EXAM PROBLEMS IN MA321

Translated by Carl Fredrik Berg, comments can be sent to carlfrbe@math.ntnu.no

Exam in MA321, 18.12.1986

Problem 1. Let $F = \mathrm{GF}(2)$ denote the field with two elements, and let G be the group $\mathbb{Z}/(2)$.

(a) Find all the idempotent and nilpotent elements in the group ring $F\langle G\rangle$.
(b) Find all the idempotent elements in the group ring $\mathbb{R}\langle G\rangle$, and find an isomorphism $\mathbb{R}\langle G\rangle \simeq \mathbb{R} \oplus \mathbb{R}$.
(c) State Maschkes' theorem.

Problem 2. Let $g(x) = (x^2 + x + 1)(x + 1)^2(x - 1)^2$.

(a) List all rational canonical forms over $\mathbb{Q}$ for 6×6 matrices with characteristic polynomial g(x) and minimal polynomial of degree 5. Find the invariant factors of each of them. (Hint: In every case the matrix is built up by two blocks.)
(b) In every case above, decompose the matrix according to the elementary divisors, and write them down, both over $\mathbb{R}$ and over $\mathbb{C}$.

Exam in MA321, 18.12.1987

Problem 3. Let R be a finite ring with unity such that $x^2 = x$ for all $x \in R$.

(a) Show that R is commutative. (Hint: Look at $(x + y)^2$ and $x^2 = (-x)^2$)
(b) Show that R is semisimple.
(c) Use Artin-Wedderburn to show that R is isomorphic as a ring to a finite product $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ of the field $\mathbb{Z}_2$.

Problem 6. Let R be the ring $\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. List all the direct summands A of R, where R is viewed as a left module over itself (i.e. $R = A \oplus B$, where A and B are left-ideals in R.)

Exam in MA321, 16.12.1988

Problem 4. Let

$$R = \{ \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} \mid a_{ij} \in \mathbb{Z}_2\}$$

be a subset of $M_3(\mathbb{Z}_2)$, 3×3 -matrices over the field $\mathbb{Z}_2$.

(a) Show that R is a subring of the ring $M_3(\mathbb{Z}_2)$, and that R is an algebra over $\mathbb{Z}_2$.
(b) Find all idempotents of R.
(c) Show that R is artin. Is R semisimple? Why?
(d) Find a ring homomorphism

$$\phi : R \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

which is onto, and find $\mathrm{Ker}(\phi)$.
(e) Find two simple and non-isomorphic submodules of R, when R is viewed as a left R-module.

Exam in MA321, 23.08.1989

Problem 1. Let GF(2) denote the field with 2 elements, and let R denote the field of 2×2 matrices over GF(2).
$(R = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{ij} \in \mathrm{GF}(2) \right\})$

(a) Show that the subset
$$S = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$
is a commutative subring of R.

(b) Show that S is a field.

(c) Show that $\phi\colon R \to R$ given by
$$\phi(M) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \forall M \in R$$
is a ring automorphism of order 2, i.e. ϕ is a ring automorphism with $\phi^2 = \mathrm{id}$.

(d) Show that the group $G = \{\phi, id\}$ sends S into S, and find the fixed point field $S^G = \{x \in S \mid g(x) = x, \forall g \in G\}$.

Problem 2. Let k be a field, and let R be a subset of all 3×3-matrices over k given by
$$R = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mid a_{ij} \in k \right\}$$

(a) Show that R is a subring of the ring of all 3×3 matrices over k, and find the center of R.

(b) Show that the subsets
$$I_1 = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} \mid a_{ij} \in k \right\}$$
and
$$I_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mid a_{ij} \in k \right\}$$
are maximal two sided ideals of R.

(c) Let $I = I_1 \cap I_2$. Show that the ring R/I are isomorphic to the subring S of R consisting of the matrices
$$\left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a_{ij} \in k \right\}$$
Is R/S semisimple? Why?

Exam in MA321, 15.12.1989

Problem 5. Let R be a commutative ring with 1.

(a) If R is artinian and J is an ideal in R, show that R/J is artinian.

(b) Find an example of a ring R which is not artinian, but where R/J is artinian for all ideals $J \neq (0)$

Is a subring of an artinian ring artinian? Why?

