow{H^3} f_2^2 R \cup f_2^2 R,
\end{align*}
\]

where \( H^3(f_1^3) = (e, v_3) \). The reader may check that \( T_4 = \emptyset = U_4 \). The induced resolution for \( M \) over \( \Lambda \) by our algorithm is

\[
\begin{align*}
  0 \to v_5 \Lambda \xrightarrow{(e)} v_4 \Lambda \cup v_5 \Lambda \xrightarrow{(b, d, e)} v_2 \Lambda \cup v_3 \Lambda \xrightarrow{(a, c)} v_1 \Lambda \to M \to 0,
\end{align*}
\]

since, for example, \( f_1^3 R / f_1^3 I \simeq v_5 \Lambda \) and \( f_2^2 R / f_2^2 I \simeq v_2 \Lambda \).

For the order \( \succ \) the reader may check that \( T_3 = \emptyset = U_3 \) and the induced resolution for \( M \) over \( \Lambda \) is

\[
\begin{align*}
  0 \to v_4 \Lambda \xrightarrow{(b, d, e)} v_2 \Lambda \cup v_3 \Lambda \xrightarrow{(a, c)} v_1 \Lambda \to M \to 0.
\end{align*}
\]

We note that the resolution for the ordering \( \succ \) is minimal whereas the resolution for the ordering \( \succ \) is not minimal. This example shows that the constructed resolution is dependent on the choice of the admissible order, since both the Gröbner basis for \( I \) and the tips are order dependent. An algorithmic method for minimizing a non-minimal projective resolution of a finite dimensional module over a finite dimensional quotient of a path algebra, is given in [13, 16].

In the next section we discuss some algorithmic aspects of the above construction. We mention that a special case of the results can be found in [1, 2], where it is shown that simple modules of the form \( vR/J \), where \( J \) is the ideal in \( R \) generated by the arrows of \( Q \) and \( v \) is a vertex.
We end this section by providing sufficient conditions for the constructed resolution to have the property that each $L^n$ is finitely generated.

**Proposition 4.2.** Let $\Lambda = R/I$, where $R = kQ$ for some quiver $Q$. Suppose that there is an admissible order $>\triangleright$ on $\mathcal{B}$ such that the Gröbner basis for $I$ with respect to $>\triangleright$ is finite. Let $M$ be a right $\Lambda$-module, which, as a right $R$-module, has a presentation

$$0 \to (\Pi_{i \in T_1} f_i^1 R) \Pi (\Pi_{j \in U_1} f_j^1 R) \xrightarrow{H^1} \Pi_{i \in T_0} f_i^0 R \xrightarrow{\pi} M \to 0$$

with $T_0$ and $T_1$ finite sets, where $H^1$ is an inclusion, $\{f_i^1\}_{i \in T_1}$ and $\{f_j^1\}_{j \in U_1}$ are right uniform and right tip-reduced sets, and $f_i^{1'}$’s are in $\Pi_{i \in T_0} f_i^0 R$. Then there is a projective resolution $(L_n,e)$ of $M$ as a $\Lambda$-module associated to (2) with the property that each $L^n$ is finitely generated.

**Proof.** Let $G$ be a finite Gröbner basis of uniform elements for $I$. Since $T_1$ and $G$ are finite sets, it follows that for each $f_i^1$ there are only a finite number of $g_j^2$ such that $\text{tip}(f_i^1)$ overlaps $\text{tip}(g_j^2)$. It follows that $T_2$ is also a finite set. Inductively we conclude that each $T_n$ is a finite set for all $n \geq 0$. \qed

Note that in the previous result the set $U_1$ can be infinite. The next result shows that all finite dimensional right $R$-modules can be chosen to finite in (2).

**Proposition 4.3.** Let $\Lambda = R/I$, where $R = kQ$ for some quiver $Q$. Let $M$ be a right $\Lambda$-module which, as a right $R$-module, is finitely presented. Suppose that there is an admissible order $>\triangleright$ on $\mathcal{B}$ such that the Gröbner basis for $I$ with respect to $>\triangleright$ is finite. Then there is a presentation of the form (1) such that the resolution $(L_n,e)$ associated to (1) has the property that each $L^n$ is a finitely generated $\Lambda$-module. The claim follows from this.

**Proof.** Every projective right $R$-module is of the form $\Pi_{i \in I} v_i R$, where $I$ is an index set and each $v_i$ is a vertex in $Q_0$. Since $R$ is a hereditary algebra and since $M$ is a finitely presented right $R$-module, we have a presentation of the form

$$0 \to \Pi_{i=1}^{n_1} w_i R \xrightarrow{\varphi} \Pi_{i=1}^{n_0} v_i R \to M \to 0,$$

where each $v_i$ and $w_i$ are vertices in $Q_0$. Let $h^i = \varphi(w_i)$, which is a right uniform element for all $i = 1, \ldots, n_1$. Right tip-reduce the set $\{h^1, \ldots, h^{n_1}\}$, and break the elements into two sets $\{f_1^1, \ldots, f_{t_1}^1\}$ and $\{f_1^{1'}, \ldots, f_{u_1}^{1'}\}$ so that each $f_j^{1'}$ is in $\Pi_{i=1}^{n_0} v_i R$. Finally set $t_0 = n_0$ and $f_j^0 = v_i$ for $i = 1, \ldots, t_0$. Thus we obtain the following presentation

$$0 \to (\Pi_{i \in T_1} f_i^1 R) \Pi (\Pi_{j \in U_1} f_j^1 R) \xrightarrow{H^1} \Pi_{i \in T_0} f_i^0 R \xrightarrow{\pi} M \to 0$$

of $M$ as a right $R$-module, where $f_i^0$’s, $f_i^1$’s, and $f_i^{1'}$’s are right uniform elements, both $T_0$ and $T_1$ are finite sets, and the set $\{f_i^1\}_{i \in T_1} \cup \{f_j^{1'}\}_{j \in U_1}$ is right tip-reduced.

