n-ANGULATED CATEGORIES

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Abstract. We define n-angulated categories by modifying the axioms of triangulated categories in a natural way. We show that Heller's parametrization of pre-triangulations extends to pre-n-angulations. We obtain a large class of examples of n-angulated categories by considering $(n-2)$-cluster tilting subcategories of triangulated categories which are stable under the $(n-2)$nd power of the suspension functor. As an application, we show how n-angulated Calabi-Yau categories yield triangulated Calabi-Yau categories of higher Calabi-Yau dimension. Finally, we sketch a link to algebraic geometry and string theory.

Introduction

0.1. **Context.** Triangulated categories were invented at the end of the 1950s by Grothendieck-Verdier [30] and, independently, Puppe [28]. Their aim was to axiomatize the properties of derived categories respectively of stable homotopy categories. In the case of the derived category, the triangles are the 'shadows' of the 3-term exact sequences of complexes. Longer exact sequences of complexes are splicings of 3-term exact sequences and thus they also have their shadows in the derived category, for example in the form of the higher octahedra described in Remark 1.1.14 of [4]. Cluster tilting theory (as developed in [2, 12, 20] and many other articles) and in particular Iyama's higher Auslander-Reiten-theory (see [16, 17]) have lead to the surprising discovery that there is a large class of categories which are naturally inhabited by shadows of n-term exact sequences without being home to shadows of 3-term exact sequences. In this paper, our main aims are to

- axiomatize this remarkable class of categories by introducing the new notion of *n-angulated category* and
- construct large classes of examples using cluster tilting theory.

The main thrust of this paper is thus foundational. However, as we will see, this apparently dry subject matter is linked to some exciting developments in algebraic geometry and string theory [25, 24, 8, 9].

0.2. **Contents.** In creating our axiomatic framework we found that mostly, the axioms of triangulated categories generalize well but that special care has to be taken with the notion of isomorphism of n-angles and with the octahedral axiom, cf. Section 1.

In Section 2, we pursue the foundational analogy with the triangulated case by generalizing a result of Heller [15]: We parametrize the set of equivalence classes of pre-n-angulations (a pre-n-angulation need not satisfy the generalized octahedral axiom) on a given additive category $\mathcal{F}$ with a given higher suspension functor Σ. Namely, we show that if it is not empty, this set has a simply transitive action by the automorphism group of the nth suspension in the stable category associated with the Frobenius category of finitely presented $\mathcal{F}$-modules.

In Section 3, we show how to construct n-angulated categories inside triangulated categories. Namely, if $\mathcal{T}$ is a triangulated category with suspension functor Σ and $\mathcal{S}$ is an

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$(n-2)$-cluster tilting subcategory in the sense of Iyama [17] which is stable under Σ^{n-2}, then $\mathcal{S}$ naturally becomes an n-angulated category with higher suspension functor Σ^{n-2}. This yields a large supply of interesting n-angulated categories and is one of the main results of this paper.

In Section 4, we show that if $\mathcal{F}$ is an n-angulated category which is Calabi-Yau of Calabi-Yau dimension d, then the stable category of finitely presented $\mathcal{F}$-modules is (triangulated and) Calabi-Yau of Calabi-Yau dimension $nd-1$. In particular, we can construct 3-Calabi-Yau categories from 4-angulated categories which are 1-Calabi-Yau. This generalizes the construction in example 8.3.3 of [21]. It was one of our original motivations for studying n-angulated categories.

We conclude by presenting several classes of examples in Section 5 and by sketching a link to algebraic geometry and string theory in 5.5.

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1. Axioms

1.1. **Definition.** Let $\mathcal{F}$ be an additive category with an automorphism Σ, and n an integer greater or equal than three. A sequence of morphisms of $\mathcal{F}$

$$X_\bullet := (X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} \Sigma X_1)$$

is an *n-Σ-sequence*. Its *left rotation* is the n-Σ-sequence

$$(X_2 \xrightarrow{\alpha_2} X_3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_n} \Sigma X_1 \xrightarrow{(-1)^n \Sigma\alpha_1} \Sigma X_2).$$

The n-Σ-sequences of the form $(TX)_\bullet := (X \xrightarrow{\mathbf{1}_X} X \to 0 \to \cdots \to 0 \to \Sigma X)$ for $X \in \mathcal{F}$, and their rotations, are called *trivial*.

An n-Σ-sequence $X_\bullet$ is *exact* if the induced sequence

$$\mathcal{F}(-, X_\bullet)\colon \quad \cdots \to \mathcal{F}(-, X_1) \to \mathcal{F}(-, X_2) \to \cdots \to \mathcal{F}(-, X_n) \to \mathcal{F}(-, \Sigma X_1) \to \cdots$$

of representable functors $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is exact. In particular, the trivial n-Σ-sequences are exact.

A *morphism* of n-Σ-sequences is given by a sequence of morphisms $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ such that the following diagram commutes:

$$\begin{tikzcd}
X_\bullet \arrow[d, "\varphi_\bullet"] & X_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"] & X_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & X_3 \arrow[r, dotted, no head] \arrow[d, "\varphi_3"] & X_n \arrow[r, "\alpha_n"] \arrow[d, "\varphi_n"] & \Sigma X_1 \arrow[d, "\Sigma\varphi_1"] \\
Y_\bullet & Y_1 \arrow[r, "\beta_1"] & Y_2 \arrow[r, "\beta_2"] & Y_3 \arrow[r, dotted, no head] & Y_n \arrow[r, "\beta_n"] & \Sigma Y_n
\end{tikzcd}$$

In this situation we call φ a *weak isomorphism* if for some $1 \le i \le n$ both φ_i and φ_{i+1} (with $\varphi_{n+1} := \Sigma\varphi_1$) are isomorphisms. Slightly abusing terminology we will say that two n-Σ-sequences are *weakly isomorphic* if they are linked by a finite zigzag of weak isomorphisms.

We call a collection $\pentagon$ of n-Σ-sequences a *(pre-) n-angulation of* $(\mathcal{F}, \Sigma)$ and its elements *n-angles* if $\pentagon$ fulfills the following axioms:

(F1) (a) $\pentagon$ is closed under direct sums and under taking direct summands.
(b) For all $X \in \mathcal{F}$, the trivial n-Σ sequence $(TX)_\bullet$ belongs to $\pentagon$.

(c) For each morphism $\alpha_1\colon X_1 \to X_2$ in $\mathcal{F}$, there exists an n-angle whose first morphism is α_1.

(F2) An n-Σ-sequence $X_\bullet$ belongs to $\pentagon$ if and only if its left rotation

$$(X_2 \xrightarrow{\alpha_2} X_3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_n} \Sigma X_1 \xrightarrow{(-1)^n \Sigma\alpha_1} \Sigma X_2)$$

belongs to $\pentagon$.

(F3) Each commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"] & X_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & X_3 \arrow[r, dotted, no head] \arrow[d, dotted, "\varphi_3"] & X_n \arrow[r, "\alpha_n"] \arrow[d, dotted, "\varphi_n"] & \Sigma X_1 \arrow[d, "\Sigma\varphi_1"] \\
Y_1 \arrow[r, "\beta_1"] & Y_2 \arrow[r, "\beta_2"] & Y_3 \arrow[r, dotted, no head] & Y_n \arrow[r, "\beta_n"] & \Sigma Y_1
\end{tikzcd}$$

with rows in $\pentagon$ can be completed to a morphism of n-Σ-sequences.

If $\pentagon$ moreover fulfills the following axiom, it is called an *n-angulation* of $(\mathcal{F}, \Sigma)$:

(F4) In the situation of (F3) the morphisms $\varphi_3, \varphi_4, \ldots, \varphi_n$ can be chosen such that the cone $C(\varphi_\bullet)$:

$$X_2 \oplus Y_1 \xrightarrow{\begin{pmatrix} -\alpha_2 & 0 \\ \varphi_2 & \beta_1 \end{pmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{pmatrix} -\alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -\alpha_n & 0 \\ \varphi_n & \beta_{n-1} \end{pmatrix}} \Sigma X_1 \oplus Y_n \xrightarrow{\begin{pmatrix} -\Sigma\alpha_1 & 0 \\ \Sigma\varphi_1 & \beta_n \end{pmatrix}} \Sigma X_2 \oplus \Sigma Y_1$$

belongs to $\pentagon$.

1.2. **Remarks.** (a) For (pre-) triangulated categories it is possible to demand in axiom (TR2), the "model" for our (F2) only one direction, see for example [14, I.1.3]. We need here however both directions. One reason for this is that for $n \geq 4$ our Axiom (F3) is weaker than its "model" for triangulated categories. For example, if φ_1 and φ_2 are isomorphisms it does *not* follow that $\varphi_3, \ldots, \varphi_n$ are isomorphisms.

(b) Our axiom (F4) is inspired by Neeman's version of "octahedral axiom (TR4)", see for example [26, 1.3, 1.8].