Exam in MA321, 14.12.1990

Problem 4. Let R be the ring $\begin{pmatrix} F & 0 \\ F & F \end{pmatrix}$, where F is a field.

(a) Show that $I = \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix}$ is an ideal in R. Show that the rings R/I and $F \times F$ are isomorphic.
(b) Check if R is a semisimple artinian ring? Find an idempotent element f in R such that Rf becomes a semisimple R-module.
(c) Let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Show that Re is a maximal left ideal in R.
(d) Find all ideals and all left ideals in R.

Exam in MA321, 13.12.1991

Problem 3. Let $\mathbb{Z}$ denote the integers, and let $R = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & 0 & e \end{pmatrix} \mid a, b, c, d, e \in \mathbb{Z} \right\}$ as a subring of all 3×3-matrices over $\mathbb{Z}$.

Let $I = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & 0 & e \end{pmatrix} \mid a, b, c, d, e \in 2\mathbb{Z} \right\}$ and $J = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & 0 & e \end{pmatrix} \mid a, b, e \in 2\mathbb{Z}, b, c \in \mathbb{Z} \right\}$

(a) Show that I and J are two-sided ideals in R.
(b) Show that R/I are isomorphic to the ring

$$S = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & 0 & e \end{pmatrix} \mid a, b, c, d, e \in \mathbb{Z}_2 \right\}$$

(c) Show that J/I is an nilpotent ideal in R/I
(d) Find all ideals A in R where $I \supseteq A \supseteq J$.

Exam in MA321, 12.12.1992

Problem 4. Let F be a field and let R be the ring

$$\begin{pmatrix} F & 0 & 0 \\ F & F & 0 \\ F & 0 & F \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & 0 & e \end{pmatrix} \mid a, b, c, d, e \in F \right\}$$

Let $I = \begin{pmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ F & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \mid a, b \in F \right\}$

(a) Show that I is an ideal in R.
(b) Show that R/I and $F \times F \times F$ are isomorphic rings.
(c) Determine if the ring R is isomorphic to a finite product of full matrix-rings over a division ring?
(d) Let $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Is Re a maximal left ideal in R.
(e) Find 5 minimal left ideals in R.

Exam in MA321, 04.12.1993

Problem 1.

(a) Find all invariant factors and rational canonical forms of 6×6-matrices over $\mathbb{Q}$ with minimal polynomial $(x-1)(x-3)^2$.

(b) Let A be the matrix
$$\begin{pmatrix} 4 & 6 & 2 \\ 2 & 4 & 2 \\ 6 & 8 & 6 \end{pmatrix}$$
Find the Smith normal form of A over $\mathbb{Z}$

Exam in MA321, 12.06.1998

Problem 3. Let R be a ring with 1 and let S be a subset of 2×2-matrices over R given by
$$\left\{ \begin{pmatrix} r & 0 \\ s & t \end{pmatrix} \mid r, s, t \in R \right\}.$$

(a) Show that S is a ring with 1 under the usual addition and multiplication of matrices.

Define a map $\varphi\colon R \to S$ by
$$\varphi(r) = \left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right)$$
for all $r \in R$. Show that φ is a homomorphism over rings. Show that a left S-module M becomes a left R-module by defining
$$r \cdot m = \varphi(r)m$$
for all $r \in R$ and $m \in M$.

(b) Show that S is an finitely generated left R-module.

(c) Assume that X is an artinian left R-module. Show that every factor module of X is an artinian left R-module.

(d) Assume that R is an left artinian ring. Show that S has to be an left artinian ring too.

Exam in MA321, 13.06.1996

Problem 3. Let R be an algebra over a field F.

(a) Let I be a left ideal in R. Show that I is a vector subspace of R.

(b) Assume that $\dim_F R$ is finite. Show that R is an left artinian ring.

(c) Let $J \subset I$ be two ideals in R. Show that there exists a natural epimorphism between rings $\varphi : R/J \to R/I$. Find $\operatorname{Ker} \varphi$.