We now apply Proposition 4.2 to obtain our desired result. \qed

The previous result raises the question: Which right $\Lambda$-modules are finitely presented as right $R$-modules? The next result shows that all finite dimensional right $\Lambda$-modules are finitely presented as right $R$-modules.
Proposition 4.4. Let $\Lambda = R/I$, where $R = kQ$ for some quiver $Q$. Let $M$ be a finite dimensional right $\Lambda$-module. Then $M$, as a right $R$-module, is finitely presented. Furthermore, if $I$ has a finite Gröbner basis, then there is a presentation of the form (1) such that the resolution $(L_n, e)$ associated to (1) has the property that each $L_n$ is a finitely generated $\Lambda$-module.

Proof. Let $M$ be a finite dimensional right $\Lambda$-module. It is enough to show that $M$ is a finitely presented right $R$-module. Let $A$ be the right annihilator of $M$ as a right $R$-module, and let $\Gamma = kQ/A$. Then the $k$-algebra $\Gamma$ is finite dimensional, and $M$ is a finitely generated right $\Gamma$-module. Let $\{m_i\}_{i=1}^{t_0}$ be a finite set of right uniform generators for $M$ as a $\Gamma$-module, and suppose that $\{f_i\}_{i=1}^{t_0}$ is a set of vertices in $Q$ such that $m_i f_i^0 = m_i$ for all $i = 1, \ldots, t_0$. Since $\Gamma$ and $M$ are finite dimensional, there is a projective $\Gamma$-presentation

$$\Pi_{n=1}^{d} w_n \gamma \rightarrow \Pi_{i=1}^{t_0} f_i^0 \Gamma \rightarrow M \rightarrow 0,$$

for some vertices $w_i$ in $Q$ and for some positive integer $d$. We also have an exact sequence of right $R$-modules

$$0 \rightarrow K \rightarrow \Pi_{i=1}^{t_0} f_i^0 R \rightarrow M \rightarrow 0.$$

It can be seen that $(\Pi_{i=1}^{d} w_i R)|_i (\Pi_{i=1}^{t_0} f_i^0 A)$ maps onto $K$. To show that $K$ is finitely generated as an $R$-module, we need to show that $\Pi_{i=1}^{t_0} f_i^0 A$ is finitely generated. By [12] $A$ has a finite Gröbner basis with respect to any admissible order. From [11, Proposition 7.1] and the fact that $\Gamma$ is finite dimensional, it follows that $\Pi_{i=1}^{t_0} f_i^0 A$ is finitely generated. This shows that $M$ is a finitely presented right $R$-module. The final statement follows from Proposition 4.3.

For finite dimensional algebras $\Lambda = kQ/I$, we have the following consequence, since $I$ has a finite Gröbner basis with respect to any admissible order [12].

Corollary 4.5. Let $\Lambda = kQ/I$ be a finite dimensional algebra. Then any finitely generated right $\Lambda$-module has a projective $\Lambda$-resolution $(L_n, e)$ which can be constructed algorithmically such that each $L_n$ is finitely generated.

5. Algorithmic aspects

In this section we discuss computational questions related to the construction presented in the previous sections. Our goal is to clarify when we have actual (finite) algorithms for constructing projective resolutions of modules over quotients of path algebras and to provide an overview of the algorithms needed. More precisely, let $Q$ be a quiver, $I$ an ideal in $R = kQ$, and $\Lambda = R/I$. Suppose $M$ is a right $\Lambda$-module. We wish to find conditions so that, given a positive integer $N$, there is an algorithm based on the construction in the earlier sections whose output is a projective $\Lambda$-resolution

$$L^N \xrightarrow{e^N} L^{N-1} \rightarrow \cdots \rightarrow L^0 \rightarrow M \rightarrow 0.$$

We also discuss the input for such an algorithm.

There are two conditions needed; one on the ideal $I$ and one on the module $M$. We begin with the condition on the ideal $I$. Let $>$ be an admissible order on $B$, and $G$ a tip-reduced Gröbner basis for $I$ with respect to $>$ consisting of uniform elements. The construction of $G$, given a finite set of generators of $I$ is discussed in [12]. For there to be a finite algorithm for constructing $G$, we must assume that
$G$ is finite. As noted earlier, if $R/I$ is finite dimensional over $k$, then there is finite tip-reduced Gröbner basis for $I$ consisting of uniform elements. We actually need something stronger than the existence of a finite Gröbner basis.

