REPRESENTATION DIMENSION OF QUASI-TILTED ALGEBRAS

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Abstract. It will be shown that any quasi-tilted algebra over an algebraically closed field has representation dimension at most three, confirming a conjecture of Assem, Platzeck and Trepode.

1. Introduction

The representation dimension of an artin algebra has been introduced by Auslander [2]. It provides a homological criterion for finite representation type. More precisely, Auslander has shown that an artin algebra is representation finite (that is, has only finitely many indecomposable modules up to isomorphism) if and only if its representation dimension is at most two. He expected that for a representation infinite algebra its representation dimension should be a measure of how far the algebra is from being of finite representation type.

However it is not reasonable to expect the representation dimension to measure the size of the module category in the sense of "number of modules". Auslander has shown that any hereditary artin algebra has representation dimension at most three, but there are examples of quotients of such algebras with arbitrarily large representation dimension (see [15, 21]). We believe that representation dimension measures how complicated the homological algebra of the module category is. One implication is supported by results of Bergh [5] and the author [21] giving lower bounds for the representation dimension in terms of certain homological behaviour of the module categories. The result presented here provides support for the other implication:

From a homological point of view, the easiest (non-trivial) categories are hereditary, such as representations of quivers or certain categories of coherent sheaves, for example on a projective line. The module categories most closely related to hereditary categories are those of quasi-tilted algebras. These algebras, introduced by Happel, Reiten, and Smalø [13], are defined to be the endomorphism rings of tilting objects in hereditary categories. In particular quasi-tilted algebras are derived equivalent to hereditary categories. Happel, Reiten, and Smalø have shown that quasi-tilted algebras also admit a simple "internal" homological characterization called almost hereditary (see [13, Chapter II], in particular Theorem 2.3).

Happel [11] has shown that the class of (connected) quasi-tilted algebras can be subdivided into those quasi-tilted algebras which are derived equivalent to

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hereditary algebras, and those which are derived equivalent to weighted projective lines. In particular any quasi-tilted algebra which is not tilted is derived equivalent to a weighted projective line (see Section 2).

For the special case of tilted algebras (that is, quasi-tilted algebras where the hereditary category is representations of a quiver) Assem, Platzeck, and Trepode [1] have proven that they always have representation dimension at most three. This, and the observations above, led them to the conjecture that any quasi-tilted algebra should have representation dimension at most three. Their proof uses the existence of projective or injective objects in the hereditary category, and hence cannot be generalized to the case where this category is coherent sheaves on a weighted projective line.

Weighted projective lines have been introduced by Geigle and Lenzing [9]. From an algebraic geometric point of view one might think of weighted projective lines as projective lines, where in certain (finitely many) points the local ring has been replaced by a semi-local ring. Geigle and Lenzing have shown that the categories of coherent sheaves over a weighted projective line behave in many ways like the categories of coherent sheaves over a smooth projective line (see [9] or Theorem 2.15).

Quasi-tilted algebras derived equivalent to weighted projective lines have been widely studied (for instance in [6, 7, 25]). In particular, the representation theory of canonical algebras (the most prominent family of such quasi-tilted algebras – introduced by Ringel [22] before weighted projective lines or quasi-tilted algebras had first appeared), and more generally concealed canonical algebras (introduced by Lenzing and Meltzer [16]) has been thoroughly investigated (see [17, 18, 19, 22, 23]). However, even for the canonical algebras the value of the representation dimension has so far not been known. We can now fill this gap:

1.1. **Theorem.** *Let Λ be a quasi-tilted algebra over an algebraically closed field. Then we have exactly one of the following:*

- *Λ is semi-simple,*
- *the representation dimension of Λ is two, and Λ is representation finite and tilted, or*
- *the representation dimension of Λ is three, and Λ is representation infinite.*

In particular, any quasi-tilted non-tilted algebra has representation dimension three.

The strategy of the proof is as follows:

Assume Λ is a connected quasi-tilted algebra which is not tilted. It follows from [11] that Λ is tilted from a hereditary category $\mathcal{H}$ which is derived equivalent to coherent sheaves on a weighted projective line. Lenzing and Skowroński [18] classified these categories $\mathcal{H}$. In Section 2 we recall this result, and use it to show that we may assume $\mathcal{H}$ to have a very specific shape (see Theorem 2.17).

The category of Λ-modules can be seen as sitting inside two consecutive copies of $\mathcal{H}$ (see Theorem 2.13). In their proof that the representation dimension of tilted algebras is at most three, Assem, Platzeck, and Trepode used the fact that in their case (where instead of from $\mathcal{H}$ one tilts from a module category of a hereditary algebra H) one can separate the two parts of the module category of Λ (in the different copies of $\operatorname{mod} H$) by a complete slice, which comes from projective or injective H-modules. Unfortunately, in our case $\mathcal{H}$ does not have any projective or injective objects, and a complete slice as in the tilted case does not exist. However we still have some control over the border between the two copies: It consist of a $\mathbb{P}^1_k$-family of tubes.

The crucial idea here is to choose a wing inside each non-homogeneous tube (see Construction 3.6). While this is not as good as the complete slices in the tilted case (there any map from the first to the second part of the module category factors through the complete slice) it still gives us some control: We show in Proposition 4.8 that the wings (in some sense) force vector bundles over the weighted projective line into a much smaller and better understood subcategory, which is equivalent to the category of vector bundles over the projective line $\mathbb{P}^1_k$.

We then show that, starting with a module in the second copy of $\mathcal{H}$ and approximating with injective modules and elements of the wings, we can find restrictions to which line bundles over $\mathbb{P}^1_k$ will occur in the cone (Proposition 6.8). This will help us show that there is a finite collection of line bundles over $\mathbb{P}^1_k$ (see Construction 5.1), which is enough to make up for the non-perfect border between the copies of $\mathcal{H}$.

2. Notation and Background

Throughout this paper we assume k to be an algebraically closed field. All categories occurring are assumed to be k-categories with finite dimensional Hom-spaces.

The main examples of such categories occurring in this paper are the category of modules over a finite dimensional algebra Λ, denoted by $\operatorname{mod}\Lambda$, the category of coherent sheaves over a weighted projective line $\mathbb{X}$, denoted by $\operatorname{coh}\mathbb{X}$ (as introduced by Geigle and Lenzing [9], see also the brief summary of properties in Theorem 2.15 below), and their bounded derived categories $D^b(\operatorname{mod}\Lambda)$ and $D^b(\operatorname{coh}\mathbb{X})$.

For an object X we denote by $\operatorname{add} X$ the category of all direct summands of finite direct sums of copies of X.

For two subcategories $\mathcal{A}$ and $\mathcal{B}$ of some category we denote by $\mathcal{A} \vee \mathcal{B}$ the full subcategory whose objects are direct sums of one object in $\mathcal{A}$ and one object in $\mathcal{B}$.

Representation dimension.

2.1. **Definition** (Auslander [2])**.** Let Λ be a finite dimensional algebra. Then the *representation dimension* of Λ is

$$\operatorname{repdim}\Lambda = \min\{\operatorname{gld}\operatorname{End}_\Lambda(G) \mid G \in \operatorname{mod}\Lambda \text{ generator and cogenerator}\}.$$

Here $\operatorname{gld}\operatorname{End}_\Lambda(G)$ denotes the global dimension of the endomorphism ring, and G being a generator and cogenerator means that all projective and injective modules are in $\operatorname{add} G$.

A generator cogenerator G realizing the minimum in the definition above is called *Auslander generator*.

Auslander's main motivation for defining this homological invariant is the following result.

2.2. **Theorem** (Auslander [2])**.** *Let Λ be a finite dimensional algebra. Then* $\operatorname{repdim}\Lambda \leq 2$ *if and only if Λ has finite representation type.*

We should also mention the following two more recent results which show that the representation dimension is reasonably well behaved.

2.3. **Theorem** (Iyama [14]). *Let Λ be a finite dimensional algebra. Then*

$$\operatorname{repdim} \Lambda < \infty.$$

2.4. **Theorem** (Rouquier [24]). *Let Λ be the exterior algebra of an n-dimensional vector space, $n \in \mathbb{N}_{\geq 1}$. Then*

$$\operatorname{repdim} \Lambda = n + 1.$$

In particular any nonnegative integer different from 1 occurs as the representation dimension of some algebra.

Since our aim in this paper is to show that certain algebras have representation dimension at most 3 we will need to find a generator cogenerator G such that $\operatorname{gld} \operatorname{End}(G) \leq 3$. Our method of verifying this (once we have a candidate G) is the following.

2.5. **Lemma** (implicit by Auslander, explicit in [8]). *Let Λ be a non-semisimple finite dimensional algebra, and $G \in \operatorname{mod} \Lambda$ a generator and cogenerator, and $n \in \mathbb{N}_{\geq 2}$. Then the following are equivalent:*

(1) $\operatorname{gld} \operatorname{End}_\Lambda(G) \leq n$, *and*
(2) *for any $X \in \operatorname{mod} \Lambda$ there is an exact sequence*

$$G_{n-2} \rightarrowtail G_{n-3} \longrightarrow \cdots \longrightarrow G_0 \twoheadrightarrow X$$

with $G_i \in \operatorname{add} G$, such that the induced sequence

$$\operatorname{Hom}_\Lambda(G, G_{n-2}) \rightarrowtail \cdots \longrightarrow \operatorname{Hom}_\Lambda(G, X) \longrightarrow 0$$

is also exact. (In this situation we say that X has a G-resolution of length $n - 2$.)

2.6. **Remark.** The sequence

$$G_{n-2} \rightarrowtail G_{n-3} \longrightarrow \cdots \longrightarrow G_0 \twoheadrightarrow X$$

being a G-resolution just means that the rightmost map is a right G-approximation of X, the next map is induced by a right G-approximation of the kernel of the first, and so on.

In particular, for $n = 3$ the claim "X has a G-resolution of length 1" just means that the kernel of a right G-approximation of X is in $\operatorname{add} G$.

Quasi-tilted algebras.

2.7. **Definition.** An abelian category $\mathcal{H}$ is called *hereditary* if $\operatorname{Ext}^2_{\mathcal{H}}(X, Y) = 0$ for any $X, Y \in \mathcal{H}$. (Here $\operatorname{Ext}^2_{\mathcal{H}}$ is understood as the Yoneda-Ext, see [26].)

