MA3201 - Problem sheet 6

Time and place: Wednesday 17rd November $14^{\underline{15}} - 16^{\underline{00}}$ in F6.

If there is some time left after having done the problems, we continue with the lectures.

Page	Exercise number
401	1, 2, 3
409	(a), (c)

Exam in MA3201 from November 30th, 2005. All problems, except Problem 3, which we have done already.

Challenge! Find an algorithm for computing the rational canonical form of a matrix.

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Exam in course MA3201 Rings and modules

English
Wednesday November 30, 2005
Time: 09.00-13.00

Permitted aids: none

Grades: 21.12.2005.

Problem 1 Let q be a fixed non-zero element in $\mathbb{C}$, the set of complex numbers. Define the subset R_q of the ring of 4×4-matrices over $\mathbb{C}$ by

$$R_q = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & 0 & a & 0 \\ d & c & -qb & a \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}.$$

a) Show that R_q is a ring.

b) For which q in $\mathbb{C}$ is R_q a commutative ring?

c) For a given element α in $\mathbb{C}$ define the subset

$$I_\alpha = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ \alpha b & 0 & 0 & 0 \\ d & \alpha b & -qb & 0 \end{pmatrix} \mid b, d \in \mathbb{C} \right\}$$

of R_q. Show that I_α is a left ideal in R_q for all α in $\mathbb{C}$.

d) Show that each of the left ideals I_α is generated by one element as a left ideal. Show that $I_\alpha \simeq R_q/I_{\alpha q}$ as left R_q-modules.

Problem 2 Let $\mathbb{Q}$ be the field of rational numbers, and let a and b in $\mathbb{Q}$ be different elements. Find all possible rational canonical forms for 4×4-matrices over $\mathbb{Q}$ having

$$(x+a)^2(x+b)$$

as a minimal polynomial.

Problem 3 Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}[x]$ the polynomial ring over $\mathbb{C}$ in one variable x. Let $\alpha \in \mathbb{C}$ be a complex number.

a) Show that the map $\varphi_\alpha \colon \mathbb{C}[x] \to \mathbb{C}$ defined by $\varphi_\alpha(f(x)) = f(\alpha)$ is a surjective ring homomorphism, and use this to show that the ideal generated by $x - \alpha$ is a maximal ideal in $\mathbb{C}[x]$.

b) For which $n \geq 1$ is the ring

$$\begin{pmatrix} \frac{\mathbb{C}[x]}{((x-\alpha)^n)} & \frac{\mathbb{C}[x]}{((x-\alpha)^n)} \\ \frac{\mathbb{C}[x]}{((x-\alpha)^n)} & \frac{\mathbb{C}[x]}{((x-\alpha)^n)} \end{pmatrix}$$

semisimple?

Problem 4 Let R be a ring, and let M be a Noetherian left R-module. Show that any surjective R-homomorphism $f \colon M \to M$ is an isomorphism. (Hint: Consider the chain $\operatorname{Ker} f \subseteq \operatorname{Ker}(f^2) \subseteq \operatorname{Ker}(f^3) \subseteq \cdots$ of submodules of M).