Let

$$rt \mathcal{G} = \{ pg \mid p \notin \text{tip}(I), g \in \mathcal{G}, \text{if } r \text{tip}(g')s = p \text{tip}(g),$$

for some $g' \in \mathcal{G}, r, s \in B$, then $s \in Q_0$ and $g = g'$

It is shown in [11] that $rt \mathcal{G}$ is a right uniform, right tip-reduced, right Gröbner basis for $I$. If $rt \mathcal{G}$ is infinite, we will not in general have a finitely terminating algorithm to right tip-reduce sets needed in the construction. For this reason, we need to assume that $rt \mathcal{G}$ is finite. Of course, $rt \mathcal{G}$ being finite implies that $\mathcal{G}$ is a finite set. We hasten to add that $rt \mathcal{G}$ is finite if $R/I$ is finite dimensional over $k$, with $|rt \mathcal{G}| \leq \dim_k(\Lambda) \cdot |\mathcal{G}|$.

We now consider the class of modules for which we have an algorithm to construct a projective resolution. Let $\Lambda = R/I$ and let $M$ be a right $\Lambda$-module. Since $\sum_{i \in Q_0} v = 1$, we see that $M$ has a projective presentation as a right $\Lambda$-module of the form

$$(3) \quad \Pi_{i \in \mathcal{I}w_i \Lambda} \cong \Pi_{i \in \mathcal{I}v_i \Lambda} \to M \to 0,$$

where $\mathcal{I}$ and $\mathcal{I}'$ are index sets and each $v_i$ and $w_i$ are vertices. The assumption on $M$ that we need is that the index sets $\mathcal{I}$ and $\mathcal{I}'$ in (3) are finite. The next result is fundamental to the existence of an algorithm.

**Proposition 5.1.** Let $\Lambda = R/I$ where $R = kQ$ for some quiver $Q$. Assume that $\mathcal{G}$ is a right tip-reduced, right Gröbner basis for $I$ consisting of uniform elements and assume further that $rt \mathcal{G}$ is finite. Let $M$ be a right $\Lambda$-module such that $M$ has a projective presentation as a $\Lambda$-module of the form (3) with $\mathcal{I}$ and $\mathcal{I}'$ finite sets. Then, there is an algorithm, whose input is (3) and output is nonnegative integers $t_0$, $t_1$, and $u_1$ and a projective presentation of $M$ as an $R$-module

$$0 \to (\Pi_{i=1}^{t_1} f_i^0 R) \Pi (\Pi_{i=1}^{u_1} f_i^1 R) \xrightarrow{H^1} \Pi_{i=1}^{t_0} f_i^0 R \to M \to 0,$$

where

1. $H^1$ is an inclusion map,
2. the $f_i^0$’s, $f_i^1$’s and $f_i^1$’s are right uniform elements,
3. $f_i^{1'} \in \Pi_{i=1}^{t_0} f_i^0 I$, for all $i = 1, \ldots, u_1$, and
4. $\{ f_i^{1'} \}_{i=1}^{u_1} \cup \{ f_i^1 \}_{i=1}^{t_1}$ is right tip-reduced.

**Proof.** By hypothesis, there exist nonnegative integers $t_0$ and $d$, vertices $w_i$, for $i = 1, \ldots, d$, and vertices $v_j$, for $j = 1, \ldots, t_0$ such that there is an exact sequence of right $\Lambda$-modules

$$\Pi_{i=1}^{d} w_i \Lambda \cong \Pi_{j=1}^{t_0} v_i \Lambda \to M \to 0.$$
theory, one is representing the $x_{i,j}$ as some $h_{i,j}$ already!) Let $h_i = (h_{i,1}, \ldots, h_{i,t_i})$ in $\Pi_{i=1}^{t_i} f_i^0 R$. By our assumptions, each $f_i^0 I$ is a finitely generated right $R$-module since the nonzero elements in $f_i^0 I$ form a right Gröbner basis of $f_i^0 I$.

We have that $\{h_i\}_{i=1}^d \cup \{f_i^0 G\}_{i=1}^u$ is a finite right uniform generating set for $K$. Let $\{f_i^*\}_{i=1}^d$ be a right tip-reduced, right uniform set obtained by right tip-reducing the set $\{h_i\}_{i=1}^d \cup \{f_i^0 G\}_{i=1}^u$. Then $\{f_i^*\}_{i=1}^d$ is a right tip-reduced and right uniform generating set for $K = \Pi_{i=1}^{t_i} f_i^* R$ by Proposition 2.6. Since right tip-reduction is algorithmic, and right tip-reduction of a right uniform set remains right uniform, taking the $f_i^*$’s to be those $f_i^*$’s not in $\Pi_{j=1}^{t_j} f_j^0 I$ and the $f_i^1$’s to be those $f_i^*$’s in $\Pi_{j=1}^{t_j} f_j^0 I$, the result follows. 

For the remainder of this section, we let $\Lambda = R/I$ where $R = kQ$ for some quiver $Q$ and assume that $G$ is a tip-reduced Gröbner basis for $I$ consisting of uniform elements. Let $M$ be a right $\Lambda$-module. We keep the following two assumptions. First, we assume that rt $G$ is finite. Second, we assume that $M$ has a projective presentation as a $\Lambda$-module of the form (3) with $\mathcal{I}$ and $\mathcal{T}$ finite sets.