(c) If $(\mathcal{F}, \Sigma, \pentagon)$ is (pre-) n-angulated, the opposite category $(\mathcal{F}^{\mathrm{op}}, \Sigma^{-1})$ is (pre-) n-angulated with n-angles

$$\Sigma^{-1} X_n \xleftarrow{(-1)^n \Sigma^{-1}\alpha_n} X_1 \xleftarrow{\alpha_1} X_2 \leftarrow \cdots \xleftarrow{\alpha_{n-1}} X_n$$

corresponding to the n-angles in $\pentagon$.

(d) In examples, n-angulated categories come frequently with a self-equivalence Σ. However, in analogy with [22, Sec. 2] we may assume without loss of generality that Σ is an automorphism. This has the advantage of a less heavy notation.

1.3. **Periodic Complexes.** Consider a complex $M_\bullet = (M_k, \delta_k)_{k\in\mathbb{Z}}$ in $\operatorname{Mod}\mathcal{F}$, the abelian category of functors $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$. For later use, we fix factorizations

$$\begin{tikzcd}
M_k \arrow[rr, "\delta_k"] \arrow[dr, two heads, "\delta'_k"'] & & M_{k+1} \\
& K_k \arrow[ur, hook, "\delta''_k"'] &
\end{tikzcd}$$

with δ'_k an epimorphism and δ''_k a monomorphism for all $k \in \mathbb{Z}$. Recall that M_k is contractible if there exists a contraction, *i.e.* a family of morphisms $\eta_k\colon M_{k+1} \to M_k$ such that $\delta_{k-1}\eta_{k-1} + \eta_k\delta_k = \mathbb{1}_{M_k}$ for all k. Equivalently, $M_\bullet$ is exact, and there exist sections η'_k for δ'_k for all k.

The automorphism Σ of $\mathcal{F}$ induces an automorphism of $\operatorname{Mod}\mathcal{F}$ defined by $M \mapsto M \circ \Sigma^{-1}$ which we denote also by Σ. We say that $M_\bullet$ is *n-Σ-periodic* if $M_{k+n+1} = \Sigma M_k$ and

$\delta_{k+n+1} = \Sigma\delta_k$ for all $k \in \mathbb{Z}$. A n-Σ-periodic complex $M_\bullet$ is *trivial* if it is exact and there exists $l \in \{1, \dots, n\}$ such that $M_k = 0$ unless $k \equiv l \pmod n$ or $k \equiv l+1 \pmod n$.

Note that if an n-Σ-periodic complex $M_\bullet$ is contractible, then it admits an n-Σ-periodic contraction. In particular, it is a finite direct sum of n-Σ-periodic trivial complexes. Indeed, in the above factorization, we may assume that $\delta'_{k+n+1} = \Sigma\delta'_k$ and $\delta''_{k+n+1} = \Sigma\delta''_k$ for all k. Since $M_\bullet$ is contractible, we find sections η'_k to δ'_k for $k = 1, 2, \dots, n$. Then, $\Sigma^l\eta'_k$ is a section to $\delta'_{l(n+1)+k}$ for all $l \in \mathbb{Z}$ and $1 \le k \le n$. This yields an n-Σ-periodic contraction.

1.4. **Lemma.** *Let $(\mathcal{F}, \Sigma, \pentagon)$ be a pre-n-angulated category. If $X_\bullet$ and $Y_\bullet$ are two weakly isomorphic exact n-Σ-sequences, then $X_\bullet$ belongs to $\pentagon$ if and only if $Y_\bullet$ belongs to $\pentagon$.*

Proof. (1) Let $X_\bullet$ be an exact n-Σ-sequence. Then it is easy to see that the cone over the identity $C(\mathbb{1}_X)$ is a finite direct sum of trivial complexes. So $C(\mathbb{1}_X)$ belongs to $\pentagon$.

(2) Let $\varphi_\bullet \colon X_\bullet \to Y_\bullet$ be a weak isomorphism between exact n-Σ-sequences. Then the cone $C(\varphi_\bullet)$ admits a periodic contraction. Equivalently, the n-Σ-periodic complex $\mathcal{F}(-, C(\varphi_\bullet))$ is a direct sum of trivial n-Σ-periodic complexes in $\mod \mathcal{F}$.

Indeed, the induced morphism of complexes $\mathcal{F}(-, \varphi_\bullet) \colon \mathcal{F}(-, X_\bullet) \to \mathcal{F}(-, Y_\bullet)$ is a homotopy equivalence since for appropriate truncations those complexes can be seen as the projective resolution of isomorphic modules with the isomorphism induced by $\varphi_\bullet$. Our claim now follows from 1.3 since the cone over a homotopy equivalence is contractible.

(3) For a weak isomorphism $\varphi_\bullet \colon X_\bullet \to Y_\bullet$, consider the following sequence of n-Σ-periodic, exact complexes:

$$0 \to \mathcal{F}(-, X_\bullet) \to \mathcal{F}(-, Y) \oplus \mathcal{F}(-, C(\mathbb{1}_X)) \xrightarrow{p} \mathcal{F}(-, C(\varphi_\bullet)) \to 0,$$

it is exact for the usual degree-wise split exact structure on the category of complexes. From (2), it follows that p admits an n-Σ-periodic section. Now, it follows from (F1) and (1) that in case $Y_\bullet \in \pentagon$ also $X_\bullet \in \pentagon$. A similar construction shows that in case $X_\bullet \in \pentagon$ implies $Y_\bullet \in \pentagon$. $\square$

1.5. **Proposition.** *Let $\mathcal{F} = (\mathcal{F}, \Sigma, \pentagon)$ be a pre-n-angulated category. Then the following hold:*

(a) *All n-angles are exact.*
(b) $\mod \mathcal{F}$*, the category of finitely presented functors $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is an abelian Frobenius category. Moreover, if $\mathcal{F}$ has split idempotents, the map $X \mapsto \mathcal{F}(-, X)$ induces an equivalence from $\mathcal{F}$ to $\operatorname{inj} \mathcal{F}$, the full subcategory of $\mod \mathcal{F}$ which consists of the injective objects.*
(c) *If $\pentagon' \subset \pentagon$ is an other pre-n-angulation of $(\mathcal{F}, \Sigma)$, then $\pentagon' = \pentagon$.*

Proof. (a) This is almost verbatim the same argument as the corresponding statement for pre-triangulated categories, see for example [26, 1.1.3 and 1.1.10].

(b) $\mod \mathcal{F}$ has clearly cokernels, and it has kernels since by (a) $\mathcal{F}$ has weak kernels [7]. In order to conclude that each projective is also injective we show $\operatorname{Ext}^i_{\mathcal{F}}(M, \mathcal{F}(-, T)) = 0$ for each $M \in \mod \mathcal{F}$ and $T \in \mathcal{F}$. Indeed, for a projective presentation

$$\mathcal{F}(-, X_{n-1}) \xrightarrow{\mathcal{F}(-, \alpha_{n-1})} \mathcal{F}(-, X_n) \to M \to 0$$

α_{n-1} can be embedded in an n-angle $X_\bullet$ by (F1c) and (F2). By part (a) of the proposition we obtain a (periodic) projective resolution $P^\bullet$ of M. By the Yoneda lemma and part (a) of the proposition applied to $\mathcal{F}^{\mathrm{op}}$ we conclude that $H^i(P^\bullet, \mathcal{F}(-, T)) = 0$ for all $i \ge 1$.

(c) Let $X_\bullet$ be in $\pentagon$. By (F1c) there exists a 4-angle $X_1 \xrightarrow{\alpha} X_2 \xrightarrow{\alpha'_2} X'_2 \xrightarrow{\alpha'_3} \cdots \xrightarrow{\alpha'_n} \Sigma X_1$ in $\pentagon'$. Since $\pentagon' \subset \pentagon$, the pair $(\mathbb{1}_{X_1}, \mathbb{1}_{X_2})$ can be completed by (F3) to a morphism of

n-angles. But this is a weak isomorphism, so the first n-angle belongs already to $\pentagon'$ by Lemma 1.4. $\square$

2. The class of n-angulations

In this section, we assume that $\mathcal{F}$ has split idempotents. Closely following [15, §16] we study the possible pre-n-angulations for $(\mathcal{F}, \Sigma)$.

2.1. **Injective Resolutions and Triangle Functors.** Let $(\mathcal{F}, \Sigma, \pentagon)$ be a pre-n-angulated category. Thus, by Proposition 1.5 $\operatorname{mod} \mathcal{F}$ is a Frobenius category and we may identify $\mathcal{F}$ with the subcategory $\operatorname{inj} \mathcal{F}$ of injectives in $\operatorname{mod} \mathcal{F}$. For each $M \in \operatorname{mod} \mathcal{F}$, we choose a short exact sequence

$$0 \to M \xrightarrow{\iota_M} I_M \xrightarrow{\pi_M} \Omega^{-1} M \to 0 \ , \tag{2.1}$$

where I_M belongs to $\mathcal{F}$ identified with $\operatorname{inj}(\mathcal{F})$. By splicing together the short exact sequences from (2.1) we obtain a standard injective resolution

$$\hat{I}_M \colon \quad I_M \xrightarrow{\nu_{M,1}} I_{\Omega^{-1}} \xrightarrow{\nu_{M,2}} I_{\Omega^{-2} M} \to \cdots \tag{2.2}$$

of M. Since the automorphism Σ of $\operatorname{mod} \mathcal{F}$ is an exact functor, we may assume $I_{\Sigma M} = \Sigma I_M$, and we obtain an isomorphism

$$\sigma_M \colon \Sigma \Omega^{-1} M \to \Omega^{-1} \Sigma M.$$

Next, the above short exact sequence induces a self equivalence Ω^{-1} of the stable category $\underline{\operatorname{mod}}\, \mathcal{F}$. Recall that $\underline{\operatorname{mod}}\, \mathcal{F}$ is a triangulated category with suspension Ω^{-1}. Moreover, (Σ, σ) and $(\Omega^{-n}, (-1)^n \mathbb{1}_{\Omega^{-n-1}})$ are triangle functors on $\underline{\operatorname{mod}}\, \mathcal{F}$.