2.8. **Definition.** An object T of a hereditary category $\mathcal{H}$ is called *tilting object* if

(1) $\operatorname{Ext}^1_{\mathcal{H}}(T, T) = 0$, and
(2) any object $X \in \mathcal{H}$ with $\operatorname{Hom}_{\mathcal{H}}(T, X) = 0$ and $\operatorname{Ext}^1_{\mathcal{H}}(T, X) = 0$ is the zero object.

Now we are ready to define quasi-tilted algebras.

2.9. **Definition** (Happel, Reiten, and Smalø [13]). A finite dimensional k-algebra Λ is called *quasi-tilted* if $\Lambda \cong \operatorname{End}_{\mathcal{H}}(T)$ for some tilting object T in a hereditary category $\mathcal{H}$. It is called *tilted* if moreover $\mathcal{H}$ can be chosen to be the module category of a hereditary algebra.

For tilted algebras the result we wish to prove here in the more general setup of quasi-tilted algebras has been shown by Assem, Platzeck, and Trepode.

2.10. **Theorem** ([1]). *Let Λ be a tilted algebra. Then* $\operatorname{repdim} \Lambda \leq 3$.

Derived categories of hereditary categories. Since tilting induces a derived equivalence $\operatorname{RHom}(T, -)$ between the categories involved we need to understand the derived categories of hereditary categories. The following description of these derived categories is well-known.

2.11. **Theorem.** *Let $\mathcal{H}$ be a hereditary category. Then the derived category of $\mathcal{H}$ is*

$$D^b(\mathcal{H}) = \vee_{i \in \mathbb{Z}} \mathcal{H}[i].$$

That is, any object in $D^b(\mathcal{H})$ is the direct sum of stalk complexes.

2.12. **Remark.** Assume $\mathcal{A}$ is derived equivalent to $\mathcal{H}$. Then we fix one derived equivalence, and identify $\mathcal{A}$ with its image under

$$\mathcal{A} \hookrightarrow D^b(\mathcal{A}) \xrightarrow{\cong} D^b(\mathcal{H}).$$

In particular, if $\mathcal{H}$ is hereditary, $\mathcal{A}$ can be described by saying which shifts of which objects in $\mathcal{H}$ lie in $\mathcal{A}$.

In the situation of a derived equivalence induced by tilting this amounts to the following:

2.13. **Theorem** ([13, Section I.4]). *Let $\mathcal{H}$ be a hereditary category, $T \in \mathcal{H}$ a tilting object, and $\Lambda = \operatorname{End}_{\mathcal{H}}(T)$. Via the equivalence $\operatorname{RHom}_{\mathcal{H}}(T, -)$ the module category of Λ is identified with*

$$\operatorname{mod} \Lambda = \operatorname{Fac}_{\mathcal{H}} T \vee (\operatorname{Sub}_{\mathcal{H}} \tau T)[1].$$

Classification of hereditary categories with tilting objects. Definition 2.9 suggests that the first step towards understanding quasi-tilted algebras might be studying possible hereditary categories $\mathcal{H}$. Happel has classified all such categories which contain a tilting object up to derived equivalence.

2.14. **Theorem** (Happel [11]). *Let $\mathcal{H}$ be a connected hereditary category with a tilting object. Then $\mathcal{H}$ is derived equivalent to one of the following:*

(1) $\operatorname{mod} H$*, where H is a finite dimensional hereditary k-algebra, or*
(2) $\operatorname{coh} \mathbb{X}$*, where $\mathbb{X}$ is a weighted projective line (introduced by Geigle and Lenzing, see* [9]*).*

Thus our investigation may be split up into these two cases. It will be shown at the end of this section that we may actually restrict ourselves to (a special subcase of) the second case.

Coherent sheaves over a weighted projective line. A weighted projective line $\mathbb{X}$ is the projective line $\mathbb{P}^1_k$ together with a finite set of points $p_1, \ldots, p_r \in \mathbb{P}^1_k$, and, for each of these, a weight $w_i \in \mathbb{N}_{\geq 2}$. For background on weighted projective lines and the coherent sheaves on them see [9]. We recall only the results needed here.

We denote by $\mathtt{VB}$ and $\mathtt{tor}$ the full subcategories of $\operatorname{coh} \mathbb{X}$ in which the objects are vector bundles and torsion sheaves, respectively.

2.15. **Theorem** (see [9])**.** *Let $\mathbb{X}$ be a weighted projective line. Then $\operatorname{coh} \mathbb{X} = \mathtt{VB} \vee \mathtt{tor}$, that is any coherent sheaf on $\mathbb{X}$ is the direct sum of a vector bundle and a torsion sheaf. Moreover $\operatorname{Hom}_{\mathbb{X}}(\mathtt{tor}, \mathtt{VB}) = 0 = \operatorname{Ext}^1_{\mathbb{X}}(\mathtt{VB}, \mathtt{tor})$.*

The category of torsion sheaves decomposes as a coproduct of categories $\mathtt{tor} = \coprod_{p \in \mathbb{P}^1_k} \mathtt{tor}_p$. For $p \notin \{p_1, \ldots, p_r\}$ the category $\mathtt{tor}_p$ is the uniserial finite length category with one simple object (= finite length modules over $k[[x]]$ – its Auslander-Reiten quiver is a homogeneous tube). For $p = p_i$ the category $\mathtt{tor}_p$ is the connected uniserial finite length category with w_i simple objects, that is the category having a tube of rank w_i as its Auslander-Reiten quiver.

The structure of $\mathtt{VB}$ depends on the weights:

- *If there are at most two weights, or if there are three weights $(w_1, 2, 2)$, $(2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$ for some w_1, then $\mathtt{VB}$ consists of only one Auslander-Reiten component.*
- *If the weights are $(2, 2, 2, 2)$, $(2, 3, 6)$, $(2, 4, 4)$, or $(3, 3, 3)$, then the weighted projective line is called tubular. In this case $\mathtt{VB} = \vee_{q \in \mathbb{Q}} \mathtt{VB}_q$. Moreover, $\mathtt{VB}_q \approx \mathtt{tor}$ for any $q \in \mathbb{Q}$, and for $q_1 < q_2$ we have $\operatorname{Hom}_{\mathbb{X}}(\mathtt{VB}_{q_2}, \mathtt{VB}_{q_1}) = 0 = \operatorname{Ext}^1_{\mathbb{X}}(\mathtt{VB}_{q_1}, \mathtt{VB}_{q_2})$.*
- *In all other cases $\mathtt{VB}$ is wild.*

The case $\mathcal{H}$ derived equivalent to $\operatorname{coh} \mathbb{X}$. The hereditary categories $\mathcal{H}$ which are derived equivalent to $\operatorname{coh} \mathbb{X}$ for some weighted projective line $\mathbb{X}$ have been studied by Lenzing and Skowroński in [18]. They obtained the following classification.

2.16. **Theorem.** *Let $\mathcal{H}$ be hereditary and derived equivalent to $\operatorname{coh} \mathbb{X}$ for some weighted projective line $\mathbb{X}$. Then $\mathcal{H}$ is equivalent to one of the following (they are described as indicated in Remark 2.12, that is by saying which shifts of which coherent sheaves on $\mathbb{X}$ are in $\mathcal{H}$, when $\mathcal{H}$ is identified with a subcategory of $D^b(\operatorname{coh} \mathbb{X})$):*

(1) *$\operatorname{mod} H$ for a tame hereditary algebra H,*
(2) *$\mathtt{tor}_{\mathcal{S}}[-1] \vee \mathtt{VB} \vee \mathtt{tor}_{\mathbb{P}^1_k \setminus \mathcal{S}}$ for some $\mathcal{S} \subseteq \mathbb{P}^1_k$. Here and in the following $\mathtt{tor}_{\mathcal{S}} = \coprod_{p \in \mathcal{S}} \mathtt{tor}_p$ denotes the category of all torsion sheaves corresponding to one of the points in $\mathcal{S}$.*
(3) *$(\vee_{q \in \mathbb{Q}_{>x}} \mathtt{VB}_q) \vee \mathtt{tor} \vee \big(\vee_{q \in \mathbb{Q}_{\leq x}} \mathtt{VB}_q[1]\big)$ for some $x \in \mathbb{R}$, if the weights of $\mathbb{X}$ are $(2, 2, 2, 2)$, $(2, 3, 6)$, $(2, 4, 4)$, or $(3, 3, 3)$.*

By Theorem 2.10 any tilted algebra has representation dimension at most three. Therefore we may disregard the first case above.

Now assume we are in the third case above, and $T \in (\vee_{q \in \mathbb{Q}_{>x}} \mathtt{VB}_q) \vee \mathtt{tor} \vee \big(\vee_{q \in \mathbb{Q}_{\leq x}} \mathtt{VB}_q[1]\big)$ is a tilting object. Then there is $\tilde{x} \in \mathbb{Q}$ such that $T \in (\vee_{q \in \mathbb{Q}_{>\tilde{x}}} \mathtt{VB}_q) \vee \mathtt{tor} \vee \big(\vee_{q \in \mathbb{Q}_{\leq \tilde{x}}} \mathtt{VB}_q[1]\big) \approx \operatorname{coh} \mathbb{X}$, where the equivalence is given in [20]. Hence any algebra tilted from a hereditary category coming up in the third case above is also tilted from a hereditary category coming up in the second case above.

Hereditary categories derived equivalent to module categories. Now we investigate the first case of Happel's Theorem (Theorem 2.14). That is we have a hereditary category derived equivalent to a module category.

If the hereditary category is derived equivalent to the module category of a representation finite algebra it is easy to see that it again is the module category of a representation finite algebra. The endomorphism ring of any tilting module (even of any tilting complex in the derived category) is representation finite, that is has representation dimension at most two. Hence we may disregard that case.

If the hereditary category is derived equivalent to a tame hereditary algebra then it is also derived equivalent to a weighted projective line (see [9, 5.4.1]). Hence this hereditary category is treated in Theorem 2.16, and we do not need to treat it here.

Finally assume we have a hereditary category $\mathcal{H}$ which is derived equivalent to the module category of a wild hereditary algebra H. If $\mathcal{H}$ contains a projective object, then by [10] it is equivalent to the module category of a hereditary algebra. Hence by Theorem 2.10 the endomorphism ring of any tilting object has representation dimension at most three. Therefore we may assume that $\mathcal{H}$ does not contain any projective objects. Then, by [10, Proposition 4.8(2)], $\mathcal{H}$ is equivalent to one of the following, where $\mathcal{P}$, $\mathcal{R}$, and $\mathcal{I}$ denote the preprojective, regular, and preinjective components of $\operatorname{mod} H$ respectively.