(d) Now let $R = \mathbb{C}[x_1, \ldots, x_n]$. Is R a left artinian ring? Let $\underline{m} = (x_1, \ldots, x_n)$ be the ideal in R generated by the variables x_i for $i = 1, \ldots, n$. Let I be an ideal in R such that $\underline{m}^t$ is contained in I for some $t \geq 1$. Show that R/I is a left artinian ring.

Exam in MA321, 09.12.1996

Problem 1. Find the invariant factors of the matrix
$$\begin{pmatrix} -x-3 & 2 & 0 \\ 1 & -x & 1 \\ 1 & -3 & -x-2 \end{pmatrix}$$
over $\mathbb{Q}[x]$, and find the rational canonical form of the matrix
$$\begin{pmatrix} -3 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & -3 & -2 \end{pmatrix}.$$

Problem 2. Let $R = C[0,1]$ denote the ring of continuous real functions defined on the interval $[0,1]$, where addition is given by $(f+g)(x) = f(x)+g(x), \forall x \in [0,1]$, and multiplication by $(fg)(x) = f(x)g(x), \forall x \in [0,1]$.

(a) Show that $I = \{f \in R \mid f(0) = 0\}$ is a maximal ideal in R.
(b) Find a zero divisor (different from zero) in R. Show that the function 0 and 1 are the only idempotent elements in R.

Problem 5.

(a) Find all ideals in the factor ring $R = \mathbb{Q}[x]/(x^2-1)(x^2+1)$. Is R an artinian ring?
(b) Let $R_n = \mathbb{Q}[x]/(x^3+3x+3)^n$ for $n \geq 1$. For which n is the matrix-ring
$$\begin{pmatrix} R_n & R_n \\ R_n & R_n \end{pmatrix}$$
a semisimple ring?
(c) Find three different left ideals in the ring $S = \begin{pmatrix} \mathbb{Q} & 0 & 0 \\ \mathbb{Q} & \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}$. Those ideals should not be minimal, but they should be isomorphic to each other as S-modules.

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MNFMA318, Rings and modules

English
Saturday, Desember 7, 2002
Time: 9-13
Permitted aids: None
Grades to be announced: Monday, January 6, 2003

Problem 1
Let $\mathbb{R}$ denote the field of real numbers and let

$$R = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & 0 & e & 0 \\ f & 0 & 0 & g \end{pmatrix} ; a,b,c,d,e,f,g \in \mathbb{R} \right\} \quad \text{and} \quad I = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ f & 0 & 0 & 0 \end{pmatrix} ; b,d,f \in \mathbb{R} \right\}$$

a) Show that R is a ring and that I is an ideal in R.

b) Show that R/I and $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ are isomorphic rings.

c) Is R a semisimple ring? Is R/I a semisimple ring?

d) Find 3 minimal left ideals in R.

Problem 2
Let R be a ring and M an R-module. Let N and L be submodules of M.

a) Show that $N \cap L$ is a submodule of M, and give an example to show that $N \cup L$ is not always a submodule of M.

b) Assume that M is a noetherian R-module. Show that then N and M/N are noetherian R-modules.

Problem 3

a) Find the possible invariant factors and rational canonical forms for 6×6-matrices over the real numbers $\mathbb{R}$, with minimal polynomial $(x^2+1)(x-3)^2$.

b) Let V be a vector space of dimension 4 over the real numbers $\mathbb{R}$, and let $T : V \to V$ be a linear transformation. Let v_1, v_2 be elements in V such that $\{v_1, v_2, Tv_2, T^2v_2\}$ is a basis for V, and assume that $Tv_1 = 2v_1$ and $T^3v_2 = 2v_2 + 3Tv_2$. In the usual way we view V (together with $T : V \to V$) as an $\mathbb{R}[x]$-module. Show that v_1 and v_2 generate V as an $\mathbb{R}[x]$-module, and find $f_1(x)$ and $f_2(x)$ in $\mathbb{R}[x]$, where $f_1(x)|f_2(x)$, such that $V \simeq \mathbb{R}[x]/\big(f_1(x)\big) \oplus \mathbb{R}[x]/\big(f_2(x)\big)$.

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EXAM IN RINGS AND MODULES (MA3201)

Thursday, 9th December 2004
Time: 09:00 – 13:00
Grades to be announced: Thursday, 6th January 2005
Permitted aids: None.