Finally, if

$$X_\bullet : X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} \Sigma X_1$$

is an exact n-Σ-sequence in $\mathcal{F}$, we may interpret it as the beginning of an injective resolution of $\operatorname{Ker}(\alpha_1) \in \operatorname{mod} \mathcal{F}$. Thus, we can lift $\mathbb{1}_{\operatorname{Ker} \alpha_1}$ to a unique, up to homotopy, morphism of complexes from $X_\bullet$ to $\hat{I}_{\operatorname{Ker} \alpha_1}$. From the factorization

$$\begin{tikzcd} X_n \arrow[rr, "\alpha_n"] \arrow[dr] & & \Sigma X_1 \\ & \Sigma \operatorname{Ker} \alpha_1 \arrow[ur] & \end{tikzcd} \ ,$$

we obtain a morphism

$$\delta_{X_\bullet} \colon \Sigma \operatorname{Ker}(\alpha_1) \to \Omega^{-n} \operatorname{Ker}(\alpha_1),$$

which becomes an isomorphism in $\underline{\operatorname{mod}}\, \mathcal{F}$.

2.2. **Lemma.** *Associated to $\pentagon$ we have a natural isomorphism*

$$\delta(\pentagon) \colon (\Sigma, \sigma) \to (\Omega^{-n}, (-1)^n \mathbb{1}_{\Omega^{-n-1}})$$

of triangle functors.

Proof. By Proposition 1.5 we find for each $M \in \operatorname{mod} \mathcal{F}$ an n-angle $X_\bullet \in \pentagon$ such that $\operatorname{Ker}(\alpha_1) = M$. We set then $\delta(\pentagon)_M = \delta_{X_\bullet} \in \underline{\operatorname{Hom}}_{\mathcal{F}}(\Sigma M, \Omega^{-n} M)$. As in [15, p.53] one verifies that this isomorphism does not depend of the particular choice of $X_\bullet \in \pentagon$.

It remains to show that we have for each M a commutative diagram

$$\begin{tikzcd}
\Sigma\Omega^{-1} \arrow[r, "\sigma_M"] \arrow[d, "\delta(\pentagon)_{\Omega^{-1}M}"'] & \Omega^{-1}\Sigma M \arrow[d, "\Omega^{-1}\delta(\pentagon)_M"] \\
\Omega^{-n-1}M \arrow[r, "(-1)^n\mathbf{1}_{\Omega^{-1-n}M}"'] & \Omega^{-1-n}M
\end{tikzcd}$$

in $\underline{\operatorname{mod}}\,\mathcal{F}$. This is mutatits mutandis the same argument as in the proof of [15, Lemma 16.3]. $\square$

2.3. **Lemma.** *Suppose* $\operatorname{mod}\mathcal{F}$ *is a Frobenius category. Let* $\Theta\colon (\Sigma,\sigma) \to (\Omega^{-n}, (-1)^n\mathbb{1}_{\Omega^{-n-1}})$ *be an isomorphism of triangle functors in* $\underline{\operatorname{mod}}\,\mathcal{F}$. *Then the collection* $\pentagon_\Theta$ *of exact* n*-*Σ*-sequences* $X_\bullet = (X_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} \Sigma X_1)$ *in* $\mathcal{F}$ *such that* $\delta_{X_\bullet} = \Theta_{\operatorname{Ker}(\alpha_1)}$ *is a pre-*n*-angulation of* $(\mathcal{F},\Sigma)$.

Proof. We verify the axioms (F1)-(F3) for $\pentagon_\Theta$.

(F1a) and (F1b) are trivially fulfilled.

(F1c): Let $\alpha_1 \in \mathcal{F}(X_1, X_2)$, and $A := \operatorname{Ker}(\alpha_1)\operatorname{mod}\mathcal{F}$. Clearly, we can find in $\operatorname{mod}\mathcal{F}$ a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, equal] & X_1 \arrow[r, "\alpha_1"] \arrow[d] & X_2 \arrow[r, "\alpha_2"] \arrow[d] & X_3 \arrow[r, "\alpha_3"] \arrow[d] & \cdots \arrow[r, "\alpha_{n-2}"] & X_{n-1} \arrow[r, "\pi_n"] \arrow[d] & C \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & A \arrow[r] & I_A \arrow[r] & I_{\Omega^{-1}A} \arrow[r] & I_{\Omega^{-2}A} \arrow[r] & \cdots \arrow[r] & I_{\Omega^{2-n}A} \arrow[r] & \Omega^{1-n}A \arrow[r] & 0
\end{tikzcd}$$

and $X_3, \dots, X_{n-1} \in \mathcal{F}$. Note that the class of φ in $\underline{\operatorname{mod}}\,\mathcal{F}$ is an isomorphism.

Next we find a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "\iota_{n-1}"] \arrow[d, equal] & X_n \arrow[r, "\pi_n"] \arrow[d] & \Sigma A \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d, "\varphi"'] & I_C \arrow[r] \arrow[d] & \Omega^{-1}C \arrow[r] \arrow[d, "\Omega^{-1}\varphi"] & 0 \\
0 \arrow[r] & \Omega^{1-n}A \arrow[r] & I_{\Omega^{1-n}A} \arrow[r] & \Omega^{-n}A \arrow[r] & 0
\end{tikzcd}$$

such that $X_n \in \mathcal{F}$ and $\Omega^{-1}\varphi\theta = \Theta_A \in \underline{\operatorname{Hom}}_{\mathcal{F}}(\Sigma A, \Omega^{-n}A)$. Then, $X_\bullet$ with $\alpha_{n-1} = \iota_{n-1}\pi_{n-1}$ and $\alpha_n = (\Sigma\iota)\pi_n$ is an n-Σ-sequence with $\delta_{X_\bullet} = \Theta_A$.

(F2): This follows since Θ is an isomorphism of *triangle* functors, see the second part of the proof of Lemma 2.2.

(F3): In the situation of (F3), let $A = \operatorname{Ker}(\alpha_1)$ and $B = \operatorname{Ker}(\beta_1)$. We clearly obtain a commutative diagram:

$$\begin{tikzcd}
& X_1 \arrow[r, "\alpha_1"] \arrow[dd, "\varphi_1"'] & X_2 \arrow[r, "\alpha_2"] \arrow[dd, "\varphi_2"'] & X_3 \arrow[r, dotted, dash] \arrow[dd, "\varphi_3"'] & X_n \arrow[rr, "\alpha_n"] \arrow[dd, "\varphi_n"'] \arrow[dr, "\pi_n"] & & \Sigma X_n \arrow[dd, "\varphi_{n+1}"] \\
A \arrow[ur, "\iota"] \arrow[dd, "\varphi"'] & & & & & \Sigma A \arrow[ur, "\Sigma\iota"'] \arrow[dd, "\psi"'] & \\
& Y_1 \arrow[r, "\beta_1"'] & Y_2 \arrow[r, "\beta_2"'] & Y_3 \arrow[r, dotted, dash] & Y_n \arrow[rr, "\beta_n"'] \arrow[dr, "\rho_n"'] & & \Sigma Y_n \\
B \arrow[ur, "\kappa"'] & & & & & \Sigma B \arrow[ur, "\Sigma\kappa"'] &
\end{tikzcd}$$

By construction, $\delta_{Y_\bullet}\psi = (\Omega^{-n}\varphi)\delta_{X_\bullet}$. On the other hand, by the naturality of Θ, we have

$$\delta_{Y_\bullet}(\Sigma\varphi) = (\Omega^{-n}\varphi)\delta_{X_\bullet} \in \underline{\operatorname{Hom}}_{\mathcal{F}}(\Sigma A, \Omega^{-n} B).$$

Thus, we have $\eta' \in \operatorname{Hom}_{\mathcal{F}}(\Sigma A, Y_n)$ and $\eta'' \in \operatorname{Hom}_{\mathcal{F}}(\Sigma X_n, \Sigma B)$ such that $\psi - \Sigma\varphi = \rho_n\eta' = \eta''(\Sigma\iota)$. As a consequence, we may replace in the above diagram ψ by $\Sigma\varphi$, φ_n by $\varphi'_n = \varphi_n - \eta'\pi_n$ and φ_{n+1} by $\varphi'_{n+1} = \varphi_{n+1} - (\Sigma\kappa)\eta''$. Now, $(\varphi'_{n+1} - \Sigma\varphi_1)(\Sigma\iota) = \Sigma(\varphi\kappa - \varphi_1\iota) = 0$, so that $(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, \varphi'_n)$ is a morphism of n-Σ-sequences. $\square$