- $\mathcal{I}[-1] \vee \mathcal{P} \vee \mathcal{R}$, or
- $\mathcal{R} \vee \mathcal{I} \vee \mathcal{P}[1]$.

In the first case one can see that for any tilting object T we have $\tau^{-n}T \in \mathcal{P} \vee \mathcal{R}$ for n sufficiently big, and hence $\operatorname{End}_{\mathcal{H}}(T) = \operatorname{End}_H(\tau^{-n}T)$ is tilted. In the second case dually $\tau^n T \in \mathcal{R} \vee \mathcal{I}$ for sufficiently large n, and again $\operatorname{End}_{\mathcal{H}}(T)$ is tilted.

Conclusion (of this section). We have shown that we may restrict ourselves to the following case.

2.17. **Theorem.** *Assume Λ is connected, quasi-tilted, and not tilted. Then there is a weighted projective line $\mathbb{X}$ and a subset $\mathcal{S} \subseteq \mathbb{P}^1_k$ such that Λ is the endomorphism ring of a tilting object in the hereditary category*

$$\mathcal{H} = \mathtt{tor}_{\mathcal{S}}[-1] \vee \mathtt{VB} \vee \mathtt{tor}_{\mathbb{P}^1_k \setminus \mathcal{S}}, \tag{2.1}$$

defined inside the derived category of $\operatorname{coh} \mathbb{X}$.

3. Construction of an Auslander generator I

From now on we assume the following setup (we may do so by Theorem 2.17 above).

We have a weighted projective line $\mathbb{X}$ with weights $w_1, \ldots, w_r$ attached in points $p_1, \ldots, p_r$, respectively. We have a hereditary category $\mathcal{H}$ inside $D^b(\operatorname{coh} \mathbb{X})$ as in formula (2.1) above. We have a tilting object $T \in \mathcal{H}$. Equivalently, T is a tilting complex in $D^b(\operatorname{coh} \mathbb{X})$ which is of the form

$$T = \bigoplus_{p \in \mathcal{S}} T_p^L \oplus T_{\mathtt{VB}} \oplus \bigoplus_{p \in \mathbb{P}^1_k \setminus \mathcal{S}} T_p^R,$$

with $T_p^L \in \mathtt{tor}_p[-1]$, $T_{\mathtt{VB}} \in \mathtt{VB}$, and $T_p^R \in \mathtt{tor}_p$. Note that $T_{\mathtt{VB}} \neq 0$, since otherwise there would be no maps or extensions from T to any of the homogeneous tubes.

We set $\Lambda = \operatorname{End}_{D^b(\operatorname{coh}\mathbb{X})}(T)$. We will think of $\operatorname{mod}\Lambda$ as a subcategory of $D^b(\operatorname{coh}\mathbb{X})$ as indicated in Remark 2.12 and Theorem 2.13. In particular we identify

$$\operatorname{proj}\Lambda = \operatorname{add}T, \text{ and}$$
$$\operatorname{inj}\Lambda = \operatorname{add}\nu T.$$

(The first identification is done since $R\operatorname{Hom}(T,-)$ maps $\operatorname{add}T$ to $\operatorname{proj}\Lambda$, the second follows from the first by applying ν, see Remark 3.3).

3.1. **Remark.** We will always work inside the triangulated category $D^b(\operatorname{coh}\mathbb{X})$. However, we will make use of the following three abelian subcategories:

- $\operatorname{coh}\mathbb{X}$,
- $\mathcal{H}$,
- $\operatorname{mod}\Lambda$.

Therefore it is always necessary to specify which of the abelian structures we are talking about when using terms like "short exact", "kernel", or "cokernel".

3.2. **Remark.** With the identification of Theorem 2.13 we have

$$\begin{aligned}\operatorname{mod}\Lambda &= \{X \in D^b(\operatorname{mod}\Lambda) \mid \operatorname{Hom}_{D^b(\operatorname{mod}\Lambda)}(\Lambda, X[i]) = 0\,\forall i \neq 0\}\\ &= \{X \in D^b(\operatorname{coh}\mathbb{X}) \mid \operatorname{Hom}_{D^b(\operatorname{coh}\mathbb{X})}(T, X[i]) = 0\,\forall i \neq 0\},\end{aligned}$$

and similarly

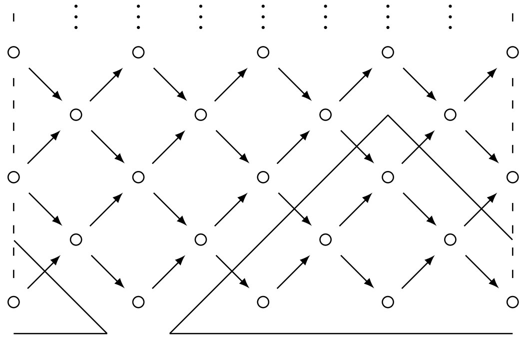
$$\operatorname{mod}\Lambda = \{X \in D^b(\operatorname{coh}\mathbb{X}) \mid \operatorname{Hom}_{D^b(\operatorname{coh}\mathbb{X})}(X, \nu T[i]) = 0\,\forall i \neq 0\}.$$

3.3. **Remark.** Here and throughout the paper ν denotes the Serre functor of $D^b(\operatorname{coh}\mathbb{X})$, and $\tau = \nu[-1]$ the Auslander-Reiten translation in $D^b(\operatorname{coh}\mathbb{X})$. Note that τ coincides with Auslander-Reiten translation in $\operatorname{coh}\mathbb{X}$ and $\mathcal{H}$ whenever the objects in question are in these categories. The Auslander-Reiten translation in $\operatorname{mod}\Lambda$ will be denoted by τ_Λ.

Wings. The rest of this section is devoted to wings in the tubes of torsion sheaves over $\mathbb{X}$. Taking these wings as part of our Auslander generator is the crucial idea of our proof of Theorem 1.1.

3.4. **Definition** (see [22])**.** Let $\mathtt{T}$ be a tube of rank w. A wing in $\mathtt{T}$ is the direct sum of all subquotients in $\mathtt{T}$ of one fixed indecomposable object of length $w-1$.

The following picture illustrates this concept on the example of a tube of rank four. Here the indecomposable direct summands of a wing correspond to the vertices in the triangle in the following picture (where the left and right side are identified):



We now want to choose a wing in every non-homogeneous tube in $\mathtt{tor}$. The following lemma ensures that we can do this in such a way that the part of T in that tube also lies in the wing.

3.5. **Lemma.** *With the notation as above, let $p \in \mathbb{P}^1_k \setminus \mathcal{S}$. Then there is a wing W_p in $\mathtt{tor}_p$ such that $T_p^R \in \operatorname{add} W_p$.*

Proof. Since $T_{\mathtt{VB}} \neq 0$, by [9] there is a simple object $S \in \mathtt{tor}_p$ such that $\operatorname{Hom}_{\mathbb{X}}(T_{\mathtt{VB}}, S) \neq 0$. We choose W_p to be the wing such that $\tau^- S \notin \operatorname{add} W_p$.

Note that $\operatorname{Hom}_{\mathbb{X}}(T_{\mathtt{VB}}, X) \neq 0$ for all indecomposable $X \in \mathtt{tor}_p$ which are not in the wing not containing S. Hence $\operatorname{Ext}^1_{\mathbb{X}}(X, T_{\mathtt{VB}}) = \operatorname{Hom}_{\mathbb{X}}(T_{\mathtt{VB}}, \tau X) \neq 0$ for all indecomposable $X \in \mathtt{tor}_p$ which are not in the wing W_p. Hence $T_p^R \in \operatorname{add} W_p$. □

This lemma tells us that it is always possible to make the following construction.

3.6. **Construction.** For $i \in \{1, \ldots . r\}$ we choose a wing W_{p_i} in $\mathtt{tor}_{p_i}$ such that

- if $p_i \in \mathcal{S}$ then $\nu T^L_{p_i} \in \operatorname{add} W_{p_i}$, and
- if $p_i \notin \mathcal{S}$ then $T^R_{p_i} \in \operatorname{add} W_{p_i}$.

This is possible by Lemma 3.5 and its dual.

Now we set $W = \bigoplus_{i=1}^r W_{p_i}$, and choose $\widetilde{W}$ such that

$$\operatorname{add} \widetilde{W} = \operatorname{mod} \Lambda \cap \operatorname{add} W.$$

Remember that we identify $\operatorname{mod} \Lambda$ with a subcategory of the derived category of $\operatorname{coh} \mathbb{X}$, so the intersection above makes sense.

4. The influence of the wings

Before we collect the remaining parts of our Auslander generator in Section 5 and we get to the more technical parts of the proof, we use this short section to explain why we want the wings to be summands of the Auslander generator, and how these wings affect resolutions.

For comparison we also remark how the complete slice, in Assem, Platzeck, and Trepode's proof (in [1]) that the representation dimension of tilted algebras is at most three, behaves opposed to the wings we have here.

4.1. **Definition.** For an object X in a triangulated category $\mathcal{T}$ we denote by

$$X^{\perp} = \{T \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(X, T[i]) = 0 \, \forall i\}$$

the *complete orthogonal* of X. (Here "complete" refers to the fact that we don't just require the morphisms to certain shifts to vanish.)

4.2. **Remark.** For a tilted algebra and a complete slice S as in the proof in [1], the category $S^\perp$ is zero.

Here we are interested in the category completely orthogonal to the wings. While it does not vanish, as in the tilted case, the following theorem shows that it is $\operatorname{coh}\mathbb{P}^1_k$, a category we know comparatively well.

4.3. **Theorem.** *With the notation above we have*

$$W^\perp = \widetilde{W}^\perp \approx D^b(\operatorname{coh}\mathbb{P}^1_k),$$

and the equivalence is induced by an equivalence $\widetilde{W}^\perp \cap \operatorname{coh}\mathbb{X} \approx \operatorname{coh}\mathbb{P}^1_k$.

4.4. **Notation.** We will identify along the equivalence of Theorem 4.3. That is, we will see coherent sheaves on $\mathbb{P}^1_k$ as special coherent sheaves on $\mathbb{X}$.

Proof of 4.3. We first construct a partial tilting object $V \in \operatorname{add}\widetilde{W}$ tube by tube as follows:

Assume first $p_i \in \mathcal{S}$. Then $\operatorname{add}\widetilde{W}_{p_i} = \{X \in \operatorname{add}W_{p_i} \mid \operatorname{Ext}^1(X, \nu T_{p_i}) = 0\}$ (see Remark 3.2). Since $\nu T_{p_i} \in \operatorname{add}W_{p_i}$ one easily verifies that the longest summand of the wing and all its subobjects are in $\operatorname{add}\widetilde{W}_{p_i}$. We choose V_{p_i} to be the direct sum over these $w_i - 1$ objects.

