MA3203 - Problem sheet 4+

Problem 1. Let Λ and Σ be k-algebras over a field k, and let $\mathcal{B}$ be a basis for Λ. Suppose $\varphi\colon \Lambda \to \Sigma$ be a k-module homomorphism such that

$$\varphi(1_\Lambda) = 1_\Sigma,$$
$$\varphi(bb') = \varphi(b)\varphi(b'), \text{ for all } b, b' \in \mathcal{B}$$

Show that φ is a k-algebra homomorphism.

Problem 2. Let Γ be a quiver and let Σ be a k-algebra over a field k. Let $f\colon \Gamma_0 \cup \Gamma_1 \to \Sigma$ be a function satisfying

(1) $\sum_{v\in\Gamma_0} f(v) = 1_\Sigma$,
(2) $qr = 0$ (which is the usual concatenation of paths) implies that $f(q)f(r) = 0$ for all $q, r \in \Gamma_0 \cup \Gamma_1$,
(3) $f(v) = f(v)^2$ for all $v \in \Gamma_0$,
(4) $f(e(\alpha))f(\alpha) = f(\alpha) = f(\alpha)f(s(\alpha))$ for all $\alpha \in \Gamma_1$.

Then f extends uniquely to a k-algebra homomorhism $\widetilde{f}\colon k\Gamma \to \Sigma$.

Fact 3. Let R be a ring and I a nil ideal in R. Then idempotents lift modulo I, that is, if f is an idempotent in R/I, then there is an idempotent e in R such that $f = e + I$ [1, Proposition 27.1].

Fact 4. Let R be a ring and I an ideal in R such that $I \subseteq J(R)$. Then the following are equivalent [1, Proposition 24.7]:

(1) Idempotents lift modulo I,
(2) Every (complete) finite orthogonal set of idempotents in R/I lifts to a (complete) orthogonal set of idempotents in R.

Here a set $\{e_i\}_{i=1}^n$ of idempotents in a ring S is *complete* if $\sum_{i=1}^n e_i = 1_S$, and a set $\{e_i\}_{i=1}^n$ of idempotents in a ring S is *orthogonal* if $e_ie_j = 0$ for all $i \neq j$.

Problem 5 - challenge. Let Λ be a finite dimensional basic algebra over an algebraically closed field k. Show that $\Lambda/\mathfrak{r} \simeq k^n$ for some $n \geqslant 1$.

References

[1] Andersen, F., W., Fuller, K., R., *Rings and categories of modules*, Graduate Texts in Mathematics, Springer Verlag, vol. 13.