2.4. **Proposition.** *Let $(\mathcal{F}, \Sigma, \pentagon)$ be a pre-n-angulated category. Then the map δ from the class of pre-n-angulations of $(\mathcal{F}, \Sigma)$ to the class of isomorphisms of triangle functors between (Σ, σ) and $(\Omega^{-n}, (-1)^n \mathbb{1}_{\Omega^{-n-1}})$, defined in Lemma 2.2, is a bijection. As a consequence, the group of automorphisms of the triangle functor $(\Omega^{-n}, (-1)^n \mathbb{1}_{\Omega^{-n-1}})$ acts simply transitively on the class of pre-n-angulations of $(\mathcal{F}, \Sigma)$.*

Proof. If $\Theta\colon (\Sigma, \sigma) \to (\Omega^{-n}, (-1)^n \mathbb{1}_{\Omega^{-n-1}})$ is an isomorphism of triangle functors, we have by Lemma 2.3 $\delta(\pentagon_\Theta) = \Theta$. If $\delta(\pentagon') = \Theta$, we have by construction $\pentagon' \subset \pentagon_\Theta$, but then $\pentagon' = \pentagon_\Theta$ by Proposition 1.5 (c). $\square$

3. Standard construction

3.1. **Definition.** Let $\mathcal{T}$ be a triangulated category with suspension Σ_3. A full subcategory $\mathcal{S} \subseteq \mathcal{T}$ is called *d-cluster tilting* if it is functorially finite [3, p.82], and

$$\begin{aligned} \mathcal{S} &= \{X \in \mathcal{T} \mid \mathcal{T}(\mathcal{S}, \Sigma_3^i X) = 0 \ \forall i \in \{1, \ldots, d-1\}\} \\ &= \{X \in \mathcal{T} \mid \mathcal{T}(X, \Sigma_3^i \mathcal{S}) = 0 \ \forall i \in \{1, \ldots, d-1\}\}. \end{aligned}$$

Note that such a category has first been called maximal $(d-1)$-orthogonal by Iyama [17].

Theorem 1. *Let $\mathcal{T}$ be a triangulated category with an $(n-2)$-cluster tilting subcategory $\mathcal{F}$, which is closed under Σ_3^{n-2}, where Σ_3 denotes the suspension in $\mathcal{T}$. Then $(\mathcal{F}, \Sigma_3^{n-2}, \pentagon)$ is an n-angulated category, where $\pentagon$ is the class of all sequences*

$$X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} \Sigma_3^{n-2} X_1$$

such that there exists a diagram

$$\begin{tikzcd}[column sep=small]
 & X_2 \arrow[rr, "\alpha_2"] \arrow[dr] & & X_3 \arrow[dr] & & \cdots & & X_{n-1} \arrow[dr, "\alpha_{n-1}"] & \\
X_1 \arrow[ur, "\alpha_1"] & & X_{2.5} \arrow[ll, "+"] \arrow[ur] & & X_{3.5} \arrow[ll, "+"] & \cdots & X_{n-1.5} \arrow[ur] & & X_n \arrow[ll, "+"]
\end{tikzcd}$$

with $X_i \in \mathcal{T}$ for $i \notin \mathbb{Z}$, such that all oriented triangles are triangles in $\mathcal{T}$, all non-oriented triangles commute, and α_n is the composition along the lower edge of the diagram.

3.2. **Remarks.** In the situation of Theorem 1 we have the following:

- The suspension in the n-angulated category $\mathcal{F}$ is given by Σ_3^{n-2}. In particular it should not be confused with Σ_3.
- If $\mathcal{F} \subset \mathcal{T}$ is an $(n-2)$-cluster tilting subcategory, then $\mathcal{F}$ is *not* closed under Σ_3^h for $1 \leq h < n-2$. If $\mathcal{T}$ has a Serre functor $\mathbb{S}$, then the assumption that $\mathcal{F}$ is closed under Σ_3^{n-2} is equivalent to $\mathcal{F}$ being closed under $\mathbb{S}$. (It follows immediately from the definition of $(n-2)$-cluster tilting that $\mathcal{F}$ is closed under $\mathbb{S}\Sigma_3^{-(n-2)}$.)

The following observation will turn out to be helpful in the proof of Theorem 1.

3.3. **Lemma.** *In the notation of Theorem 1, we have*

$$\mathcal{T}(X_{i+1.5}, \Sigma_3^j \mathcal{F}) = 0 \ \textit{for} \ 0 < j < i < n-2$$

Proof. We use downward induction on i. For $i = n-3$ consider the triangle
$$X_{n-1.5} \to X_{n-1} \to X_n \to \Sigma_3 X_{n-1.5}.$$
Since $\mathcal{T}(X_{n-1}, \Sigma_3^j \mathcal{F}) \subseteq \mathcal{T}(\mathcal{F}, \Sigma_3^j \mathcal{F}) = 0$ we have a monomorphism
$$\mathcal{T}(X_{n-1.5}, \Sigma_3^j \mathcal{F}) \to \mathcal{T}(X_n, \Sigma_3^{j+1} \mathcal{F}) = 0,$$
so the claim of the lemma holds for $i = n-3$.

For arbitrary $i < n-3$ we use the triangle
$$X_{i+1.5} \to X_{i+2} \to X_{i+2.5} \to \Sigma_3 X_{i+1.5}.$$
As above we get a monomorphism
$$\mathcal{T}(X_{i+1.5}, \Sigma_3^j \mathcal{F}) \to \mathcal{T}(X_{i+2.5}, \Sigma_3^{j+1} \mathcal{F}).$$
Now the latter space is 0 inductively, so the claim holds. □

Proof of Theorem 1. We verify the axioms.

(F1) (a) The class $⬠$ is closed under direct summands, as one easily shows using the fact that, up to isomorphism, the construction of a triangle over a morphism commutes with the passage to direct summands. Part (b) is clear.

For (F1)(c) let α_1 be a morphism in $\mathcal{F}$. Let $X_{2.5}$ be the cone of α_1 in $\mathcal{T}$. Since $\mathcal{F} \subseteq \mathcal{T}$ is $(n-2)$-cluster tilting $X_{2.5}$ has an $\mathcal{F}$-coresolution of length at most $n-2$. Combining this coresolution with the original triangle we obtain an n-angle starting with α_1.

Next we verify axiom (F4). So assume we have the following commutative diagram.

$$\begin{tikzcd}
 & X_2 \arrow[rr, "\alpha_2"] \arrow[dr, "\alpha_2'"] \arrow[dd, "\varphi_2"] & & X_3 \arrow[dr, "\alpha_3'"] & & \cdots & & X_{n-1} \arrow[dr, "\alpha_{n-1}"] & \\
X_1 \arrow[ur, "\alpha_1"] \arrow[dd, "\varphi_1"] & & X_{2.5} \arrow[ll, "\alpha_n^2", "|" marking] \arrow[ur, "\alpha_2''"] & & X_{3.5} \arrow[ll, "\alpha_n^3", "|" marking] & \cdots & X_{n-1.5} \arrow[ur, "\alpha_{n-2}''"] & & X_n \arrow[ll, "\alpha_n^{n-1}", "|" marking] \\
 & Y_2 \arrow[rr, "\beta_2"] \arrow[dr, "\beta_2'"] & & Y_3 \arrow[dr, "\beta_3'"] & & \cdots & & Y_{n-1} \arrow[dr, "\beta_{n-1}"] & \\
Y_1 \arrow[ur, "\beta_1"] & & Y_{2.5} \arrow[ll, "\beta_n^2", "|" marking] \arrow[ur, "\beta_2''"] & & Y_{3.5} \arrow[ll, "\beta_n^3", "|" marking] & \cdots & Y_{n-1.5} \arrow[ur, "\beta_{n-2}''"] & & Y_n \arrow[ll, "\beta_n^{n-1}", "|" marking]
\end{tikzcd}$$

DIAGRAM 3.1. Given morphisms

The aim is to obtain a diagram

$$\begin{tikzcd}
 & X_3 \oplus Y_2 \arrow[rr, "\begin{pmatrix} -\alpha_3 & \\ \varphi_3 & \beta_2 \end{pmatrix}"] \arrow[dr] & & X_4 \oplus Y_3 \arrow[dr] & & \cdots & & X_n \oplus Y_{n-1} \arrow[dr, "\begin{pmatrix} -\alpha_n & \\ \varphi_n & \beta_{n-1} \end{pmatrix}"] & \\
X_2 \oplus Y_1 \arrow[ur, "\begin{pmatrix} -\alpha_2 & \\ \varphi_2 & \beta_1 \end{pmatrix}"] & & H_{2.5} \arrow[ll, "|" marking] \arrow[ur] & & H_{3.5} \arrow[ll, "|" marking] & \cdots & H_{n-1.5} \arrow[ur] & & \Sigma_3^{n-2} X_1 \oplus Y_n \arrow[ll, "|" marking]
\end{tikzcd}$$

DIAGRAM 3.2. Target diagram

such that the composition along the lower row is $\begin{pmatrix} -\Sigma_3^{n-2}\alpha_1 & \\ \Sigma_3^{n-2}\varphi_1 & \beta_n \end{pmatrix}$.