Dually one can construct V_{p_i} for $p_i \in \mathbb{P}^1_k \setminus \mathcal{S}$. Then we set $V = \bigoplus_{i=1}^r V_{p_i}$. It is clear that $\operatorname{Ext}^1_{\mathbb{X}}(V,V) = 0$, and by construction $V \in \operatorname{add}\widetilde{W}$.

Since $\operatorname{Ext}^1_{\mathbb{X}}(V,V) = 0$, by applying [12, Corollary 2.8] repeatedly, one obtains that $V^\perp \cap \operatorname{coh}\mathbb{X}$ is a hereditary category with a tilting object.

Since this hereditary category contains a $\mathbb{P}^1_k$-family of homogeneous tubes, and there are no maps from these tubes to anything else, the classification results (Theorems 2.14 and 2.16) imply that $V^\perp \cap \operatorname{coh}\mathbb{X} \approx \operatorname{coh}\mathbb{P}^1_k$. Extending this equivalence to shifts we obtain $V^\perp \approx D^b(\operatorname{coh}\mathbb{P}^1_k)$.

Finally note that any indecomposable direct summand of W occurs in a short exact sequence, where the other two terms are direct summands of V. Therefore $V^\perp \subseteq W^\perp$. Since moreover $\operatorname{add}V \subseteq \operatorname{add}\widetilde{W} \subseteq \operatorname{add}W$ we have $V^\perp = \widetilde{W}^\perp = W^\perp$. □

Now remember that $\operatorname{mod}\Lambda$ lies in $\mathcal{H} \vee \mathcal{H}[1]$, and that the largest (and least controlled) part of $\mathcal{H}$ is $\mathtt{VB}$. Since the wings lie in $\mathtt{tor}$ there are no maps to $\mathtt{VB}$, and (as coherent sheaves over $\mathbb{X}$) the approximation of an object in $\mathtt{VB}[1]$ by W is a universal extension. Hence we have to understand such universal extensions.

4.5. **Definition.** Let $\mathcal{A}$ be an abelian category, and $X, Y \in \mathcal{A}$. A short exact sequence $Y' \rightarrowtail E \twoheadrightarrow X$ is called a *universal extension* of X by objects in $\operatorname{add}Y$, if

(1) $Y' \in \operatorname{add}Y$, and
(2) any other short exact sequence $Y'' \rightarrowtail F \twoheadrightarrow X$ with $Y'' \in \operatorname{add}Y$ is a pushout of the first short exact sequence.

It is the *minimal universal extension* if moreover

(3) the map $E \twoheadrightarrow X$ is right minimal, or, equivalently, the sequence does not contain a direct summand of the form $Y'' \rightarrowtail Y'' \twoheadrightarrow 0$ with $Y'' \neq 0$.

The minimal universal extension of X by objects in $\operatorname{add} Y$ is unique up to isomorphism. Whenever we refer to *the* universal extension we mean the minimal one.

Dually one defines universal extensions of objects in $\operatorname{add} X$ by Y.

Note that if $\dim_k \operatorname{Ext}^1_{\mathcal{A}}(X, Y) < \infty$ (so in particular in all categories we consider in this paper) then minimal universal extensions exist.

4.6. **Remark.** For $\mathcal{H} = \operatorname{mod} H$ with H a hereditary algebra, and the complete slice $DH \in D^b(\mathcal{H})$ as in [1], for any $X \in \mathcal{H}$ the universal extension of an object in $\operatorname{add} DH$ by X lies in $\operatorname{add} DH$.

Here we look at universal extensions with wings, and where they lie.

4.7. **Construction.** Let $X \in \mathtt{VB}$. We denote by $E(X)$ the universal extension (in $\operatorname{coh} \mathbb{X}$) of objects in $\operatorname{add} \widetilde{W}$ by X.

4.8. **Proposition.** *For any $X \in \mathtt{VB}$ we have $E(X) \in \mathtt{VB}_{\mathbb{P}^1_k} \vee \operatorname{add} \widetilde{W}$. Here $\mathtt{VB}_{\mathbb{P}^1_k}$ denotes the vector bundles over $\mathbb{P}^1_k$, which are identified with certain vector bundles over $\mathbb{X}$ via the equivalence of Theorem 4.3.*

Proof. Let $E(X)_{\mathtt{VB}}$ and $E(X)_{\mathtt{tor}}$ be the vector bundle and torsion part of $E(X)$, respectively. Then we can construct the following pushout diagram in $\operatorname{coh} \mathbb{X}$.

$$\begin{array}{ccccc}
 & & E(X)_{\mathtt{tor}} & \longrightarrow & W' \\
 & & \downarrow & & \downarrow \\
X & \rightarrowtail & E(X) & \twoheadrightarrow & \widetilde{W}' \\
\downarrow & & \downarrow & \text{PO} & \downarrow \\
K & \rightarrowtail & E(X)_{\mathtt{VB}} & \twoheadrightarrow & W''
\end{array}$$

Here $\widetilde{W}' \in \operatorname{add} \widetilde{W}$, and K and W' are kernels. Since W' and W'' are sub- and factor objects of $\widetilde{W}'$ they are in $\operatorname{add} W$. By the snake lemma the kernel of the left vertical map is a subobject of $E(X)_{\mathtt{tor}}$, hence a torsion sheaf. Since X has no non-zero torsion sheaves as subsheaves we have $X = K$, and the pushout is actually exact. Therefore also the top horizontal map is an isomorphism.

Now we decompose $\widetilde{W}' = \widetilde{W}'_{\mathcal{S}} \oplus \widetilde{W}'_{\mathbb{P}^1_k \setminus \mathcal{S}}$ with $\widetilde{W}'_{\mathcal{S}} \in \mathtt{tor}_{\mathcal{S}}$ and $\widetilde{W}'_{\mathbb{P}^1_k \setminus \mathcal{S}} \in \mathtt{tor}_{\mathbb{P}^1_k \setminus \mathcal{S}}$. We decompose W' and W'' similarly.

We now factor the map above as indicated in the following diagram.

$$\begin{array}{ccccc}
X & \rightarrowtail & E(X) & \twoheadrightarrow & \widetilde{W}'_{\mathcal{S}} \oplus \widetilde{W}'_{\mathbb{P}^1_k \setminus \mathcal{S}} \\
\| & & \downarrow & \text{exact} & \downarrow \\
X & \rightarrowtail & ? & \twoheadrightarrow & \widetilde{W}'_{\mathcal{S}} \oplus W''_{\mathbb{P}^1_k \setminus \mathcal{S}} \\
\| & & \downarrow & \text{exact} & \downarrow \\
K & \rightarrowtail & E(X)_{\mathtt{VB}} & \twoheadrightarrow & W''_{\mathcal{S}} \oplus W''_{\mathbb{P}^1_k \setminus \mathcal{S}}
\end{array}$$

Since $\operatorname{mod} \Lambda$ is closed under quotients in $\mathcal{H}$ it is also closed under quotients in the tubes $\mathtt{tor}_p$ with $p \in \mathbb{P}^1_k \setminus \mathcal{S}$. Hence $\widetilde{W}'_{\mathcal{S}} \oplus W''_{\mathbb{P}^1_k \setminus \mathcal{S}} \in \operatorname{add} \widetilde{W}$. Since the top sequence in the diagram is a universal extension this means $\widetilde{W}'_{\mathbb{P}^1_k \setminus \mathcal{S}} = W''_{\mathbb{P}^1_k \setminus \mathcal{S}}$, and hence $W'_{\mathbb{P}^1_k \setminus \mathcal{S}} = 0$.

Now note that (dually to the argument above) $\operatorname{mod} \Lambda$ is closed under subobjects in the tubes $\mathtt{tor}_p$ with $p \in \mathcal{S}$. Hence $E(X)_{\mathtt{tor}} = W' = W'_{\mathcal{S}} \in \operatorname{add} \widetilde{W}$.

It remains to see that $E(X)_{\mathtt{VB}} \in \mathtt{VB}_{\mathbb{P}^1_k}$. Note that $\operatorname{Hom}_{D^b(\operatorname{coh} \mathbb{X})}(\widetilde{W}, \mathtt{VB}[i]) = 0$ for all $i \neq 1$. Hence it suffices to show $\operatorname{Ext}^1_{\mathbb{X}}(\widetilde{W}, E(X)) = 0$. Let $E(X) \rightarrowtail H \twoheadrightarrow \widetilde{W}$ be any element of this Ext-group. We construct the pushout as indicated in the following diagram.

$$\begin{array}{ccccc} X & \rightarrowtail & E(X) & \twoheadrightarrow & \widetilde{W}' \\ \| & & \downarrow & \text{PO} & \downarrow \\ X & \rightarrowtail & H & \twoheadrightarrow & \widetilde{W}'' \\ & & \downarrow & & \downarrow \\ & & \widetilde{W} & = & \widetilde{W} \end{array}$$

Since $\widetilde{W}''$ is an extension of $\widetilde{W}$ by $\widetilde{W}'$ we have $\widetilde{W}'' \in \operatorname{add} \widetilde{W}$. Since the upper sequence is a universal extension it is a direct summand of the middle horizontal sequence. In particular the short exact sequence $E(X) \rightarrowtail H \twoheadrightarrow \widetilde{W}$ splits. Therefore $\operatorname{Ext}^1_{\mathbb{X}}(\widetilde{W}, E(X)) = 0$, and hence $E(X)_{\mathtt{VB}} \in \mathtt{VB} \cap \widetilde{W}^{\perp} = \mathtt{VB}_{\mathbb{P}^1_k}$. □

5. Construction of an Auslander Generator II

We now collect the remaining, more technical summands of our Auslander generator. The reason for these parts to be necessary are the following limitations of Proposition 4.8 opposed to Remark 4.6.

- The universal extension can (and will) include line bundles on $\mathbb{P}^1_k$, and
- Proposition 4.8 only applies to $\mathtt{VB}$, not to all of $\mathcal{H}$.

Line bundles orthogonal to the wings.