Problem 1 Let

$$R = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & d & e \end{pmatrix} \mid a, b, c, d, e \in \mathbb{C} \right\}.$$

a) Show that R is a ring under the usual addition and multiplication of matrices.

b) Let

$$I = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ c & d & 0 \end{pmatrix} \mid b, c, d \in \mathbb{C} \right\}.$$

Show that I is a two-sided ideal in R, and that I is nilpotent.

c) Show that R/I and $\mathbb{C} \oplus \mathbb{C}$ are isomorphic rings. Is R/I a semisimple ring?

d) How can the two-sided ideals in the ring R/I be described in terms of two-sided ideals in R? Find two maximal two-sided ideals in R.

Problem 2

a) Let $\varphi\colon R \to S$ be a homomorphism of rings. Show that any left S-module M becomes a left R-module by defining

$$r \cdot m = \varphi(r)m$$

for all r in R and m in M.

Recall the following: Let F be a field. Suppose A is an algebra over F; that is, there is a map $F \times A \to A$, written $(\alpha, r) \mapsto \alpha \cdot r$, such that A is a vector space over F and

$$\alpha \cdot (rr') = (\alpha \cdot r)r' = r(\alpha \cdot r')$$

for all α in F, and all r and r' in A.

Assume that $0 \neq 1_A$ in A, where 1_A is the identity in A.

b) Show that $\psi\colon F \to A$ given by $\psi(\alpha) = \alpha \cdot 1_A$, is a homomorphism of rings with $\operatorname{Im}\psi \subseteq Z(A)$. Here

$$Z(A) = \{z \in A \mid za = az \text{ for all } a \in A\}.$$

Also, show that ψ is injective.

c) Suppose that A is a finite dimensional algebra over F; that is, $\dim_F A$ is finite. Show that A is both left artinian and left noetherian.

Let M be a finitely generated left A-module. Show that M is both an artinian and a noetherian A-module.

Problem 3 Let V be a vector space over a field F with $\dim_F V = n < \infty$. Let $T\colon V \to V$ be a non-zero linear transformation. Then V becomes an $F[x]$-module by letting

$$x^i \cdot v = T^i(v)$$

for all v in V and $i \geq 0$. It is not necessary to prove this.

a) Let $\operatorname{Ann}_{F[x]} V = \{g(x) \in F[x] \mid g(x) \cdot v = 0 \text{ for all } v \in V\}$. Show that $\operatorname{Ann}_{F[x]} V$ is an ideal in $F[x]$.

Let $f(x)$ be the minimal polynomial of T. Show that $\operatorname{Ann}_{F[x]} V = (f(x))$.

b) Suppose that T is a non-zero nilpotent linear transformation; that is, $T^l = 0$ for some positive integer l. Show that the minimal polynomial $f(x)$ of T is equal to x^m for some integer m with $0 < m \leq n$.

c) Suppose also here that T is a non-zero nilpotent linear transformation. What is the smallest possible dimension of the kernel of T? And what is the largest possible dimension of the kernel of T?

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Exam in course MA3201 Rings and modules

English
Wednesday November 30, 2005
Time: 09.00-13.00

Permitted aids: none

Grades: 21.12.2005.

Problem 1 Let q be a fixed non-zero element in $\mathbb{C}$, the set of complex numbers. Define the subset R_q of the ring of 4×4-matrices over $\mathbb{C}$ by

$$R_q = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & 0 & a & 0 \\ d & c & -qb & a \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}.$$

a) Show that R_q is a ring.

b) For which q in $\mathbb{C}$ is R_q a commutative ring?

c) For a given element α in $\mathbb{C}$ define the subset

$$I_\alpha = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ \alpha b & 0 & 0 & 0 \\ d & \alpha b & -qb & 0 \end{pmatrix} \mid b, d \in \mathbb{C} \right\}$$

of R_q. Show that I_α is a left ideal in R_q for all α in $\mathbb{C}$.

d) Show that each of the left ideals I_α is generated by one element as a left ideal. Show that $I_\alpha \simeq R/I_{\alpha q}$ as left R-modules.