By Proposition 5.1, there is an algorithm, which we call LiftPresentation, whose input is a projective $\Lambda$-presentation of $M$ of form (3) with $\mathcal{I}$ and $\mathcal{T}$ finite, and whose output is nonnegative integers $t_0, t_1$, and $u_1$ and a projective presentation of $M$ as an $R$-module

$$0 \rightarrow (\Pi_{i=1}^{t_i} f_i^1 R) \Pi (\Pi_{i=1}^{u_1} f_i^{t_1} R) \xrightarrow{H^1} \Pi_{i=1}^{t_i} f_i^0 R \rightarrow M \rightarrow 0,$$

where

1. $H^1$ is an inclusion map,
2. the $f_i^0$s, $f_i^1$s and $f_i^{t_1}$s are right uniform elements,
3. $f_i^{t_1} \in \Pi_{i=1}^{t_i} f_i^0 I$, for all $i = 1, \ldots, u_1$, and
4. $\{f_i^1 R\}_{i=1}^{t_1} \cup \{f_i^{t_1} R\}_{i=1}^{u_1}$ is right tip-reduced.

Let $T$ be some finite set, for $i \in T$, let $\{f_i\}_{i \in T}$ be a set of right uniform elements in $R$. If $h_1, \ldots, h_m, h_1', \ldots, h_n'$ is a right tip-reduced, right uniform subset of $\Pi_{i \in T} f_i R$ and $x \in (\Pi_{i=1}^{t_i} h_i R) \Pi (\Pi_{i=1}^{u_i} f_i^{t_1} R)$ is right uniform, let FirstPart be the algorithm that takes as input $x$, $\{h_1, \ldots, h_m\}$, and $\{h_1', \ldots, h_n'\}$ and outputs $\sum_{i=1}^{m} h_i r_i$ where $x = (h_1 r_1, \ldots, h_m r_m, h_1' s_1, \ldots, h_n' s_n)$ where the $r_i$ and $s_i$ are uniform elements of $R$. Note that FirstPart is an algorithm, since the $r_i$’s and the $s_i$’s can be obtained by right tip-reducing $x$ by $\{h_1, \ldots, h_m, h_1', \ldots, h_n'\}$.

If $\{h_1, \ldots, h_m\}$ is a right uniform, right tip-reduced subset of $\Pi_{i=1}^{t_i} f_i R$ where $\{f_i\}$ is a right uniform, right tip-reduced set, let CreateMatrix be the algorithm with input $\{h_1, \ldots, h_m\}$ and $\{f_1, \ldots, f_u\}$ and output the $n \times m$ matrix $(h_{i,j})$ with uniform entries given by $h_{i,j} = (f_1 h_{1,j}, \ldots, f_u h_{n,j})$. Note that in CreateMatrix writing $h_{i,j} = (f_1 h_{1,j}, \ldots, f_u h_{n,j})$ can be done algorithmically by right tip-reducing $h_{i,j}$ by the set $\{f_i\}$.

We now give an algorithmic description of the construction of a projective resolution of a module $M$ given is the preceding sections. We are given a field $k$, quiver $Q$, an admissible order $> \in B$, and a finite generating set $F$ for an ideal $I$ in $kQ$. Set $R = kQ$ and $\Lambda = kQ/I$. We find a tip-reduced reduced Gröbner basis of uniform elements for $I$ with respect to $>$ and compute rt $G$ which must be finite. We also use the sets $O(p)$ and $N(p)$ defined in Section 3. We note that, by the assumption that rt $G$ is a finite set, therefore $X(p)$ is a finite set and hence both $O(p)$ and $N(p)$ are finite sets.
We input $M$ in the algorithm as a matrix. In particular, suppose that
$$
\Pi_{i=1}^d w_i \Lambda \xrightarrow{\varphi} \Pi_{i=1}^{t_0} v_i \Lambda \to M \to 0
$$
is projective $\Lambda$-presentation of $M$. Then we represent $M$ as the matrix $(s_{ij})_{i=1,j=1}^{t_0,d}$, where $\varphi(w_j) = (s_{1j}, \ldots, s_{t_0,j})$. Note that $s_{ij}$ is in $v_i \Lambda w_j$.

**INPUT:** Nonnegative integers $N, t_0$ and $d$, vertices $v_1, \ldots, v_n$, $w_1, \ldots, w_d$, and a $t_0 \times d$-matrix $D$ whose $(i, j)$-th entry is in $v_i \Lambda w_j$.

**OUTPUT:** For $0 \leq n \leq N$, nonnegative integers $t_n$ and $u_n$, $\{f_n^{i_1}\}_{i_1=1}^{t_n}$, $\{f_n^{i_1'}\}_{i_1=1}^{u_n}$ and, if $n \geq 1$, $h_{n,b}^{i,n}$ for $1 \leq a \leq t_{n-1}$ and $1 \leq b \leq t_n$ as in Section 2.

1. Set $t_0 = 0$. **LiftPresentation**(D) outputs $t_0$, $t_1$, $u_1$, $\{f_1^{i_1}\}_{i_1=1}^{t_1}$, $\{f_1^{i_1'}\}_{i_1=1}^{u_1}$. **CreateMatrix**($\{f_1^{i_1}\}_{i_1=1}^{t_1}, \{f_1^{i_1'}\}_{i_1=1}^{u_1}$) outputs $(h_{0,1}^{i,1})$.