5.1. **Construction.** With E as in Construction 4.7 we set

$$\begin{aligned} d_{\min} &= \min\{i \in \mathbb{Z} \mid \mathcal{O}_{\mathbb{P}^1_k}(i) \text{ is isomorphic to a direct} \\ &\qquad \text{summand of } E(T_{\mathtt{VB}})\} \\ d_{\max} &= \max(\{i \in \mathbb{Z} \mid \mathcal{O}_{\mathbb{P}^1_k}(i) \text{ is isomorphic to a direct} \\ &\qquad \text{summand of } E(T_{\mathtt{VB}} \oplus \tau^2 T_{\mathtt{VB}})\} \cup \{d_{\min} + 1\}) \end{aligned}$$

By choosing a special equivalence in Theorem 4.3 we may (and will for the rest of the paper) assume $d_{\min} = 0$. We then write $d = d_{\max}$.

We set

$$L = \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^1_k}(i).$$

To see that this is a legal ingredient for the Auslander generator we need the following result.

5.2. **Lemma.** *The line bundle* $\mathcal{O}_{\mathbb{P}^1_k}(i)$ *is in* $\operatorname{mod} \Lambda$ *if and only if* $i \geq 0$.

Proof. Since $\mathcal{O}_{\mathbb{P}^1_k}$ is a direct summand of an extension of $T_{\mathtt{VB}}$ with an object in $\operatorname{add} \widetilde{W}$ it is in $\operatorname{mod} \Lambda$. Now recall (Theorem 2.13) that $\operatorname{mod} \Lambda \cap \mathcal{H}$ is closed under quotients. Then the "if"-part of the lemma follows from the fact that $\mathcal{O}_{\mathbb{P}^1_k}(i) \in \operatorname{Fac}_{\mathcal{H}} \mathcal{O}_{\mathbb{P}^1_k}$ for $i \geq 0$.

For the converse assume $\mathcal{O}_{\mathbb{P}^1_k}(i) \in \operatorname{mod} \Lambda$. Again by Theorem 2.13 this means that $\mathcal{O}_{\mathbb{P}^1_k}(i) \in \operatorname{Fac}_{\mathcal{H}} T$. In particular there is non-zero map $T_{\mathtt{VB}} \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(i)$. Since $\operatorname{Hom}(\widetilde{W}, \mathcal{O}_{\mathbb{P}^1_k}(i)[1]) = 0$ this map factors through $T_{\mathtt{VB}} \rightarrowtail E(T_{\mathtt{VB}})$. Now $i \geq 0$ follows from the convention in Construction 5.1 and the fact that there are no non-zero maps $\mathcal{O}_{\mathbb{P}^1_k}(i) \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(j)$ for $i > j$. □

Left and right ends.

5.3. **Construction.** We choose $\overleftarrow{T}$ and $\overrightarrow{T}$ such that

$$\operatorname{add} \overleftarrow{T} = \{X \in \operatorname{mod} \Lambda \mid \tau_\Lambda X \in \operatorname{mod} \Lambda \cap \mathtt{tor}[-1]\}, \text{ and}$$

$$\operatorname{add} \nu \overrightarrow{T} = \{X \in \operatorname{mod} \Lambda \mid \tau_\Lambda^- X \in \operatorname{mod} \Lambda \cap \mathtt{tor}[1]\}.$$

Choosing such a $\overleftarrow{T}$ is possible since $\operatorname{mod} \Lambda \cap \mathtt{tor}[-1] = \prod_{p \in \mathcal{S}} \operatorname{Fac}_{\mathtt{tor}_p[-1]} T_p^L$ has only finitely many indecomposables. Dually it is possible to choose $\overrightarrow{T}$.

Note that $T \in \operatorname{add} \overleftarrow{T}$ and $T \in \operatorname{add} \overrightarrow{T}$.

The Auslander generator.

5.4. **Theorem.** *Let $\mathcal{H}$ be a hereditary category and $T \in \mathcal{H}$ tilting as described at the beginning of this section. Then for the generator cogenerator*

$$G = \overleftarrow{T} \oplus L \oplus \widetilde{W} \oplus \nu \overrightarrow{T}$$

of $\operatorname{mod} \operatorname{End}(T)$ *we have*

$$\operatorname{gld} \operatorname{End}(G) = 3,$$

and hence

$$\operatorname{repdim} \operatorname{End}(T) \leq 3.$$

5.5. **Remark.** One easily sees that in the setup of Theorem 5.4 above all homogeneous tubes in $\mathtt{tor}$ lie in $\operatorname{mod} \operatorname{End}(T)$. Hence $\operatorname{End}(T)$ is representation infinite, and we have

$$\operatorname{repdim} \operatorname{End}(T) = 3.$$

We conclude this section by pointing out that Theorem 5.4 implies Theorem 1.1.

Proof of Theorem 1.1. Clearly it suffices to prove the theorem for connected algebras. By Theorem 2.10 and Auslander's characterization of finite representation type the theorem holds for tilted algebras. In Theorem 2.17 we have seen that any quasi-tilted algebra which is not tilted comes up in the setup of Theorem 5.4, hence has representation dimension at most three. Finally, either by Remark 5.5 or by [13, Corollary II.3.6], any quasi-tilted non-tilted algebra is representation infinite, and hence has representation dimension exactly three. □

Since Assem, Platzeck, and Trepode have shown (see [1, Theorem 4.1]) that any connected Laura-algebra which is not quasi-tilted has representation dimension at most three, the following is an immediate consequence of Theorem 1.1.

5.6. **Corollary.** *Let Λ be a Laura-algebra. Then the representation dimension of Λ is at most three.*

6. Proof of Theorem 5.4

By Lemma 2.5 we have to show that any Λ-module has a G-resolution of length 1, that is that the kernel of its right minimal $\operatorname{add} G$-approximation is again in $\operatorname{add} G$. We may restrict to indecomposable modules.

Clearly we may disregard the case of an indecomposable module in $\operatorname{add} G$. In particular we do not have to look at indecomposable modules in $\operatorname{mod}\Lambda \cap \mathtt{tor}[-1]$ or $\operatorname{mod}\Lambda \cap \mathtt{tor}[1]$. The following three cases remain:

- An indecomposable module in $\operatorname{mod}\Lambda \cap \mathtt{VB}$,
- an indecomposable module in $\operatorname{mod}\Lambda \cap \mathtt{tor}$, or
- an indecomposable module in $\operatorname{mod}\Lambda \cap \mathtt{VB}[1]$.

We will gather the result for these three cases in Propositions 6.5, 6.7, and 6.10. Before we distinguish the cases we have the following general observations on approximations by $\nu\overrightarrow{T}$ and $\overleftarrow{T}$.

6.1. **Lemma.** *Assume X is an indecomposable Λ-module, and $X \notin \operatorname{add}\nu\overrightarrow{T}$. Then any map $\nu\overrightarrow{T} \longrightarrow X$ factors through an injective module.*

Proof. Assume $X \in \operatorname{mod}\Lambda$ is indecomposable, such that $\overline{\operatorname{Hom}}_\Lambda(\nu\overrightarrow{T}, X) \neq 0$. (Here $\overline{\operatorname{Hom}}_\Lambda$ denotes morphisms modulo such morphisms that factor through injective modules. Dually $\underline{\operatorname{Hom}}_\Lambda$ denotes morphisms modulo ones factoring through projective modules.) By the Auslander-Reiten formula (see for instance [4])

$$\underline{\operatorname{Hom}}_\Lambda(\underbrace{\tau_\Lambda^- \nu\overrightarrow{T}}_{\in \operatorname{mod}\Lambda\cap\mathtt{tor}[1]}, \tau_\Lambda^- X) = \overline{\operatorname{Hom}}_\Lambda(\nu\overrightarrow{T}, X) \neq 0.$$

Since $\operatorname{mod}\Lambda \cap \mathtt{tor}[1]$ is closed under successors also $\tau_\Lambda^- X \in \operatorname{mod}\Lambda \cap \mathtt{tor}[1]$, and hence $X \in \operatorname{add}\nu\overrightarrow{T}$. □

6.2. **Lemma.** *Assume there is an exact sequence of Λ-modules*

$$G'' \rightarrowtail G' \xrightarrow{f} X \twoheadrightarrow C$$

with $G', G'' \in \operatorname{add} G$, $C \in \operatorname{mod}\Lambda \cap \operatorname{coh}\mathbb{X}$ and such that the map

$$\operatorname{Hom}(L \oplus \widetilde{W} \oplus \nu\overrightarrow{T}, G') \xrightarrow{f_*} \operatorname{Hom}(L \oplus \widetilde{W} \oplus \nu\overrightarrow{T}, X)$$

is onto. Then G-resol.dim $X \leq 1$.

Before we can prove this, we need the following two results.

6.3. **Lemma.** *Let $X \in \operatorname{mod}\Lambda \cap (\mathtt{tor}[-1] \vee \mathtt{VB} \vee \mathtt{tor})$. Then $\Omega X \in \operatorname{add}\overleftarrow{T}$. (Here Ω denotes the syzygy as Λ-module.)*

Dually, for $X \in \operatorname{mod}\Lambda \cap (\mathtt{tor} \vee \mathtt{VB}[1] \vee \mathtt{tor}[1])$ we have $\mho X \in \nu\overrightarrow{T}$.

Proof. Let $X \in \operatorname{mod}\Lambda \cap (\mathtt{tor}[-1] \vee \mathtt{VB} \vee \mathtt{tor})$. Let Ω' be an indecomposable direct summand of ΩX. By the Auslander-Reiten formula

$$\begin{aligned}\overline{\operatorname{Hom}}_\Lambda(\tau_\Lambda\Omega', \tau_\Lambda\Omega X) &= D\operatorname{Ext}^1(\Omega X, \tau_\Lambda\Omega') = D\operatorname{Ext}^2(X, \tau_\Lambda\Omega') \\ &= D\operatorname{Hom}(X[-2], \tau_\Lambda\Omega').\end{aligned}$$

(Here D denotes the duality $\operatorname{Hom}(-,k)$.) If $\tau_\Lambda \Omega' = 0$ then $\Omega' \in \operatorname{add} \overleftarrow{T}$ as claimed. Otherwise all the terms above are non-zero. Hence $X \in \mathtt{tor}$ and $\tau_\Lambda \Omega' \in \mathtt{tor}[-1]$. Therefore $\Omega' \in \operatorname{add} \overleftarrow{T}$ as claimed also in this case.