Problem 2 Let $\mathbb{Q}$ be the field of rational numbers, and let a and b in $\mathbb{Q}$ be different elements. Find all possible rational canonical forms for 4×4-matrices over $\mathbb{Q}$ having

$$(x+a)^2(x+b)$$

as a minimal polynomial.

Problem 3 Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}[x]$ the polynomial ring over $\mathbb{C}$ in one variable x. Let $\alpha \in \mathbb{C}$ be a complex number.

a) Show that the map $\varphi_\alpha \colon \mathbb{C}[x] \to \mathbb{C}$ defined by $\varphi_\alpha(f(x)) = f(\alpha)$ is a surjective ring homomorphism, and use this to show that the ideal generated by $x - \alpha$ is a maximal ideal in $\mathbb{C}[x]$.

b) For which $n \geq 1$ is the ring

$$\begin{pmatrix} \frac{\mathbb{C}[x]}{((x-\alpha)^n)} & \frac{\mathbb{C}[x]}{((x-\alpha)^n)} \\ \frac{\mathbb{C}[x]}{((x-\alpha)^n)} & \frac{\mathbb{C}[x]}{((x-\alpha)^n)} \end{pmatrix}$$

semisimple?

Problem 4 Let R be a ring, and let M be a Noetherian left R-module. Show that any surjective R-homomorphism $f \colon M \to M$ is an isomorphism. (Hint: Consider the chain $\operatorname{Ker} f \subseteq \operatorname{Ker}(f^2) \subseteq \operatorname{Ker}(f^3) \subseteq \cdots$ of submodules of M).

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EXAM IN RINGS AND MODULES (MA3201)

English
Friday 15th December 2006
Time: 09:00–13:00
Permitted aids: None

Grades: 15.01.2007.

Problem 1 Let A be the 3×3 matrix

$$\begin{pmatrix} 1 & 2 & -4 \\ 1 & 2 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

over $\mathbb{C}$, the complex numbers.

a) Find the Smith normal form of the matrix $A - xI_3$ over the ring $\mathbb{C}[x]$, where $\mathbb{C}[x]$ is the polynomial ring in one variable x over $\mathbb{C}$ and I_3 is the 3×3 identity matrix.

b) Find the rational canonical form of the matrix A over $\mathbb{C}$.

c) Find the Jordan canonical form of the matrix A over $\mathbb{C}$.

Problem 2 Let R and S be two rings. An abelian group M is called a S-R-bimodule if M is a left S-module and a right R-module, such that

$$s(mr) = (sm)r$$

for all s in S, for all r in R and for all m in M. Let

$$\Lambda = \begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$$

where M is a S-R-bimodule different from (0). Let $\begin{pmatrix} r & 0 \\ m & s \end{pmatrix}$ and $\begin{pmatrix} r' & 0 \\ m' & s' \end{pmatrix}$ be two elements in Λ. The set Λ becomes an abelian group under the binary operation, $+$, given by

$$\begin{pmatrix} r & 0 \\ m & s \end{pmatrix} + \begin{pmatrix} r' & 0 \\ m' & s' \end{pmatrix} = \begin{pmatrix} r+r' & 0 \\ m+m' & s+s' \end{pmatrix}.$$

Define a binary operation, $\cdot$, on Λ by letting

$$\begin{pmatrix} r & 0 \\ m & s \end{pmatrix} \cdot \begin{pmatrix} r' & 0 \\ m' & s' \end{pmatrix} = \begin{pmatrix} rr' & 0 \\ mr'+sm' & ss' \end{pmatrix}.$$

a) Show that Λ is a ring with 1, when addition, $+$, and multiplication, $\cdot$, is defined as above.

b) Find

(i) an idempotent element different from 0 and 1 in Λ,
(ii) a nilpotent element different from 0 i Λ.

c) Let $I = \{\begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} \mid m \in M\}$. Show that I is a twosided ideal in Λ. Show that $\Lambda/I \simeq R \oplus S$ as rings.