2. Set $j = 1$.

3. While ($j < N$)
   1. Let $T_{j+1} = \{(i, q) \mid 1 \leq i \leq t_j, \text{ and } q \in O(\text{tippath}(f_i^1)) \}$ and $t_{j+1} = |T_{j+1}|$. Choosing $(i, q) \in T_{j+1}$, one at a time, indexing by $l = 1, \ldots, t_{j+1}$, output
      $$
      f_{i}^{j+1} = f_i^{j} p - \text{FirstPart}((f_i^j p - \epsilon_i q' g_u^{2}), \{f_i^{1}\}, \{f_i^{j'}\}),
      $$
      where $q = \text{tippath}(f_i^j p) = q' \text{tip}(g_u^2)$ and $c = \text{coefficient of } \text{tip}(f_i^j p)$ in $k$.
   2. Let $U_{j+1} = \{(i, q) \mid 1 \leq i \leq t_j, \text{ and } q \in N(\text{tippath}(f_i^j)) \}$ and $u_{j+1} = |U_{j+1}|$. Choosing $(i, q) \in U_{j+1}$, one at a time, indexing by $l = 1, \ldots, u_{j+1}$, output
      $$
      f_{i}^{j+1'} = f_i^{j} z g_u^2,
      $$
      where $q = \text{tippath}(f_i^j p) z \text{tip}(g_u^2)$.
   3. **CreateMatrix**($\{f_{i}^{j+1}\}_{i=1}^{t_{j+1}}, \{f_{i}^{j+1'}\}_{i=1}^{u_{j+1}}$) outputs $(h_{j,j+1}^{i,1})$.
   4. $j \leftarrow j + 1$.

The above algorithm outputs the $f_i^{n}$’s, the $f_i^{n'}$’s and the $h_{j,j+1}^{i,n}$’s. Next we note that reducing an element $x$ of $R$ by $G$ uses a noncommutative division algorithm [12]. The output of this algorithm is called the normal form of $x$, which we denote by **NormalForm**$(x)$. We now obtain the desired first $N$ steps of a projective $\Lambda$-resolution of the cokernel of $\varphi$: $\Pi_{i=1}^d w_i \Lambda \xrightarrow{\varphi} \Pi_{i=1}^{t_0} v_i \Lambda$ for the algorithm above as follows. Since each $f_i^j$ is right uniform, we let $v_{ij}$ be the vertex so that $f_i^j = f_i^{v_{ij}}$. For $n = 0, \ldots, N$, let $L^n = \Pi_{i=1}^d v_i \Lambda$ and, for $n = 1, \ldots, N$, define $e^n : L^n \to L^{n-1}$ by $e^n(v^n_{i})$ is **NormalForm**($h_{j,j+1}^{i,n-1,n}$) in the $v_{ij}^{n-1}$-th component.

Summarizing, we have the following result.

**Theorem 5.2.** Let $\Lambda = R/I$ where $R = kQ$ for some quiver $Q$. Assume that $G$ is a tip-reduced Gröbner basis for $I$ consisting of uniform elements and assume further that $R \Lambda$ is finite. Let $M$ be a right $\Lambda$-module such that $M$ has a projective presentation as a $\Lambda$-module of the form
$$
\Pi_{i \in \mathcal{I}} w_i \Lambda \xrightarrow{\varphi} \Pi_{i \in \mathcal{I}} v_i \Lambda \to M \to 0
$$
with $\mathcal{I}$ and $\mathcal{I}'$ finite sets.

Then there is an algorithm to construct a projective resolution of $M$ over $\Lambda$ associated to the $R$-presentation obtained by our algorithm **LiftPresentation**.
6. Resolutions of linear modules over Koszul algebras

In this section we modify our construction to produce minimal projective resolutions of linear modules over Koszul algebras. We obtain an algorithm to do this and point out that the assumption that the Gröbner basis is finite is no longer needed. For completeness, we provide some background.

Recall that if \( R = kQ \) and \( I \) is an ideal generated by length homogeneous elements, then the length grading on \( R \) induces a positive \( \mathbb{Z} \)-grading on \( \Lambda = R/I \); namely, \( \Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots \), where \( \Lambda_0 \) is isomorphic to a finite product of copies of \( k \). Let \( \tau = \Pi_{i \geq 1} \Lambda_i \), which is the graded radical of \( \Lambda \). If \( M \) is a graded right \( \Lambda \)-module with \( M_n = 0 \) for \( n < 0 \), then \( M \) has a minimal graded projective \( \Lambda \)-resolution \( (\Lambda, e) \), where minimal means that \( e^n(L^n) \subseteq L^{n-1} \tau \) for all \( n \geq 1 \). We say that \((\Lambda, e)\) is a linear resolution and that \( M \) is a linear module if, for each \( n \geq 0 \), the graded module \( L^n \) is finitely generated in degree \( n \). The algebra \( \Lambda \) is a Koszul algebra if \( \Lambda_0 \) is a linear right \( \Lambda \)-module. Koszul algebras were introduced in [19] and we refer the reader to [4, 14, 15] for further details. Let \( J \) denote the ideal in \( kQ \) generated by the arrows.

**Theorem 6.1.** Let \( \Lambda = kQ/\mathcal{I} \) be a Koszul algebra with \( \mathcal{I} \) in \( J^2 \), and let \( M \) be a linear right \( \Lambda \)-module. Suppose that a start of a minimal projective linear resolution

\[
\Pi_{i=1}^{t_1} w_i \Lambda \rightarrow \Pi_{i=1}^{t_0} v_i \Lambda \rightarrow M \rightarrow 0
\]

is given for \( M \), where \( v_i \) and \( w_i \) are vertices in \( Q \) and \( t_0 \) and \( t_1 \) are positive integers.