We construct further morphisms φ_i, and at the same time the objects H_i from left to right. (We actually also construct φ_i for $i \in \{2.5, 3.5, \ldots, n-1.5\}$, even though they are not visible in the target diagram.)

We choose $\varphi_{2.5} \colon X_{2.5} \to Y_{2.5}$ to be a "good cone morphism", that is a cone morphism such that the sequence

$$X_2 \oplus Y_1 \xrightarrow{\begin{pmatrix} -\alpha'_2 & \\ \varphi_2 & \beta_1 \end{pmatrix}} X_{2.5} \oplus Y_2 \xrightarrow{\begin{pmatrix} -\alpha_n^2 & \\ \varphi_{2.5} & \beta'_2 \end{pmatrix}} \Sigma_3 X_1 \oplus Y_{2.5} \xrightarrow{\begin{pmatrix} -\Sigma_3\alpha_1 & \\ \Sigma_3\varphi_1 & \beta_n^2 \end{pmatrix}} \Sigma_3 X_2 \oplus \Sigma_3 Y_1$$

is a triangle in $\mathcal{T}$.

Now the left-most triangle of the target diagram is the lower horizontal triangle in following octahedron.

$$\begin{tikzcd}
& \Sigma_3^{-1} X_{3.5} \arrow[r, equal] \arrow[d, "\begin{pmatrix} \Sigma_3^{-1}\alpha_n^3 \\ 0 \end{pmatrix}"] & \Sigma_3^{-1} X_{3.5} \arrow[d] & \\
X_2 \oplus Y_1 \arrow[r, "\begin{pmatrix} -\alpha'_2 & \\ \varphi_2 & \beta_1 \end{pmatrix}"] \arrow[d, equal] & X_{2.5} \oplus Y_2 \arrow[r, "\begin{pmatrix} -\alpha_n^2 & \\ \varphi_{2.5} & \beta'_2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \alpha''_2 & \\ & \mathbf{1} \end{pmatrix}"] & \Sigma_3 X_1 \oplus Y_{2.5} \arrow[r, "\begin{pmatrix} -\Sigma_3\alpha_1 & \\ \Sigma_3\varphi_1 & \beta_n^2 \end{pmatrix}"] \arrow[d, dashed] & \Sigma_3 X_2 \oplus \Sigma_3 Y_1 \arrow[d, equal] \\
X_2 \oplus Y_1 \arrow[r, "\begin{pmatrix} -\alpha_2 & \\ \varphi_2 & \beta_1 \end{pmatrix}"] & X_3 \oplus Y_2 \arrow[r, dashed] \arrow[d, "(-\alpha'_3\ 0)"] & H_{2.5} \arrow[r, dashed] \arrow[d, dashed] & \Sigma_3 X_2 \oplus \Sigma_3 Y_1 \\
& X_{3.5} \arrow[r, equal] & X_{3.5} &
\end{tikzcd}$$

DIAGRAM 3.3

We now iteratedly construct the further triangles of the target diagram (Diagram 3.2). We assume to have a triangle

$$H_{i+1.5} \longrightarrow X_{i+2.5} \xrightarrow{\begin{pmatrix} -\Sigma_3^i\alpha_n^2 \cdots \Sigma_3\alpha_n^{i+1}\alpha_n^{i+2} \\ \varphi_{i+1.5}\alpha_n^{i+2} \end{pmatrix}} \Sigma_3^{i+1} X_1 \oplus \Sigma_3 Y_{i+1.5} \longrightarrow \Sigma_3 H_{i+1.5}.$$

For $i = 1$ this can be found in the right column of the octahedron in Diagram 3.3, for $i > 1$ it is in the right column of the octahedron in Diagram 3.5 for $i - 1$. We use this triangle to construct the following octahedron

$$\begin{tikzcd}
& Y_{i+2} \arrow[r, equal] \arrow[d, dashed, "\begin{pmatrix} 0 \\ \mathbf{1} \end{pmatrix}"'] & Y_{i+2} \arrow[d, "\begin{pmatrix} 0 \\ \beta'_{i+2} \end{pmatrix}"] & \\
H_{i+1.5} \arrow[r, dashed] \arrow[d, equal] & X_{i+2.5} \oplus Y_{i+2} \arrow[r, dashed, "(*)"] \arrow[d, dashed, "(\mathbf{1}\ 0)"'] & \Sigma_3^{i+1} X_1 \oplus Y_{i+2.5} \arrow[r] \arrow[d, "\begin{pmatrix} \mathbf{1} & \\ & \beta_n^{i+2} \end{pmatrix}"] & \Sigma_3 H_{i+1.5} \arrow[d, equal] \\
H_{i+1.5} \arrow[r] & X_{i+2.5} \arrow[r, "\begin{pmatrix} -\Sigma_3^i\alpha_n^2 \cdots \Sigma_3\alpha_n^{i+1}\alpha_n^{i+2} \\ \varphi_{i+1.5}\alpha_n^{i+2} \end{pmatrix}"] \arrow[d, "0"'] & \Sigma_3^{i+1} X_1 \oplus \Sigma_3 Y_{i+1.5} \arrow[r] \arrow[d, "(0\ -\Sigma_3\beta''_{i+1})"] & \Sigma_3 H_{i+1.5} \\
& \Sigma_3 Y_{i+2} \arrow[r, equal] & \Sigma_3 Y_{i+2} &
\end{tikzcd}$$

DIAGRAM 3.4

Note that the map $X_{i+2.5} \to \Sigma_3 Y_{i+2}$ vanishes by Lemma 3.3. Hence the left vertical triangle splits, and thus can be assumed to have the form above.

We dermine the shape of the map $(*)$: By commutativity of the upper square we know that the right column of the matrix $(*)$ is $\left(\begin{smallmatrix} 0 \\ \beta'_{i+2} \end{smallmatrix}\right)$, and by commutativity of the central square we know that the left upper entry of the matrix $(*)$ is $-\Sigma_3^i \alpha_n^2 \cdots \alpha_n^{i+2}$. Hence

$$(*) = \begin{pmatrix} -\Sigma_3^i \alpha_n^2 \cdots \alpha_n^{i+2} & 0 \\ \varphi_{i+2.5} & \beta'_{i+2} \end{pmatrix}$$

for some $\varphi_{i+2.5} \colon X_{i+2.5} \to Y_{i+2.5}$.

Now we take the upper horizontal triangle of the above octahedron to construct the following.

$$\begin{tikzcd}
 & \Sigma_3^{-1} X_{i+3.5} \arrow[r, equal] \arrow[d, "\left(\begin{smallmatrix} \Sigma_3^{-1}\alpha_n^{i+3} \\ 0 \end{smallmatrix}\right)"'] & \Sigma_3^{-1} X_{i+3.5} \arrow[d] & \\
H_{i+1.5} \arrow[r] \arrow[d, equal] & X_{i+2.5} \oplus Y_{i+2} \arrow[r, "\left(\begin{smallmatrix} -\Sigma_3^i \alpha_n^2 \cdots \alpha_n^{i+2} & 0 \\ \varphi_{i+2.5} & \beta'_{i+2} \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} \alpha''_{i+2} & \\ & \mathbf{1} \end{smallmatrix}\right)"'] & \Sigma_3^{i+1} X_1 \oplus Y_{i+2.5} \arrow[r] \arrow[d, dashed] & \Sigma_3 H_{i+1.5} \arrow[d, equal] \\
H_{i+1.5} \arrow[r] & X_{i+3} \oplus Y_{i+2} \arrow[r, dashed] \arrow[d, "(-\alpha'_{i+3}\ 0)"'] & H_{i+2.5} \arrow[r, dashed] \arrow[d, dashed] & \Sigma_3 H_{i+1.5} \\
 & X_{i+3.5} \arrow[r, equal] & X_{i+3.5} &
\end{tikzcd}$$

DIAGRAM 3.5

The lower row of this diagram is supposed to be the $(i+1)$-st triangle of the target diagram. We need to check that the composition

$$(\dagger) \qquad X_{i+2} \oplus Y_{i+1} \to H_{i+1.5} \to X_{i+3} \oplus Y_{i+2}$$

is of the form $\left(\begin{smallmatrix} -\alpha_{i+2} & \\ \varphi_{i+2} & \beta_{i+1} \end{smallmatrix}\right)$ for some $\varphi_{i+2} \colon X_{i+2} \to Y_{i+2}$. First note that by commutativity of the left square of Diagram 3.5 the above composition is

$$= \begin{pmatrix} \alpha''_{i+2} & \\ & \mathbf{1} \end{pmatrix} \circ \Big[X_{i+2} \oplus Y_{i+1} \to H_{i+1.5} \to X_{i+2.5} \oplus Y_{i+2} \Big]$$

