MA3201 - Problem sheet 2

Time and place: Wednesday 22th September $14^{\underline{15}} - 16^{\underline{00}}$ in F6.

If there is some time left after having done the problems, we continue with the lectures.

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Problem 1. Find all left, right and two-sided ideals in the ring $\begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$.

Problem 2. Let A be an F-algebra, where F is a field. In particular, there is a map $F \times A \to A$, where we write the image of (α, a) as $\alpha \cdot a$.

(a) Define $\varphi\colon F \to A$ by $\varphi(\alpha) = \alpha \cdot 1_A$. Show that φ is a homomorphism of rings with $\operatorname{Im}\varphi = \{\varphi(\alpha) \mid \alpha \in F\} \subseteq Z(A)$.

(b) Suppose that $\varphi\colon F \to A$ is a homomorphism of rings with $\operatorname{Im}\varphi \subseteq Z(A)$. Show that A is an A is an F-algebra when $\alpha \cdot a = \varphi(\alpha)a$ for all α in F and all a in A.

Problem 3. Let R be a ring with 1, and let $f\colon \mathbb{Z} \to R$ be given by $f(n) = n \cdot 1_R$ for n in $\mathbb{Z}$.

(a) Show that f is a homomorphism of rings, and that it is the only one from $\mathbb{Z} \to R$.

(b) What is $\operatorname{Ker} f$?

Challenge: Let R be the ring given by $\begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}$.

(a) Find all the left ideals in R.

(b) Show that any left ideal can be generated by at most 2 elements.

(c) Consider a ascending chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ of left ideals in R. Can one have an infinite number of proper inclusions?

(d) Find/classify all the right ideals in R. Are they all finitely generated as right ideals?

(e) Consider a ascending chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ of right ideals in R. Can one have an infinite number of proper inclusions?