MA3203 - Problem sheet 1

Problem 1.

(a) Given a quiver $\Gamma\colon\ 1 \longrightarrow 2 \longrightarrow 3$ and a field k, let $\Lambda = k\Gamma$. Let S be the k-algebra $\left(\begin{smallmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{smallmatrix}\right)$. Show that Λ and S are isomorphic as k-algebras.

(b) Given the representation $(V, f)\colon\ k^2 \xrightarrow{(1\ 1)} k \xrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} k^2$ and the representation $(V', f')\colon\ k \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} k^2 \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} k^2$ of Γ over k, describe homomorphisms from (V, f) to (V', f').

(c) Describe homomorphisms from (V', f') to (V, f) where (V, f) and (V', f') are as in (b).

(d) We say that a representation (V, f) of some quiver Γ over some field k is a subrepresentation of a representation (V', f') of Γ over k if for each vertex i the vectorspace $V(i)$ is a subspace of $V'(i)$ and for each arrow $\alpha\colon i \to j$ in Γ we have that $f_\alpha = f'_\alpha|_{V(i)}\colon V(i) \to V(j)$.

Let (V', f') be the representation $k \xrightarrow{1} k \xrightarrow{1} k$ of the quiver Γ from (a) over the field k. Find all subrepresentations of (V', f').

(e) *Challenge*: For a subrepresentation (V, f) of a representation (V', f') define *the factor representation* $(V', f')/(V, f)$.

Let (V', f') be as in (d). Find all factor-representations of (V', f').

Problem 2. Let $R = \left(\begin{smallmatrix} k & 0 \\ k[x]/(x^2) & k \end{smallmatrix}\right) \subset \left(\begin{smallmatrix} k[x]/(x^2) & 0 \\ k[x]/(x^2) & k[x]/(x^2) \end{smallmatrix}\right)$, where k is a field and addition and multiplication are the usual addition and multiplication of matrices.

(a) Show that R is a ring, and that it is an algebra over k when $\alpha \cdot r = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right) r$ for $\alpha \in k$ and $r \in R$. Is R a left artinian ring?

(b) Let $I = \left(\begin{smallmatrix} 0 & 0 \\ k[x]/(x^2) & 0 \end{smallmatrix}\right)$. Show that I is an ideal in R. Is R a semisimple ring? Show that $R/I \cong k \bigoplus k$ as rings.

(c) Show that R is isomorphic to the path algebra of the quiver $\Gamma\colon\ 1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2$.

(d) Let M_λ be the representation of the quiver Γ over k given by $V(1) = k, V(2) = k, f_\alpha = 1$ and $f_\beta = \lambda$ (i.e. $f_\alpha(a) = a, f_\beta(a) = \lambda a, \forall a \in k$) where $\lambda \in k$.

Show that $M_\lambda \ncong M_{\lambda'}$ if $\lambda \neq \lambda'$.

(e) Show that M_λ is indecomposable for each λ in k.

Problem 3.

(a) Given the quiver $\Gamma\colon\ 1 \circlearrowright \alpha$ with a relation $\sigma = \alpha^2$ and a field k, let $\Lambda = k\Gamma/\langle \alpha^2 \rangle$. Show that Λ and $k[x]/(x^2)$ are isomorphic as k-algebras.

(b) Find all homomorphisms from the representation $(V, f)\colon\ k^3 \circlearrowright \left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ to itself.

(c) *Challenge*: Find all indecomposable representations of (Γ, σ) over k. Hint: Only two!