We first focus on the components of $(\dagger)$ starting in Y_{i+1}. By commutativity of the central squares of Diagrams 3.3 (for $i = 1$) and 3.5 (for $i > 1$) we have that these components can be rewritten

$$\begin{pmatrix} \alpha''_{i+2} & \\ & \mathbf{1} \end{pmatrix} \circ \Big[\Sigma_3^i X_1 \oplus Y_{i+1.5} \to H_{i+1.5} \to X_{i+2.5} \oplus Y_{i+2} \Big] \circ \begin{pmatrix} 0 \\ \beta'_{i+1} \end{pmatrix},$$

and further, by anti-commutativity of the outer square of Diagram 3.4, as

$$\begin{pmatrix} \alpha''_{i+2} & \\ & \mathbf{1} \end{pmatrix} \circ \left(\begin{smallmatrix} 0 \\ \mathbf{1} \end{smallmatrix}\right) \circ (0\ \beta''_{i+1}) \circ \begin{pmatrix} 0 \\ \beta'_{i+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \beta_{i+1} \end{pmatrix}.$$

Next we look at the components of $(\dagger)$ ending in X_{i+3}. That is

$$\begin{aligned} &(\alpha''_{i+2}\ 0) \circ \Big[X_{i+2} \oplus Y_{i+1} \to H_{i+1.5} \to X_{i+2.5} \oplus Y_{i+2} \Big] \\ =&\alpha''_{i+2} \circ \Big[X_{i+2} \oplus Y_{i+1} \to H_{i+1.5} \to X_{i+2.5} \Big]. \end{aligned}$$

By commutativity of the lower squares of Diagrams 3.3 (for $i = 0$) and 3.5 (for $i > 1$) this is

$$\alpha''_{i+2} \circ (-\alpha'_{i+2}\ 0) = (-\alpha_{i+2}\ 0).$$

Thus we have now shown that the composition $(\dagger)$ is of the form $\left(\begin{smallmatrix} -\alpha_{i+2} & \\ \varphi_{i+2} & \beta_{i+1} \end{smallmatrix}\right)$ as claimed.

This iteration works for $i = 1, \dots, n-4$ (for $i = n-4$ we set $X_{n-0.5} := X_n$ and $\alpha'_{n-1} := \alpha_{n-1}$ in Diagram 3.5). With these (and similar) identifications Diagram 3.4 can still be constructed for $i = n-3$. Its upper horizontal triangle is

$$\begin{tikzcd}
H_{n-1.5} \arrow[r] & X_{n-0.5} \oplus Y_{n-1} \arrow[rr, "\left(\begin{smallmatrix} -\Sigma_3^{n-3}\alpha_n^2\cdots\alpha_n^{n-1} & 0 \\ \varphi_{n-0.5} & \beta'_{n-1} \end{smallmatrix}\right)"] \arrow[d, equal] & & \Sigma_3^{n-2}X_1 \oplus Y_{n-0.5} \arrow[r] \arrow[d, equal] & \Sigma_3 H_{n-1.5} \\
& X_n \oplus Y_{n-1} \arrow[rr, "\left(\begin{smallmatrix} -\alpha_n & 0 \\ \varphi_n & \beta_{n-1} \end{smallmatrix}\right)"'] & & \Sigma_3^{n-2}X_1 \oplus Y_n &
\end{tikzcd}$$

This is the final triangle of the target diagram (Diagram 3.2).

It remains to check that the composition

$$\Sigma_3^{n-2}X_1 \oplus Y_n \to \Sigma_3 H_{n-1.5} \to \Sigma_3^2 H_{n-2.5} \to \cdots \to \Sigma_3^{n-3}H_{2.5} \to \Sigma_3^{n-2}X_2 \oplus \Sigma_3^{n-2}Y_1$$

is $\left(\begin{smallmatrix} -\Sigma_3^{n-2}\alpha_1 & 0 \\ \Sigma_3^{n-2}\varphi_1 & \beta_n \end{smallmatrix}\right)$. By commutativity of the right squares of Diagrams 3.4 and 3.3 (for various i) we iteratedly obtain

$$\begin{aligned}
&= \left[\Sigma_3^{n-2}X_1 \oplus \Sigma_3 Y_{n-1.5} \to \Sigma_3 H_{n-1.5} \to \cdots \to \Sigma_3^{n-3}H_{2.5} \to \Sigma_3^{n-2}X_2 \oplus \Sigma_3^{n-2}Y_1\right] \circ \left(\begin{smallmatrix} \mathbf{1} & \\ & \beta_n^{n-1} \end{smallmatrix}\right) \\
&= \left[\Sigma_3^{n-2}X_1 \oplus \Sigma_3 Y_{n-1.5} \to \Sigma_3^2 H_{n-2.5} \cdots \to \Sigma_3^{n-3}H_{2.5} \to \Sigma_3^{n-2}X_2 \oplus \Sigma_3^{n-2}Y_1\right] \circ \left(\begin{smallmatrix} \mathbf{1} & \\ & \beta_n^{n-1} \end{smallmatrix}\right) \\
&= \left[\Sigma_3^{n-2}X_1 \oplus \Sigma_3^2 Y_{n-2.5} \to \Sigma_3^2 H_{n-2.5} \cdots \to \Sigma_3^{n-3}H_{2.5} \to \Sigma_3^{n-2}X_2 \oplus \Sigma_3^{n-2}Y_1\right] \circ \left(\begin{smallmatrix} \mathbf{1} & \\ & \Sigma_3\beta_n^{n-2}\beta_n^{n-1} \end{smallmatrix}\right) \\
&\quad\vdots \\
&= \left[\Sigma_3^{n-2}X_1 \oplus \Sigma_3^{n-3}Y_{2.5} \to \Sigma_3^{n-3}H_{2.5} \to \Sigma_3^{n-2}X_2 \oplus \Sigma_3^{n-2}Y_1\right] \circ \left(\begin{smallmatrix} \mathbf{1} & \\ & \Sigma_3^{n-4}\beta_n^3\cdots\beta_n^{n-1} \end{smallmatrix}\right)
\end{aligned}$$

and by commutativity of the right square of Diagram 3.3 this is

$$\begin{aligned}
&= \left(\begin{smallmatrix} -\Sigma_3^{n-2}\alpha_1 & \\ \Sigma_3^{n-2}\varphi_1 & \Sigma_3^{n-3}\beta_n^2 \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} \mathbf{1} & \\ & \Sigma_3^{n-4}\beta_n^3\cdots\beta_n^{n-1} \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} -\Sigma_3^{n-2}\alpha_1 & \\ \Sigma_3^{n-2}\varphi_1 & \beta_n \end{smallmatrix}\right)
\end{aligned}$$

This completes the verification of axiom (F4).

Axiom (F3) follows from (F4) immediately.

To see that axiom (F2) holds first note that the $\pentagon$ is clearly closed under shifts by $\pm 2n$. Hence it suffices to show the "only if"-part of the claim. This follows from (F4) for $Y_i = 0$. $\square$

4. Calabi-Yau Properties

4.1. **Serre functor and Calabi-Yau categories.** Let k be a field and $\mathcal{F}$ a k-category. For a self-equivalence $\mathbb{S}$ of $\mathcal{F}$ and $M \in \operatorname{mod}\mathcal{F}$ we denote by $\mathbb{S}M$ the twisted module $M \circ \mathbb{S}^{-1}$. Let us consider the $\mathcal{F}$-bimodules $D\mathcal{F}$ and $\mathbb{S}\mathcal{F}$ which are defined as follows:

$$D\mathcal{F}(x,y) := \operatorname{Hom}_k(\mathcal{F}(y,x),k) \text{ and } \mathbb{S}\mathcal{F}(x,y) := \mathcal{F}(\mathbb{S}^{-1}x,y) \text{ for all } x,y \in \mathcal{F}.$$

We say that $\mathbb{S}$ is a *Serre-functor* if $\mathbb{S}\mathcal{F} \cong D\mathcal{F}$ as $\mathcal{F}$-bimodules.

Suppose that $\mathcal{T}$ is a triangulated category with suspension Σ such that the k-linear category $\mathcal{T}$ admits a Serre functor $\mathbb{S}$ as above. Then, as shown in [6] and [5, Appendix], $\mathbb{S}$ becomes a canonically a triangle functor $\mathcal{T} \to \mathcal{T}$. We say that $\mathcal{T}$ is *n-Calabi-Yau* if $\mathbb{S}$ is isomorphic to Σ^n as a triangle functor.

4.2. **Serre functor for stable module categories.** Let $\mathcal{F}$ be a k-category with a Serre functor $\mathbb{S}$ as above, and suppose that the category $\operatorname{mod}\mathcal{F}$ of finitely presented functors $\mathcal{F} \to \operatorname{mod} k$ is abelian. Then it is automatically a Frobenius category and $- \otimes_{\mathcal{F}} D\mathcal{F} \cong \mathbb{S}$ is a self-equivalence of $\operatorname{mod}\mathcal{F}$. In fact, $\mathbb{S}$ is a Nakayama functor *i.e.* we have a functorial homomorphism

$$\psi_Y : D\operatorname{Hom}_{\mathcal{F}}(Y, ?) \to \operatorname{Hom}_{\mathcal{F}}(?, \mathbb{S}Y)$$

which is an isomorphism for $Y \in \operatorname{mod}\mathcal{F}$ projective. It follows then from the lemma below that the triangle functor

$$\Omega_{\underline{\operatorname{mod}}\mathcal{F}} \circ \mathbb{S}$$

is a Serre functor for the stable category $\underline{\operatorname{mod}}\mathcal{F}$ which is naturally a triangulated category [15, 13] with suspension functor $\Omega^{-1}_{\underline{\operatorname{mod}}\mathcal{F}}$.