Clearly the second part of the lemma is dual to the first. $\square$

6.4. **Lemma.** *Let $X \twoheadrightarrow \overleftarrow{T}'$ be a right minimal epimorphism in* $\operatorname{mod} \Lambda$, *with $\overleftarrow{T}' \in \operatorname{add} \overleftarrow{T}$. Then $X \in \operatorname{add} \overleftarrow{T}$.*

Dually, if $\nu \overrightarrow{T}' \rightarrowtail X$ is a left minimal monomorphism in $\operatorname{mod} \Lambda$, *with $\overrightarrow{T}' \in \operatorname{add} \overrightarrow{T}$, then $X \in \operatorname{add} \nu \overrightarrow{T}$.*

Proof. Before we start the actual proof we need the following observation on Λ-modules in $\mathtt{tor}[-1]$:

The indecomposable objects in $\operatorname{mod} \Lambda \cap \mathtt{tor}[-1]$ are partially ordered by

$$X \preceq Y \iff \exists X \xrightarrow{\neq 0} X_1 \xrightarrow{\neq 0} \cdots \xrightarrow{\neq 0} X_n \xrightarrow{\neq 0} Y \quad \text{with } X_1, \ldots, X_n \text{ indecomposable.}$$

This follows from the facts that the indecomposable objects in $\operatorname{mod} \Lambda \cap \mathtt{tor}_p[-1]$ all lie in the wing $\operatorname{add} \nu^{-1} W_p$, and that modules in different tubes are incomparable.

Now we prove the lemma. We may assume $\overleftarrow{T}'$ is indecomposable, and the map $X \twoheadrightarrow \overleftarrow{T}'$ not split epi. We assume further, contrary to the claim of the lemma, that $X = X' \oplus X''$ with X' indecomposable and $X' \notin \operatorname{add} \overleftarrow{T}$. This setup gives rise to a short exact sequence

$$K \rightarrowtail X' \oplus X'' \twoheadrightarrow \overleftarrow{T}'$$

without any split components. In particular $\operatorname{Ext}^1(\overleftarrow{T}, K') \twoheadleftarrow \operatorname{Hom}(K', \tau \overleftarrow{T}) \neq 0$ for any indecomposable summand K' of K. Hence $K \in \operatorname{mod} \Lambda \cap \mathtt{tor}[-1]$.

We may assume this sequence to be maximal in the following sense: For any $K' \in \operatorname{mod} \Lambda \cap \mathtt{tor}[-1]$ and $X''' \in \operatorname{mod} \Lambda$ such that there is an exact sequence

$$K' \rightarrowtail X' \oplus X''' \twoheadrightarrow \overleftarrow{T}'$$

there is at least one indecomposable direct summand of K' which is no proper successor of any indecomposable summand of K.

Now we denote by $K \rightarrowtail \vartheta^- K \twoheadrightarrow \tau^- K$ the almost split sequence starting in K (if K is not indecomposable we add up the almost split sequences). Then we obtain the following diagram, where the center map exists by the factorization property of almost split sequences.

$$\begin{array}{ccccc} K & \rightarrowtail & X' \oplus X'' & \twoheadrightarrow & \overleftarrow{T}' \\ \| & & \uparrow & & \uparrow \\ K & \rightarrowtail & \vartheta^- K & \twoheadrightarrow & \tau^- K \end{array}$$

The right square gives rise to a short exact sequence

$$\vartheta^- K \rightarrowtail X' \oplus X'' \oplus \tau^- K \twoheadrightarrow \overleftarrow{T}'.$$

By general Auslander-Reiten theory (see [4]) we have $\vartheta^- K \in \operatorname{add} \overleftarrow{T}$. Hence the component $\vartheta^- K \longrightarrow X'$ is a radical map. We split off all direct summands of

$\vartheta^- K$ which are mapped isomorphically to a direct summand of the middle term of the above sequence, and obtain a short exact sequence

$$K' \rightarrowtail X' \oplus X''' \twoheadrightarrow \overleftarrow{T}',$$

with K' a direct summand of $\vartheta^- K$. This contradicts the maximality of our choice of K.

Therefore all direct summands of X have to be in add $\overleftarrow{T}$.

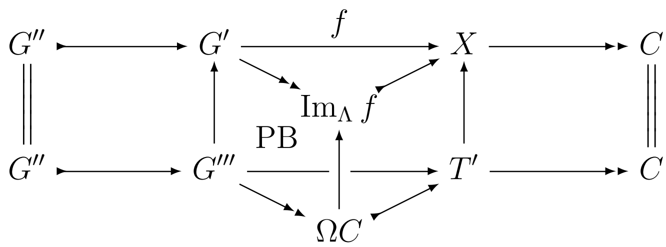
The second part of the lemma is dual. □

Now we are ready to show Lemma 6.2.

Proof of 6.2. In the setup of the lemma we need to extend f such that also any map from $\overleftarrow{T}$ to X factors through it. We do so in two steps. First we show that we may assume f to be onto:

We take a projective cover T' of C. Its kernel (as Λ-modules) ΩC is in add $\overleftarrow{T}$ by Lemma 6.3.

We now have the following diagram



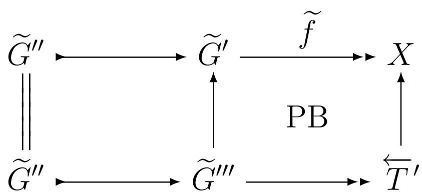
where the map $T' \longrightarrow X$ exists since T' is projective, the map $\Omega C \longrightarrow \operatorname{Im}_\Lambda f$ is the kernel morphism, and G''' is the pullback of the square to its upper right. By Lemma 6.4 we have $G''' \in \operatorname{add} G$.

The central square can be seen as a short exact sequence. By setting $\widetilde{G}' = G' \oplus T'$ and $\widetilde{G}'' = G'''$ we obtain

$$\widetilde{G}'' \rightarrowtail \widetilde{G}' \overset{\tilde{f}}{\twoheadrightarrow} X$$

as in the lemma, but with the additional condition that $\tilde{f}$ is an epimorphism of Λ-modules.

Now we choose a right $\overleftarrow{T}$-approximation $\overleftarrow{T}' \longrightarrow X$. Then we consider the following pullback diagram.



By Lemma 6.4 we have $\widetilde{G}''' \in \operatorname{add} G$. Moreover the right square turns into the short exact sequence

$$\widetilde{G}''' \rightarrowtail \widetilde{G}' \oplus \overleftarrow{T}' \twoheadrightarrow X$$

which is a (not necessary minimal) G-resolution of X. □

We now distinguish the three cases according to where indecomposable modules could lie.

Modules in $\operatorname{mod}\Lambda \cap \mathtt{VB}$**.** We assume $X \in \operatorname{mod}\Lambda \cap \mathtt{VB}$. Then a right G-approximation of X is a right $(\overleftarrow{T} \oplus L)$-approximation.

We denote by H the trace of $\mathcal{O}_{\mathbb{P}^1_k}$ in X as coherent sheaves on $\mathbb{X}$. Hence, in $\operatorname{coh}\mathbb{X}$, we have a short exact sequence

$$H \overset{\iota}{\rightarrowtail} X \twoheadrightarrow C_0$$

with $C_0 \in \operatorname{coh}\mathbb{X}$. Since $H \in \operatorname{Fac}_{\mathbb{X}} \mathcal{O}_{\mathbb{P}^1_k}$ and $H \in \mathtt{VB} \subset \mathcal{H}$ we have $H \in \operatorname{Fac}_{\mathcal{H}} \mathcal{O}_{\mathbb{P}^1_k} \subset \operatorname{mod}\Lambda$. Applying $\operatorname{Hom}(\widetilde{W}, -)$ to a short exact sequence $\operatorname{Ker} \rightarrowtail \mathcal{O}^?_{\mathbb{P}^1_k} \twoheadrightarrow H$ of coherent sheaves over $\mathbb{X}$ one sees that $\operatorname{Ext}^1_{\mathbb{X}}(\widetilde{W}, H) = 0$. Since $H \in \mathtt{VB}$ this means $H \in \mathtt{VB}_{\mathbb{P}^1_k}$, and moreover since $H \in \operatorname{mod}\Lambda$ we have that H is the direct sum of copies of objects of the form $\mathcal{O}_{\mathbb{P}^1_k}(i)$ with $i \geq 0$ (Lemma 5.2). In particular there is a short exact sequence of coherent sheaves on $\mathbb{X}$

$$L'' \rightarrowtail L' \twoheadrightarrow H$$

with L' and $L'' \in \operatorname{add} L$, where the map $L' \twoheadrightarrow H$ can be chosen to be a right L-approximation of H. Since all three objects are Λ-modules, this is also a short exact sequence in $\operatorname{mod}\Lambda$.

We now consider ι as a map of Λ-modules. We denote its kernel and cokernel by K and C, respectively.

Since ι is a monomorphism in $\operatorname{coh}\mathbb{X}$ we have $K \in \mathtt{tor}_{\mathcal{S}}[-1]$. Now let $\widetilde{K}$ be the kernel of the composition $L' \twoheadrightarrow H \longrightarrow X$ of maps of Λ-modules. The following diagram of Λ-modules shows that $\widetilde{K}$ is an extension of K by L''.

$$\begin{array}{ccccc}
K & \rightarrowtail & H & \twoheadrightarrow & \operatorname{Im}_\Lambda \iota \\
\uparrow & & \uparrow & & \| \\
\widetilde{K} & \rightarrowtail & L' & \twoheadrightarrow & \operatorname{Im}_\Lambda \iota \\
\uparrow & & \uparrow & & \\
L'' & = & L'' & &
\end{array}$$

Since $\operatorname{Ext}^1_\Lambda(K, L'') = 0$ (because $K \in \mathtt{tor}_{\mathcal{S}}[-1]$ and $L'' \in \mathtt{VB}$) we have $\widetilde{K} = K \oplus L''$.

Now we focus on the cokernel C. Note that there is a triangle

$$K[1] \xrightarrow{h} \underbrace{\operatorname{Cone}(\iota)}_{=C_0} \longrightarrow C \longrightarrow K[2].$$

Since $C_0 \in \operatorname{coh}\mathbb{X}$ the map h can only be non-zero on direct summands of C_0 which lie in $\mathtt{tor}_{\mathcal{S}}$. Hence $C \in \operatorname{mod}\Lambda \cap (\operatorname{coh}\mathbb{X} \vee \mathtt{tor}_{\mathcal{S}}[1]) = \operatorname{mod}\Lambda \cap \operatorname{coh}\mathbb{X}$.

Now the exact sequence

$$K \oplus L'' \rightarrowtail L' \longrightarrow X \twoheadrightarrow C$$

of Λ-modules satisfies the conditions of Lemma 6.2. Therefore we have shown

6.5. **Proposition.** *Assume we are in the setup if Theorem 5.4, and* $X \in \operatorname{mod}\Lambda \cap \mathtt{VB}$. *Then* G-resol.dim $X \leq 1$.