Problem 3 Let k be a field. The map $\varphi\colon k[x]/(x^2) \to k$ given by

$$\varphi(f(x) + (x^2)) = f(0)$$

is a homomorphism of rings. Let $R = k$ and $S = k[x]/(x^2)$.

a) Let M be a left R-module. Show that M becomes a left S-module by defining

$$s \cdot m = \varphi(s)m$$

for all s in S and for all m in M.

b) Let $M = k^2 = \{(a, b) \mid a, b \in k\}$. Then is M a left k-module by letting

$$\alpha(a, b) = (\alpha a, \alpha b)$$

and a right k-module by letting

$$(a, b)\alpha = (a\alpha, b\alpha)$$

for all α in k and for all (a, b) in M. With these module structures M becomes a k-k-bimodule (Do not need to show this). By a) we have that the left k-module M is a left S-module by letting $(f(x) + (x^2)) \cdot m = \varphi(f(x) + (x^2))m$. Show that M is a S-R-bimodule, when $R = k$ and $S = k[x]/(x^2)$.

c) Now let $\Lambda = \left(\begin{smallmatrix} R & 0 \\ M & S \end{smallmatrix}\right)$, where M is as in b), and Λ is a ring as given in Problem 2. Show that Λ is an algebra over k. What is $\dim_k \Lambda$? Decide if Λ is

(i) a left artinian ring,

(ii) a left noetherian ring,

(iii) a semisimple ring.

d) Let J be the left ideal $\{\left(\begin{smallmatrix} 0 & 0 \\ (0,a) & bx+(x^2) \end{smallmatrix}\right) \mid a, b \in k\}$. Consider the left Λ-module $X = \Lambda/J$. Show that $f\colon X \to X$ given by

$$f(\lambda + J) = \lambda \left(\begin{smallmatrix} 0 & 0 \\ (0,0) & 1+(x^2) \end{smallmatrix}\right) + J$$

is a Λ-homomorphism. Find the image $\operatorname{Im} f$ of f. Show that $X = \operatorname{Im} f \oplus Y$ for a submodule Y of X.

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EXAM IN RINGS AND MODULES (MA3201)

Tuesday, 11.th December 2007
Time: 09:00 – 13:00
Grades to be announced: Friday, 21 December 2007
Permitted aids: None.

You should give a reason for all answers.

Problem 1

Let F be a field , R the matrix ring

$$\mathrm{R} = \begin{pmatrix} F & 0 & 0 \\ F & F & 0 \\ F & F & F \end{pmatrix}$$

and

$$\mathrm{I} = \begin{pmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ F & F & 0 \end{pmatrix}$$

a) Show that I is an ideal in R, and that I is nilpotent. Is R a semisimple ring?

b) Show that the factor ring R/I is a semisimple ring.

c) Find 2 different minimal left ideals in R which are isomorphic as R-modules.

d) Find all the ideals in R which contain the ideal I.

Problem 2

Let $\mathbb{Z}$ be the ring of integers, and let R be the ring $\begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$

a) Find all idempotent elements in R, and describe the left ideals of the form Re for an idempotent element e in R.

b) Let I be the left ideal $\begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & 0 \end{pmatrix}$. Find an infinite number of left ideals J in R such that $R = I \oplus J$.

Problem 3

a) Show that the ring of integers $\mathbb{Z}$ is noetherian, and not artinian.

b) Give a proof of the fact that if M is a noetherian module over a ring R, then M is a finitely generated R-module.

Problem 4

a) Denote by $\mathbb{R}$ the real numbers. Find the Smith normal form over $\mathbb{R}[x]$ for the matrix

$$\begin{pmatrix} -3-x & 2 & 0 \\ 1 & -x & 1 \\ 1 & -3 & -2-x \end{pmatrix}$$

Let $V = \mathbb{R}^3$, and let $T = T_A : V \to V$ be the linear transformation given by the matrix $A = \begin{pmatrix} -3 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & -3 & -2 \end{pmatrix}$ with respect to the standard basis for $V = \mathbb{R}^3$. Describe the $\mathbb{R}[x]$-module V (defined using $T : V \to V$) in terms of cyclic $\mathbb{R}[x]$-modules.

b) Let A be a 7 x 7 matrix over $\mathbb{R}$, with characteristic polynomial $c(x) = -(x-1)^2(x-2)^3(x^2+1)$ and with minimal polynomial $m(x)$ of degree 5. Find all the possibilities for the invariant factors for A, (that is, for $xI - A$), and in each case, the associated rational canonical form for A.