4.3. **Lemma.** *Let $\mathcal{E}$ be a Hom-finite Frobenius category and denote by Ω^{-1} the suspension functor for the stable category $\underline{\mathcal{E}}$. Let $\mathcal{P} \subset \mathcal{E}$ be the full subcategory of projective-injectives. Suppose that $\nu \colon \mathcal{P} \xrightarrow{\sim} \mathcal{P}$ is a Serre functor for $\mathcal{P}$ such that $D\mathcal{E}(P, X) \cong \mathcal{E}(X, \nu P)$ holds functorially for $P \in \mathcal{P}$ and $X \in \mathcal{E}$. Then ν induces a self-equivalence of $\underline{\mathcal{E}}$ which we denote also by ν and the composition $S_{\underline{\mathcal{E}}} := \Omega \circ \nu$ is a Serre functor for the stable category $\underline{\mathcal{E}}$ as a triangle functor.*

Proof. Let $\mathcal{C}_{\mathrm{ac}}(\mathcal{P})$ be the dg-category of acyclic complexes of projectives from $\mathcal{P}$, and $\mathcal{H}_{\mathrm{ac}}(\mathcal{P}) = H^0(\mathcal{C}_{\mathrm{ac}}(\mathcal{P}))$ its homotopy category. It follows from [22] that we have an equivalence of triangulated categories

$$Z^0 \colon \mathcal{H}_{\mathrm{ac}}(\mathcal{P}) \to \underline{\mathcal{E}}$$

which sends a complex $P^\bullet = (\cdots \to P^{-1} \to P^0 \to P^1 \to \cdots)$ to $Z^0 P^\bullet$. Since $\mathcal{E}$ has enough projectives, a complex $P^\bullet$ in $\mathcal{E}$ is acyclic if and only if $\mathcal{E}(Q, P^\bullet)$ is acyclic for any $Q \in \mathcal{P}$. Dually, it is acyclic if and only if $\mathcal{E}(P^\bullet, I)$ is acyclic for any $I \in \mathcal{P}$. It follows, that ν defines a selfequivalence of $\mathcal{H}_{\mathrm{ac}}(\mathcal{P})$. We denote also by ν the induced selfequivalence of $\underline{\mathcal{E}}$. Now, using the isomorphism $D\mathcal{E}(P, X) \cong \mathcal{E}(X, \nu P)$, it is easy to verify for $P^\bullet, Q^\bullet \in \mathcal{H}_{\mathrm{ac}}(\mathcal{P})$ the following isomorphisms

$$\begin{aligned} D\mathcal{H}_{\mathrm{ac}}(\mathcal{P})(P^\bullet, Q^\bullet) &\cong DH^0\mathcal{E}(P^\bullet[-1], Z^0(Q^\bullet)) \\ &\cong H^0\mathcal{E}(Z^0(Q^\bullet), \nu P^\bullet[-1]) \cong \mathcal{H}_{\mathrm{ac}}(\mathcal{P})(Q^\bullet, \nu P^\bullet[-1]), \end{aligned}$$

which imply our claim. □

4.4. **Construction of Calabi-Yau categories from n-angulated categories.** Consider now an n-angulated k-category $\mathcal{F} = (\mathcal{F}, \Sigma_n, \pentagon)$ which has a Serre-functor $\mathbb{S}$ with $\mathbb{S} = \Sigma_n^d$ as k-linear functors. Then the triangulated category $\underline{\operatorname{mod}}\mathcal{F}$ is $(nd-1)$-Calabi-Yau. In fact, by Lemma 2.2 in $\underline{\operatorname{mod}}\mathcal{F}$, we have an isomorphism of triangle functors $\Omega^{-n}_{\underline{\operatorname{mod}}\mathcal{F}} \cong \Sigma_n$. Finally, by Lemma 4.3, $\underline{\operatorname{mod}}\mathcal{F}$ has as a Serre functor as a triangulated category:

$$\Omega_{\underline{\operatorname{mod}}\mathcal{F}} \circ \mathbb{S} \cong \Omega_{\underline{\operatorname{mod}}\mathcal{F}} \Sigma_n^d \cong \Omega^{1-nd}_{\underline{\operatorname{mod}}\mathcal{F}}$$

with all isomorphisms being isomorphisms of triangle functors.

5. Examples

5.1. **n-angulated subcategories of derived categories.** Let Λ be a finite dimensional algebra of global dimension at most $n-2$. We denote the Serre functor of $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$ by $\mathbb{S}$.

5.2. **Definition** (see [18]). The algebra Λ is called $(n-2)$-representation finite if the module category $\operatorname{mod}\Lambda$ has an $(n-2)$-cluster tilting object. (Cluster tilting subcategories of abelian categories are defined in the same way as cluster tilting subcategories for triangulated categories in Definition 3.1.)

In this case, by [16, Theorem 1.23], the subcategory

$$\mathcal{U} = \operatorname{add}\{\mathbb{S}^i\Lambda[-(n-2)i] \mid i \in \mathbb{Z}\}$$

of the bounded derived category is $(n-2)$-cluster tilting. Moreover, by [19, Corollary 3.7], $\mathbb{S}\mathcal{U} = \mathcal{U}$.

Thus, by Theorem 1, $\mathcal{U}$ is an n-angulated category for any $(n-2)$-representation finite algebra Λ.

5.3. **n-angulated categories in Calabi-Yau categories.** Let $\mathcal{T}$ be a triangulated d-Calabi-Yau-category with Serre functor $\mathbb{S}$. By our standard construction, an $(n-2)$-cluster tilting subcategory $\mathcal{F} \subset \mathcal{T}$ which is closed under Σ^{n-2} has a structure of a n-angulated category with suspension $\Sigma_n := \Sigma^{n-2}$. By the remarks in 3.2 we conclude that $d = d'(n-2)$ for some integer $d' \in \mathbb{Z}$, and in particular $\Sigma_n^{d'} = \mathbb{S}$. By Remark 4.4 we conclude that $\underline{\operatorname{mod}}\,\mathcal{F}$ is $(d + 2d' - 1)$-Calabi-Yau.

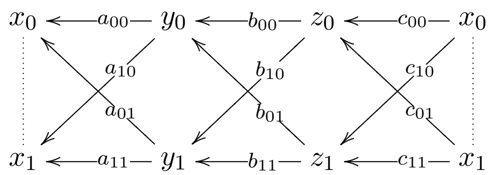
By arguments in [29] for $d = 2$ we have that the endomorphism ring of a 2-cluster tilting object T is selfinjective if and only if $\mathbb{S}T \in \operatorname{add}(T)$. Thus we see the following.

5.4. **Proposition.** *Let Λ be a selfinjective 2-Calabi-Yau tilted algebra. Then* $\operatorname{proj}\Lambda$ *is a 4-angulated category, with the Nakayama functor ν as suspension.*

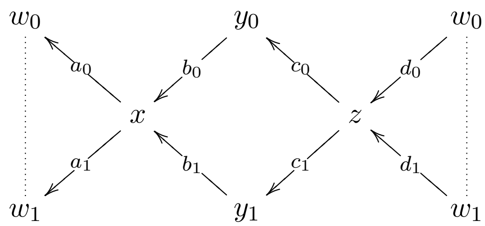
There are many examples for this proposition which are closely related to cluster algebras:

- By definition, a selfinjective cluster tilted algebra A is a selfinjective algebra of the form $A = \operatorname{End}_{\mathcal{C}_H}(T)$ for T a cluster tilting object T in a cluster category $\mathcal{C}_H$ associated to an hereditary algebra H. It is well-known, that A is selfinjective if and only if $T = \Sigma^2 T$. Thus, by our standard construction, the category $\operatorname{add}(T) \cong \operatorname{proj}(A)$ is a 4-angulated category with suspension $\Sigma_4 = \Sigma^2$ being the Serre functor. Note that the selfinjective cluster tilted algebras were classified by Ringel [29].
- Quite similarly, one may study selfinjective cluster quasi-tilted algebras of the form $A = \operatorname{End}_{\mathcal{C}_{\mathbb{X}}}(T)$ for a cluster tilting object T in the cluster category $\mathcal{C}_{\mathbb{X}}$ of a weighted projective line $\mathbb{X}$ in the sense of Geigle and Lenzing [10]. A similar analysis as in [29] shows, that this can occur only if $\mathbb{X}$ is of one of the following tubular types: $(6,3,2)$, $(4,4,2)$ and $(2,2,2,2)$. Note that since in $\mathcal{C}_{\mathbb{X}}$ we have $\tau = \Sigma$, for the tubular type $(3,3,3)$ no cluster tilting object T can fulfill $\Sigma^2 T = T$ because $\tau^3 = \mathbb{1}$. In contrast, by the same token in type $(2,2,2,2)$ all cluster tilting objects fulfill $\Sigma^2 T = T$. In this case, one obtains 4 families of weakly symmetric cluster quasi-tilted algebras which are best described using quivers with potential, cf. figure 1.