(a) Then there exists a projective presentation

\[
0 \rightarrow (\Pi_{i=1}^{t_1} f_i^1 R) \oplus (\Pi_{j \in T_0} f_j^1' R) \xrightarrow{H} \Pi_{i=1}^{t_0} f_i^0 R \bar{z} \rightarrow M \rightarrow 0
\]

of \( M \) as a right \( R \)-module, where the elements \( \{f_i^0\}_{i \in T_0} \) are vertices, the sets \( \{f_i^1\}_{i \in T_1} \) and \( \{f_j^1'\}_{j \in U_0} \) are right uniform and right tip-reduced and contained in \( \Pi_{i=1}^{t_0} f_i^0 R \), and can be chosen such that every coordinate of each \( f_i^1 \) as an element in \( \Pi_{i=1}^{t_0} f_i^0 R \) is a sum of elements of length \( 1 \) in \( R \) and each \( f_j^1' \) is in \( \Pi_{i \in T_0} f_i^0 R \).

(b) There is an algorithm to construct a finite set of elements \( \{f_i^2\}_{i \in T_2} \) in \( \Pi_{i=1}^{t_1} f_i^1 R \) with \( f_i^2 = \sum_{l=1}^{t_0} f_i^1 r_l \) for some linear elements \( r_l \) in \( R \) such that

\[
\Pi_{i \in T_2} f_i^2 R / \Pi_{i \in T_1} f_i^1 R \xrightarrow{e^2} \Pi_{i=1}^{t_1} f_i^1 R / \Pi_{i=1}^{t_0} f_i^1 I \rightarrow \Omega_{\Lambda}^1(M) \rightarrow 0
\]

is a start of a minimal projective linear resolution of \( \Omega_{\Lambda}^1(M) \), where \( \Omega_{\Lambda}^1(M) = \text{Ker}((\Pi_{i=1}^{t_1} v_i \Lambda \rightarrow M) \oplus \Lambda^1) \) and the map \( e^2 \) is induced by the inclusion \( \Pi_{i \in T_2} f_i^2 R \hookrightarrow \Pi_{i=1}^{t_1} f_i^1 R \) as in our earlier construction.

**Proof.** (a) The presentation

\[
\Pi_{i=1}^{t_1} w_i \Lambda \rightarrow \Pi_{i=1}^{t_0} v_i \Lambda \rightarrow M \rightarrow 0
\]

of \( M \) gives rise to the exact sequence

\[
0 \rightarrow K \xrightarrow{\bar{z}} \Pi_{i=1}^{t_0} v_i R \rightarrow M \rightarrow 0
\]

of right \( R \)-modules. Then \( K \) is a projective \( R \)-module which maps onto \( \Omega_{\Lambda}^1(M) \). It is easy to see that the natural map \( \Pi_{i=1}^{t_1} w_i R \rightarrow \Omega_{\Lambda}^1(M) \) is a projective cover in the category of graded right \( R \)-modules and degree 0 homomorphisms, hence...
there are degree zero maps $\alpha: \Pi_{i=1}^{t_0} w_i R \to K$ and $\beta: K \to \Pi_{i=1}^{t_0} w_i R$ such that $\beta \alpha = \text{id}_{\Pi_{i=1}^{t_0} w_i R}$. In particular

$$K = \text{Im}\, \alpha \,\Pi \, \text{Ker}\, \beta \simeq (\Pi_{i=1}^{t_0} w_i R) \,\Pi \, \text{Ker}\, \beta.$$ 

Since $\Pi_{i=1}^{t_0} v_i I$ is the kernel of the map $\Pi_{i=1}^{t_0} v_i R \to \Pi_{i=1}^{t_0} v_i \Lambda$, we have that $\text{Ker}\, \beta$ is contained in $\Pi_{i=1}^{t_0} v_i I$. As $\text{Ker}\, \beta$ is a projective $R$-module, there are vertices $(w_i^t)_{i \in U_t'}$ in $Q_0$ for some index set $U_t'$ such that $\text{Ker}\, \beta \simeq \Pi_{i \in U_t'} w_i R$.

Now let

$$f_0^t = v_i, \quad \text{for } i = 1, \ldots, t_0,$$

$$h_1^t = \varphi(\alpha(w_i)) \quad \text{for } i = 1, \ldots, t_1$$

and

$$h_i^t' = \varphi(w_i'), \quad \text{for } i \in U_t'.$$

Since $w_i$ and $w_i'$ are vertices in $Q$, the elements $h_0^t$ and $h_i^t'$ are clearly right uniform. Right tip-reduce each of the sets $(h_1^t)_{i=1}^{t_1}$ and $(h_i^t')_{i \in U_t'}$ and denote the result by $(f_1^t)_{i=1}^{t_1}$ and $(f_i^t')_{i \in U_t'}$, respectively. The elements are still right uniform.

Since the map $\alpha$ has degree zero and $M$ is a linear $A$-module, each of the coordinates of the elements $(f_1^t)_{i=1}^{t_1}$ as elements in $\Pi_{i=1}^{t_0} f_0^t R$ are all a sum of elements of length 1 in $R$. The elements $(f_i^t')_{i \in U_t'}$ are in $\Pi_{i=1}^{t_0} f_i^t R$, so that each of the coordinates of an element $f_i^t'$ as an element of $\Pi_{i=1}^{t_0} f_i^t R$ is a sum of elements of length at least 2 in $R$. This completes the proof of (a).