Modules in $\operatorname{mod}\Lambda \cap \mathtt{tor}$**.** Now we assume X to be indecomposable in $\operatorname{mod}\Lambda \cap \mathtt{tor}_p$ for some $p \in \mathbb{P}^1_k$. We may assume that $X \notin \operatorname{add} W_p$. As before we denote by H the trace of $\mathcal{O}_{\mathbb{P}^1_k}$ in X as coherent sheaves over $\mathbb{X}$. Moreover we denote by H_0 the trace of W_p in X. Note that H_0 is the trace of the unique longest summand of W_p. In particular it belongs to $\operatorname{Fac}_{\mathbb{X}} \mathcal{O}_{\mathbb{P}^1_k} \cap \operatorname{Fac}_{\mathbb{X}} \widetilde{W_p}$.

6.6. **Lemma.** *In the setup above H and H_0 are in* $\operatorname{mod}\Lambda \cap \mathtt{tor}_p$. *Moreover H_0 is a subsheaf (as coherent sheaves over $\mathbb{X}$) of H.*

Proof. We have to distinguish two cases:

Case $p \in \mathcal{S}$: Then $\operatorname{mod}\Lambda$ is closed under subobjects in the tube. Hence H and H_0 are in $\operatorname{mod}\Lambda$.

Case $p \in \mathbb{P}^1_k \setminus \mathcal{S}$: Now all objects involved are in $\mathcal{H}$, and $H, H_0 \in \operatorname{mod}\Lambda$ follows from the fact that $\operatorname{mod}\Lambda$ is closed under factors in $\mathcal{H}$.

The second claim follows from the fact that $H_0 \in \operatorname{Fac}_{\mathbb{X}} \mathcal{O}_{\mathbb{P}^1_k}$. □

As before, by applying $\operatorname{Hom}(W, -)$ to a short exact sequence $\operatorname{Ker} \rightarrowtail \mathcal{O}^{?}_{\mathbb{P}^1_k} \twoheadrightarrow H$, we see $\operatorname{Ext}^1(W, H) = 0$. Similarly $\operatorname{Ext}^1(W, H_0) = 0$. Finally we set $\overline{H} = H/H_0$. Applying $\operatorname{Hom}(W, -)$ to the short exact sequence $H_0 \rightarrowtail H \twoheadrightarrow \overline{H}$ we see that $\overline{H}$ is a torsion sheaf in $\operatorname{coh}\mathbb{P}^1_k = W^{\perp}$. Hence it has an L-resolution

$$\mathcal{O}_{\mathbb{P}^1_k}(d-1)^n \rightarrowtail \mathcal{O}_{\mathbb{P}^1_k}(d)^n \twoheadrightarrow \overline{H}$$

for some $n \in \mathbb{N}$ (as coherent sheaf over $\mathbb{X}$).

Since $\operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^1_k}(d), H_0) = 0$ we obtain the following diagram,

$$\begin{array}{ccccc}
H_0 & \rightarrowtail & H & \twoheadrightarrow & \overline{H} \\
\uparrow & & \uparrow & & \| \\
\mathcal{O}_{\mathbb{P}^1_k}(d-1)^n & \rightarrowtail & \mathcal{O}_{\mathbb{P}^1_k}(d)^n & \twoheadrightarrow & \overline{H}
\end{array}$$

and hence the short exact sequence

$$\mathcal{O}_{\mathbb{P}^1_k}(d-1)^n \rightarrowtail H_0 \oplus \mathcal{O}_{\mathbb{P}^1_k}(d)^n \twoheadrightarrow H$$

of coherent sheaves over $\mathbb{X}$. Since all three objects are modules this is also a short exact sequence in $\operatorname{mod}\Lambda$.

Now we denote the kernel and cokernel of the map $H \xrightarrow{\iota} X$ (in $\operatorname{mod}\Lambda$) by K and C, respectively. We obtain the following diagram, where L denotes the kernel of the map to its right.

$$\begin{array}{ccccccc}
K & \rightarrowtail & H & \xrightarrow{\iota} & X & \twoheadrightarrow & C \\
\uparrow & & \uparrow & \searrow & \nearrow & & \\
L & \rightarrowtail & H_0 \oplus \mathcal{O}_{\mathbb{P}^1_k}(d)^n & \twoheadrightarrow & \operatorname{Im}_{\Lambda}\iota & & \\
\uparrow & & \uparrow & & & & \\
\mathcal{O}_{\mathbb{P}^1_k}(d-1)^n & = & \mathcal{O}_{\mathbb{P}^1_k}(d-1)^n & & & &
\end{array}$$

As in the case $X \in \operatorname{mod}\Lambda \cap \mathtt{VB}$ one sees that $K \in \mathtt{tor}_p[-1]$ and $C \in \mathtt{tor}_p$. In particular the left vertical sequence splits. Hence we get an exact sequence

$$K \oplus \mathcal{O}_{\mathbb{P}^1_k}(d-1)^n \rightarrowtail H_0 \oplus \mathcal{O}_{\mathbb{P}^1_k}(d)^n \xrightarrow{f} X \twoheadrightarrow C$$

of Λ-modules. Any map from L or $\widetilde{W}$ to X factors through f. Moreover, by Lemma 6.1 any map from $\nu\overrightarrow{T}$ to X factors through an injective module, hence through $\widetilde{W}_p$, and hence through f. Therefore the following proposition follows from Lemma 6.2.

6.7. **Proposition.** *Assume we are in the setup of Theorem 5.4, and* $X \in \operatorname{mod}\Lambda \cap \mathtt{tor}$. *Then* $G\text{-resol.dim}\, X \leq 1$.

Modules in $\operatorname{mod}\Lambda \cap \mathtt{VB}[1]$**.** Finally we assume X to be indecomposable in $\operatorname{mod}\Lambda \cap \mathtt{VB}[1]$. We may assume that $X \notin \operatorname{add}\nu\overrightarrow{T}$.

We start by approximating X by modules in $\operatorname{add}\nu\overrightarrow{T}$. By Lemma 6.1 is suffices to approximate X by $\operatorname{add}\nu T$. We denote such an approximation by $f\colon \nu T' \longrightarrow X$, and its cone by H. The following proposition gives a more precise location of H.

6.8. **Proposition.** *Let* $X \in \operatorname{mod}\Lambda \cap \mathtt{VB}[1]$ *indecomposable,* $X \notin \operatorname{add}\nu\overrightarrow{T}$*, and let* $f\colon \nu T' \longrightarrow X$ *be a right minimal approximation by injective modules. Then* $\operatorname{Cone}(f) \in \mathtt{VB}[1]$*, and* $\operatorname{Cone}(f)[-1] \in \operatorname{Sub}_{\mathcal{H}} \tau^2 T_{\mathtt{VB}}$.

We postpone the proof of this proposition to the end of the section. The following consequence of this proposition will be essential to the discussion here.

6.9. **Corollary.** *Let* H *as above. Then*

$$E(H[-1]) \in \operatorname{add}\{\mathcal{O}_{\mathbb{P}^1_k}(i) \mid i \leq d\} \vee \operatorname{add}\widetilde{W}.$$

(For the definition of $E(-)$ *see Construction 4.7.)*

Proof. By Proposition 4.8 we have

$$E(H[-1]) \in \operatorname{add}\{\mathcal{O}_{\mathbb{P}^1_k}(i) \mid i \in \mathbb{Z}\} \vee \operatorname{add}\widetilde{W}.$$

Now let $H[-1] \rightarrowtail \tau^2 T_{\mathtt{VB}}^n$ be a monomorphism in $\mathcal{H}$, which exists by Proposition 6.8. Then it is also a monomorphism in $\operatorname{coh}\mathbb{X}$. This map can be extended to the following diagram, where the rows are exact sequences of coherent sheaves over $\mathbb{X}$ coming up in the definition of $E(-)$.

$$\begin{array}{ccccc} H[-1] & \rightarrowtail & E(H[-1]) & \twoheadrightarrow & \widetilde{W}' \\ \downarrow & & \downarrow & & \downarrow \\ \tau^2 T_{\mathtt{VB}}^n & \rightarrowtail & E(\tau^2 T_{\mathtt{VB}})^n & \twoheadrightarrow & \widetilde{W}'' \end{array}$$

By the snake lemma the kernel of the central vertical map is a subsheaf of the kernel of the right vertical map. Hence it has finite length. Now the claim follows from the fact that

$$E(\tau^2 T_{\mathtt{VB}})^n \in \operatorname{add}\{\mathcal{O}_{\mathbb{P}^1_k}(i) \mid i \leq d\} \vee \operatorname{add}\widetilde{W}$$

(which is part of the definition of d in Construction 5.1). □

Now we ready to add a $\widetilde{W}$-approximation to the $\nu\overrightarrow{T}$-approximation f of X we chose above. Note that any map $\widetilde{W} \longrightarrow X$ that does not factor through f is mapped to a nonzero map in $\operatorname{Hom}(\widetilde{W}, H)$. Conversely any map in $\operatorname{Hom}(\widetilde{W}, H)$ lifts to a map in $\operatorname{Hom}(\widetilde{W}, X)$, since $\operatorname{Ext}^1(\widetilde{W}, \nu T) = 0$. Hence we may take the $\widetilde{W}$-approximation $\widetilde{W}' \longrightarrow H$ and complete the following diagram.

$$\begin{array}{ccccccc}
H[-1] & \longrightarrow & \nu T' & \stackrel{f}{\longrightarrow} & X & \longrightarrow & H \\
\| & & \uparrow & & \uparrow & & \| \\
H[-1] & \longrightarrow & E(H[-1]) & \longrightarrow & \widetilde{W}' & \longrightarrow & H
\end{array}$$

This gives rise to the triangle

$$E(H[-1]) \longrightarrow \nu T' \oplus \widetilde{W}' \stackrel{\tilde{f}}{\longrightarrow} X \longrightarrow E(H[-1])[1]$$

where any map from $\nu\overrightarrow{T} \oplus \widetilde{W}$ to X factors through $\tilde{f}$.

Next we add an L-approximation $L' \longrightarrow X$. We obtain the following diagram, where K is an object completing the lower triangle.

$$\begin{array}{ccccccc}
E(H[-1]) & \longrightarrow & \nu T' \oplus \widetilde{W}' & \stackrel{\tilde{f}}{\longrightarrow} & X & \longrightarrow & E(H[-1])[1] \\
\| & & \uparrow & & \uparrow & & \| \\
E(H[-1]) & \longrightarrow & K & \longrightarrow & L' & \longrightarrow & E(H[-1])[1]
\end{array}$$

Since $E(H[-1])$ and L' are both in

$$\operatorname{add}\{\mathcal{O}_{\mathbb{P}^1_k}(i) \mid i \leq d\} \vee \operatorname{add} \widetilde{W}$$

(by Corollary 6.9 and the construction of L below 5.1, respectively) this also holds for K.