(b) First we look at the construction of $f_i^t$'s given in Section 3. By linearity, all the coordinates of the elements $f_i^t$'s occurring in a minimal projective linear resolution of $M$ as elements in $\Pi_{i=1}^{t_0} f_i^t R$ are a sum of elements of length 2 in $R$. Since $I$ is generated by length homogeneous elements of degree 2, there is a tip-reduced uniform Gröbner basis consisting of length homogeneous elements of degree at least 2. An element $g_j^2$ of degree 2 in $G$, occurring in the construction of a $f_i^t$, gives rise to a homogeneous $f_i^t$ of degree 2. Tip-reduction does not change the degree, so that to obtain all the $f_i^t$'s to continue the minimal projective linear resolution of $M$, we only need to consider the elements of degree 2 in $G$. There are only a finite number of such elements, since $G$ is tip-reduced.

Let $s = (i, q)$ be in $T_2$. Then there is a $j$ such that

$$q = \text{tipath}(f_i^t)p = q' \text{ tip}(g_j^2)$$

for some paths $p$ and $q'$ and $g_j^2$ in $G$ is the end relation of $q$, where $\text{tip}(g_j^2)$ and $\text{tipath}(f_i^t)$ overlap. Suppose $\text{tip}(g_j^2)$ is a path of length 2. It overlaps $\text{tipath}(f_i^t)$, hence $p$ must be a path of length 1 (an arrow), and $q'$ is a vertex. The element $f_i^t p - e_i \cdot c q' g_j^2$ is in $(\Pi_{j \in T_1} f_j^t R) \Pi (\Pi_{j \in U_t} f_j^t' R)$ with $c = \frac{\text{coefficient of $\text{tip}(f_i^t)$}}{\text{coefficient of $\text{tip}(g_j^2)$}}$ in $k$.

but since the set $(f_i^t)_{i \in T_1} \cup (f_i^t')_{i \in U_t}$ is not necessarily right tip-reduced there is no apparent algorithm to express $f_i^t p - e_i \cdot c q' g_j^2$ in this direct sum. Since all the coordinates of this element have degree 2 as an element in $\Pi_{i \in T_0} f_i^t R$; an element $f_i^t'$ of degree at least 3 does not occur in this expression, so that we only need to consider the $f_i^t$'s of degree 2. Since there is a finite number of paths of length 2 and since the set of elements of homogeneous degree 2 in $(f_i^t)_{i \in U_t}$ is right tip-reduced,
there are only a finite number of such elements. Say they are \( \{ f_i^{12} \}_{i \in U_1(2)} \) for some set finite \( U_1(2) \).

In the construction in Section 3 of \( f_s^2 \) we are assuming that the set \( \{ f_i^{12} \}_{i \in T_1} \cup \{ f_i^{12} \}_{i \in U_1(2)} \) is right tip-reduced. In our case we only have that each of the sets \( \{ f_i^{12} \}_{i \in T_1} \) and \( \{ f_i^{12} \}_{i \in U_1(2)} \) is right tip-reduced, but not necessarily the union. A tip of an \( f_i^{12} \) cannot reduce a tip of an \( f_i^{12} \) by length arguments. So in order to right tip-

reduce the set \( \{ f_i^{12} \}_{i \in T_1} \cup \{ f_i^{12} \}_{i \in U_1(2)} \) the elements \( \{ f_i^{12} \}_{i \in T_1} \) stay unchanged, while the elements \( \{ f_i^{12} \}_{i \in U_1(2)} \) might change. We need only the elements obtained from the set \( \{ f_i^{12} \}_{i \in U_1(2)} \). Denote these new elements by \( \{ f_i^{12} \}_{i \in U_1(2)} \) for some finite set \( U_1(2) \), where we record how \( f_i^{12} \) is expressed in terms of \( f_s^2 \)'s and \( f_s^2 \)'s.

Furthermore the right-tip-reduction of some \( f_i^{12} \) is obtained by subtracting elements of the form \( df_j^2 a \), where \( a \) is an arrow and \( d \) is in \( k \) and elements of the form \( df_j^2 b \), where \( j \) is in \( U_1(2) \) and \( d \) is in \( k \). Therefore each \( f_i^{12} \) is still homogeneous of degree 2.

When constructing \( f_s^2 \) for \( s = (i, q) \) in \( T_2 \) we can algorithmically find a presentation

\[
f_i^2 - p - \varepsilon_i \cdot c q g_j^2 = \sum_{i \in T_1} f_i^1 \gamma_i + \sum_{i \in U_1(2)} f_i^2 s_i
\]

for some elements \( \gamma_i \) and \( s_i \) in \( R \). The left hand side has all coordinates being a sum of elements of length 2. If some path of length at least 2 occurs in some \( \gamma_i \), then \( \text{tippath}(\sum_{i \in T_1} f_i^1 \gamma_i) \) is equal to \( \text{tippath}(\sum_{i \in U_1(2)} f_i^2 s_i) \). As we have seen before this contradicts the fact that \( \{ f_i^1 \}_{s \in T_1} \cup \{ f_i^2 \}_{s \in U_1(2)} \) is right tip-reduced. By length arguments no vertex can occur in any \( \gamma_i \). Hence each \( \gamma_i \) is a sum of elements of length 1 and each \( s_i \) is a vertex. Substituting \( f_i^2 \) with the expressions in \( f_s^2 \)'s and \( f_j^2 \)'s, we obtain as before

\[
f_i^2 = f_i^2 - \sum_{i \in T_1} f_i^1 \gamma_i + \sum_{i \in U_1(2)} f_i^2 s_i
\]

for some linear elements \( \gamma_i \) in \( R \) and some elements \( s_i \) in \( R \) of degree 0, and therefore all the coordinates of \( f_i^2 \) as elements in \( \sum_{i = 1}^{t_0} f_i^2 R \) are a sum of elements of degree 2.

Now let \( x \) be a homogeneous element of degree 2 in \((\sum_{i = 1}^{t_0} f_i^2 R) \cap (\sum_{i = 1}^{t_0} f_i^1 I) \), where \( \text{tip}(x) \) is smallest possible such that \( x \) is not in \( \sum_{i \in T_2} f_i^2 R \). Hence \( x = \sum_{i = 1}^{t_0} f_i^1 b_i \), where \( b_i \) is a sum of elements of degree 1 in \( R \) and \( g_j^2 \) is homogeneous elements of degree 2 in \( G \). Then \( \text{tip}(x) = \text{tip}(f_i^1 b_i) = f_i^1 \text{tip}(g_j^2) \) for some \( i \) and some \( l \). Then \( \text{tip}(x) \) is equal to \( \text{tip}(f_j^2) \) for some \( j \) in \( T_2 \). By the choice of \( x \) the element \( x - c f_j^2 \) is in \( \sum_{i \in T_2} f_i^2 R \) for some element \( c \) in \( k \). Hence all homogeneous elements of degree 2 in \((\sum_{i = 1}^{t_0} f_i^2 R) \cap (\sum_{i = 1}^{t_0} f_i^1 I) \) are in \( \sum_{i \in T_2} f_i^2 R \). Therefore we have constructed all \( f_j^2 \)'s of degree 2, and by construction the elements \( \{ f_j^2 \}_{i \in T_2} \) are right uniform and right tip-reduced. These elements give rise to the minimal projective cover of \( \Omega_\lambda^2(M) \). Then there is a natural map \( \Pi_{\epsilon_2} f_{\epsilon_2} I \rightarrow \Omega_\lambda^2(M) \), which is a projective cover. Then we have a start of minimal projective linear resolution of \( \Omega_\lambda^2(M) \)

\[
\Pi_{\epsilon_2} f_{\epsilon_2} I \xrightarrow{\epsilon_2} \Pi_{\epsilon_2} w_1 I \rightarrow \Omega_\lambda^2(M) \rightarrow 0.
\]

since the elements \( p \) and \( r_j^2 \) are linear elements. The construction of the map \( \epsilon_2 \) is the same as given in Section 3. \( \square \)
To see that the above arguments actually give rise to an algorithm, we first note that one need not right tip-reduce the whole set \( \{ h_i^{e_i} \}_{e \in E_i} \), which maybe infinite. We only need to right tip-reduce those elements of homogeneous degree 2 in this set. This subset of \( \{ h_i^{e_i} \}_{e \in E_i} \) can be chosen to be finite, since the subspace of elements of homogeneous degree 2 of each \( v_i \) has basis the elements of homogeneous degree 2 of \( v_i \langle G \rangle \), which is a finite set. We are also not assuming that the Gröbner basis \( G \) is finite. But we only need the homogeneous elements of degree 2 in \( G \), which is a finite set and may be computed by right tip-reducing a set of right uniform generators of \( I \). It follows that the construction of the elements \( f_2^j \) of homogeneous degree 2 is algorithmic.

We remark that there is another method for constructing the \( f_2^j \)'s of homogeneous degree 2. Namely, let \( A \) be the \( k \)-span of \( \{ f_1^i a | i = 1, \ldots, t_1, a \in Q_1 \} \) and \( B \) the \( k \)-span of \( \{ f_0^g | g \in G \) and length of \( g = 2 \}. \) Then one may use linear algebra to find a basis \( \{ b_1, \ldots, b_t \} \) of \( A \cap B \). Viewing the \( b_i \)'s as elements in \( \mathbb{Q}_{i=1}^{\infty} f_1^i R \), the set \( \{ f_2^j \}_{j=1}^{t} \) can be found by right tip-reducing \( \{ b_i \}_{i=1}^{t} \). This may be a faster way of finding the \( f_2^j \)'s than the method presented in the above proof.

In general the construction in Section 4 does not produce a minimal projective resolution of a linear module. The algorithm of this section differs from the algorithm in Section 3 in that one only considers elements of a Gröbner basis of length 2 when constructing \( T_2 \). We illustrate this in the following example.

**Example 6.2.** We continue with Example 4.1. We note that \( A \) is a Koszul algebra and that \( M = v_1 A / r \) is a linear module. We saw that the resolution constructed by the algorithm in Section 4 gave a non-minimal projective resolution of \( M \) for the ordering \( >_1 \). For this ordering, recall that the Gröbner basis \( G = \{ ab - cd, be, cde \} \) for \( I \). Referring back to Example 3.2 in Section 3 we now construct \( T_2 \) using the algorithm described above, that is; only using \( ab - cd \) and \( be \) from \( G \). In this way we only produce \( f_2^j \) as in Example 3.2. We obtain the same resolution as given in Example 4.1 for the ordering \( >_2 \) in this way, and hence producing a minimal projective resolution of \( M \) over \( A \).

**References**


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