As before, the diagram turns into a triangle

$$K \longrightarrow \nu T' \oplus \widetilde{W}' \oplus L' \longrightarrow X \longrightarrow K[1].$$

We now show that $K \in \operatorname{mod} \Lambda$. To do so we have to see that any map from T to X factors through $\operatorname{add} L \oplus \widetilde{W}$.

- For $p \in \mathcal{S}$ there are no maps $T_p^L \longrightarrow X$.
- Applying $\operatorname{Hom}(-, X)$ to the triangle

$$T_{\mathtt{VB}} \longrightarrow E(T_{\mathtt{VB}}) \longrightarrow \widetilde{W}'' \longrightarrow T_{\mathtt{VB}}[1]$$

  one finds that the map $\operatorname{Hom}(E(T_{\mathtt{VB}}), X) \longrightarrow \operatorname{Hom}(T_{\mathtt{VB}}, X)$ is onto. Hence (by construction of L and Proposition 4.8) any map from $T_{\mathtt{VB}}$ to X factors through $\operatorname{add} L \oplus \widetilde{W}$.
- For $p \in \mathbb{P}^1_k \setminus \mathcal{S}$ we have $T_p^R \in \operatorname{add} \widetilde{W}$.

Therefore the first map in the following exact sequence (induced by the triangle above) is onto.

$$\begin{array}{l}
\operatorname{Hom}(T, \nu T' \oplus \widetilde{W}' \oplus L') \twoheadrightarrow \operatorname{Hom}(T, X) \longrightarrow \\
\longrightarrow \operatorname{Hom}(T, K[1]) \longrightarrow \operatorname{Hom}(T, (\nu T' \oplus \widetilde{W}' \oplus L')[1]).
\end{array}$$

Since $\nu T' \oplus \widetilde{W}' \oplus L' \in \operatorname{mod}\Lambda$ the last term above vanishes, and therefore so does $\operatorname{Hom}(T, K[1])$. This shows $K \in \operatorname{mod}\Lambda$, and the triangle gives rise to a short exact sequence

$$K \rightarrowtail \nu T' \oplus \widetilde{W}' \oplus L' \twoheadrightarrow X$$

of Λ-modules.

Now we are done by Lemma 6.2.

6.10. **Proposition.** *Assume we are in the setup if Theorem 5.4, and* $X \in \operatorname{mod}\Lambda \cap \mathtt{VB}[1]$. *Then* G-resol.dim $X \leq 1$.

It remains to prove Proposition 6.8. We recall the claim for the convenience of the reader.

6.8. **Proposition.** *Let* $X \in \operatorname{mod}\Lambda \cap \mathtt{VB}[1]$ *be indecomposable,* $X \notin \operatorname{add} \nu \overrightarrow{T}$, *and let* $f \colon \nu T' \longrightarrow X$ *be a right minimal approximation by injective modules. Then* $\operatorname{Cone}(f) \in \mathtt{VB}[1]$, *and* $\operatorname{Cone}(f)[-1] \in \operatorname{Sub}_{\mathcal{H}} \tau^2 T_{\mathtt{VB}}$.

Proof. We first show that $f[-1]$, seen as a map in $\mathcal{H}$, is a monomorphism. We denote by K and I the kernel and image of $f[-1]$ in $\mathcal{H}$. Since $(\operatorname{mod}\Lambda)[-1] \cap \mathcal{H} = \operatorname{Sub}_{\mathcal{H}} \tau T$ we have $I[1] \in \operatorname{mod}\Lambda$. Since $\nu T[-1] = \tau T$ is a tilting object in $\mathcal{H}$ we have that $K \in \operatorname{add} \tau T$. The triangle $K[1] \longrightarrow \nu T' \longrightarrow I[1] \longrightarrow K[2]$ therefore comes from a short exact sequence

$$K[1] \rightarrowtail \nu T' \twoheadrightarrow I[1]$$

in $\operatorname{mod}\Lambda$, which is split, since $K[1] \in \operatorname{add} \nu T = \operatorname{inj}\Lambda$. By the assumption that f is right minimal this means $K = 0$.

Hence we have a short exact sequence

$$\tau T' \overset{f[-1]}{\rightarrowtail} X[-1] \twoheadrightarrow H[-1]$$

in $\mathcal{H}$, where as before $H = \operatorname{Cone}(f)$. In particular $H \in \mathcal{H}[1]$.

We decompose $H = H_{\mathtt{VB}} \oplus H_{\mathtt{tor}}$ with $H_{\mathtt{VB}} \in \mathtt{VB}[1]$ and $H_{\mathtt{tor}} \in \mathtt{tor}_{\mathbb{P}^1_k \setminus \mathcal{S}}[1]$. Now we denote the pullback in the following diagram (in $\mathcal{H}$) by $Y[-1]$.

$$\begin{array}{ccccc}
\tau T' & \overset{f_Y[-1]}{\rightarrowtail} & Y[-1] & \twoheadrightarrow & H_{\mathtt{tor}}[-1] \\
\| & & \downarrow & \text{PB} & \downarrow \text{split} \\
\tau T' & \overset{f[-1]}{\rightarrowtail} & X[-1] & \twoheadrightarrow & H[-1]
\end{array}$$

Since $Y[-1]$ is a subobject of $X[-1]$ we have $Y \in \operatorname{mod}\Lambda$. Note that since X is assumed to be in $\mathtt{VB}[1]$ the objects Y and $H_{\mathtt{tor}}$ have no common summands. Hence f_Y is left minimal. We denote by K and C the kernel and cokernel of the map f_Y in $\operatorname{mod}\Lambda$, respectively. Then we have a triangle

$$K[1] \longrightarrow \underbrace{H_{\mathtt{tor}}}_{=\operatorname{Cone}(f_Y)} \longrightarrow C \longrightarrow K[2].$$

Clearly C and $K[2]$ cannot have any common direct summands, hence the map $H_{\mathtt{tor}} \longrightarrow C$ is left minimal. Therefore $C \in \mathtt{tor}_{\mathbb{P}^1_k \setminus \mathcal{S}}[1]$. Since $K[2]$ is the cone of the map $H_{\mathtt{tor}} \longrightarrow C$ we have $K[2] \in \mathtt{tor}_{\mathbb{P}^1_k \setminus \mathcal{S}}[1] \vee \mathtt{tor}_{\mathbb{P}^1_k \setminus \mathcal{S}}[2]$. But $K \in \operatorname{mod}\Lambda$, so $K \in \mathtt{tor}_{\mathbb{P}^1_k \setminus \mathcal{S}}$.

We now denote the image of f_Y as Λ-morphism by $\operatorname{Im}_\Lambda f_Y$. Since $\operatorname{Im}_\Lambda f_Y = \mho K$ Lemma 6.3 says that $\operatorname{Im}_\Lambda f_Y \in \operatorname{add} \nu \overrightarrow{T}$. Since the inclusion $\operatorname{Im}_\Lambda f_Y \rightarrowtail Y$ is left minimal Lemma 6.4 implies that $Y \in \operatorname{add} \nu \overrightarrow{T}$.

Now, by Lemma 6.1, the map $Y \longrightarrow X$ factors through an injective module, hence through f. Therefore the sequence $\tau T' \rightarrowtail Y[-1] \twoheadrightarrow H_{\mathtt{tor}}[-1]$ splits, and hence $H_{\mathtt{tor}} = 0$.

For the proof of the second claim we apply $\operatorname{Hom}(-, \tau^2 T)$ to the short exact sequence $\tau T' \rightarrowtail X[-1] \twoheadrightarrow H[-1]$ in $\mathcal{H}$. We obtain

$$\begin{array}{ccccc}
\overbrace{\operatorname{Hom}(\tau T', \tau^2 T)}^{=0} & \longrightarrow & \operatorname{Ext}^1(H[-1], \tau^2 T) & \longrightarrow & \\
\longrightarrow & \operatorname{Ext}^1(X[-1], \tau^2 T) & \twoheadrightarrow & \operatorname{Ext}^1(\tau T', \tau^2 T) & \\
 & \| & & \| & \\
 & D \operatorname{Hom}(\tau T, X[-1]) & \xrightarrow{f^*} & D \operatorname{Hom}(\tau T, \tau T') &
\end{array}$$

Since f is an injective approximation the map f^* above is into, and hence $\operatorname{Ext}^1(H[-1], \tau^2 T) = 0$. Now note that $\tau^2 T$ is cotilting in $\mathcal{H}$. Therefore this vanishing of Ext implies $H[-1] \in \operatorname{Sub}_{\mathcal{H}} \tau^2 T$.

Since $H[-1] \in \mathtt{VB}$ there is a monomorphism $H[-1] \rightarrowtail \tau^2 T'_{\mathtt{VB}} \oplus R$ in $\mathcal{H}$, with $T'_{\mathtt{VB}} \in \operatorname{add} T_{\mathtt{VB}}$ and $R \in \mathtt{tor}_{\mathbb{P}^1_k \setminus \mathcal{S}}$. We obtain the following diagram in $\mathcal{H}$, where φ is the composition of the inclusion above with projection to the direct summand $\tau^2 T'_{\mathtt{VB}}$.

$$\begin{array}{ccccc}
\operatorname{Ker}_{\mathcal{H}} \varphi & \rightarrowtail & H[-1] & \twoheadrightarrow & \operatorname{Im}_{\mathcal{H}} \varphi \\
\downarrow & & \downarrow & \searrow^{\varphi} & \downarrow \\
R & \rightarrowtail & \tau^2 T'_{\mathtt{VB}} \oplus R & \twoheadrightarrow & \tau^2 T'_{\mathtt{VB}}
\end{array}$$

Since $\operatorname{Ker}_{\mathcal{H}} \varphi$ is a subobject of R it is in $\mathtt{tor}_{\mathbb{P}^1_k \setminus \mathcal{S}}$. Therefore it admits no non-zero maps to $H[-1]$, and hence $\operatorname{Ker}_{\mathcal{H}} \varphi = 0$. That shows $H[-1] \in \operatorname{Sub}_{\mathcal{H}} \tau^2 T_{\mathtt{VB}}$. □

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