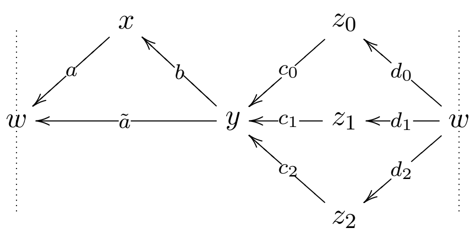
  Here, the parameter λ runs through $k \setminus \{0,1\}$ and the left and right borders (marked by a dotted line) of each quiver have to be identified. Recall that for weight type $(2,2,2,2)$ the cluster category $\mathcal{C}_{\mathbb{X}}$ depends on a parameter. In the corresponding 4-angulated categories with four isomorphism classes of indecomposable objects, the suspension Σ_4 as well as the Serre functor are the identity (which is a bit untypical).
- Let Λ be the preprojective algebra of Dynkin type $\mathsf{A}, \mathsf{D}, \mathsf{E}$. It is well-known that the category of projective modules $\operatorname{proj}\Lambda$ is a 1-Calabi-Yau triangulated category. Thus, the stable module category $\underline{\operatorname{mod}}\,\Lambda$ is a 2-Calabi-Yau triangulated category. In fact, it is a generalized cluster category in the sense of Amiot associated to a stable

$Q_1 =$ 

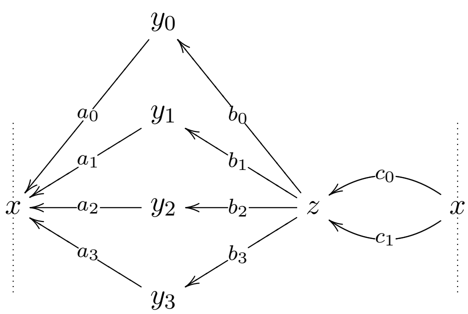
$$\begin{aligned} W_1 = {} & \lambda a_{00}b_{00}c_{00} - a_{01}b_{10}c_{00} \\ & + a_{11}b_{10}c_{01} - a_{10}b_{00}c_{01} \\ & + a_{01}b_{11}c_{10} - a_{00}b_{01}c_{10} \\ & + a_{10}b_{01}c_{11} - a_{11}b_{11}c_{11}, \end{aligned}$$

$Q_2 =$ 

$$\begin{aligned} W_2 = {} & \lambda a_0b_0c_0d_0 - a_0b_1c_1d_0 \\ & + a_1b_1c_1d_1 - a_1b_0c_0d_1, \end{aligned}$$

$Q_3 =$ 

$$\begin{aligned} W_3 = {} & \lambda \tilde{a}c_0d_0 - abc_0d_0 \\ & + abc_1d_1 - \tilde{a}c_1d_1 + \tilde{a}c_2d_2, \end{aligned}$$

$Q_4 =$ 

$$\begin{aligned} W_4 = {} & \lambda a_0b_0c_0 + a_1b_1c_0 - a_2b_2c_0 \\ & - a_0b_0c_1 - a_1b_1c_1 + a_3b_3c_1. \end{aligned}$$

FIGURE 1. Quivers with potentials for 4 families of weakly symmetric cluster quasi-tilted algebras

Auslander algebra of the corresponding Dynkin type [1]. With the description in [11] it is easy to check that for type A_n, the standard cluster tilting object T_n associated to the linear orientation of A_n fulfills $\Omega_\Lambda^{-2}T_n \cong T_n$. Thus, $A_n := \operatorname{End}_\Lambda(T_n)$ is selfinjective, and $\operatorname{proj} A_n \cong \operatorname{add}(T_n)$ is a 4-angulated category. Note, that for $n \geq 6$ the algebras A_n are not cluster (quasi-)tilted.

For $d > 2$ it is in general insufficient to check selfinjectivity of the cluster tilted algebra in order to find out if the cluster tilting object is closed under the Serre functor.

If Λ is $(n-2)$-representation finite then we still know (by [19, Corollary 3.7]) that the canonical $(n-2)$-cluster tilting object $\pi\Lambda$ in the corresponding $(n-2)$-Amiot cluster category $\mathcal{C}$ is closed under the Serre functor. Thus we have that $\operatorname{add}(\pi\Lambda) = \operatorname{proj}\operatorname{End}_{\mathcal{C}}(\pi\Lambda)$ is an n-angulated category.

These n-angulated categories (in particular for the case of higher Auslander algebras of linearly oriented A_n) are studied in [27, Sections 5 and 6].

5.5. **Link to algebraic geometry and string theory.** The theory of non-commutative Donaldson-Thomas invariants developed by Kontsevich-Soibelman in [25, 24] allows one to associate refined DT-invariants to suitable 3-Calabi-Yau categories. In particular, for each finite quiver Q endowed with a potential W which is a *finite* linear combination of cycles of length at least three, this theory provides us with a refined DT-invariant $A_{Q,W}$, which is an element of a quantized power series algebra Λ_Q depending only on Q. In string theory, the element $A_{Q,W}$ is also called [8] the *Kontsevich-Soibelman half-monodromy* and its square the *Kontsevich-Soibelman monodromy*. The determination of its spectrum (in representations of Λ_Q) is of considerable interest. An easy case is the one where $A_{Q,W}$ is of finite order. Many examples where this occurs are related to 4-angulated categories. Indeed, let us suppose that (Q, W) satisfies suitable finiteness conditions and in particular that its Jacobi algebra is finite-dimensional. Then one can show (see [23]) that the Kontsevich-Soibelman half-monodromy (more precisely: its adjoint action composed with the antipodal map) is 'categorified' by the suspension functor Σ of the generalized cluster-category $\mathcal{C}_{Q,W}$ associated with (Q, W) in [1]. Thus, the Kontsevich-Soibelman monodromy is categorified by Σ^2. If the cluster category $\mathcal{T} = \mathcal{C}_{Q,W}$ contains a 2-cluster tilting subcategory $\mathcal{F}$ invariant under Σ^2, then the action of Σ^2 on the set of isoclasses of indecomposables in $\mathcal{F}$ defines a faithful permutation representation of the cyclic group generated by the Kontsevich-Soibelman monodromy. In particular, in this case, the order of the Kontsevich-Soibelman monodromy is finite. The case where $\mathcal{F}$ is stable under Σ^2 is precisely the one where $\mathcal{F}$ inherits a natural structure of 4-angulated category with 4-suspension functor given by Σ^2. Thus, the action of the 4-suspension functor in the 4-angulated category $\mathcal{F}$ categorifies the Kontsevich-Soibelman monodromy.

As a particularly nice example (already mentioned in Section 5.3), let us consider the quiver Q below endowed with the potential W which is the signed sum over all small triangles, where adjacent triangles have different signs.

$$\begin{tikzcd}
& & & 0 \arrow[dr] & & & \\
& & 1 \arrow[ur] \arrow[dr] & & 2 \arrow[ll] \arrow[dr] & & \\
& 3 \arrow[ur] \arrow[dr] & & 4 \arrow[ll] \arrow[ur] \arrow[dr] & & 5 \arrow[ll] \arrow[dr] & \\
6 \arrow[ur] & & 7 \arrow[ll] \arrow[ur] & & 8 \arrow[ll] \arrow[ur] & & 9 \arrow[ll]
\end{tikzcd}$$

As shown in [11], the category of projective modules over the associated Jacobi algebra is equivalent to a 2-cluster tilting subcategory $\mathcal{F}$ of the stable module category over the preprojective algebra of type A_5. By [1], this category can also be described as the generalized cluster category associated with the above pair (Q, W). The suspension functor of this category is of order 6 and its square leaves $\mathcal{F}$ invariant. The associated permutation of the indecomposable projectives corresponds to the rotation by 120 degrees of the above quiver. We conclude that the 4-angulated category $\mathcal{F}$ has a 4-suspension functor of order 3 and that this is also the order of the Kontsevich-Soibelman monodromy associated with (Q, W).

We refer to [23] for a more detailed survey of the ideas sketched above and for many more examples.

References

[1] Claire Amiot, *Cluster categories for algebras of global dimension* 2 *and quivers with potential*, Annales de l'institut Fourier **59** (2009), no. 6, 2525–2590.

[2] Aslak Bakke Buan, Robert J. Marsh, Markus Reineke, Idun Reiten, and Gordana Todorov, *Tilting theory and cluster combinatorics*, Advances in Mathematics **204 (2)** (2006), 572–618.

[3] M. Auslander, S. Smalø: *Preprojective modules over Artin algebras*, J. Algebra **66** (1980), no. 1, 61–122.

[4] Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne, *Analyse et topologie sur les espaces singuliers*, Astérisque, vol. 100, Soc. Math. France, 1982 (French).

[5] Raf Bocklandt, *Graded Calabi Yau algebras of dimension 3*, J. Pure Appl. Algebra **212** (2008), no. 1, 14–32.

[6] A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, 1183–1205, 1337, translation in Math. USSR-Izv. 35 (1990), no. 3, 519–541.

[7] H. Cartan, *Variétés analytiques complexes et cohomologie*, Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, Georges Thone, Liège, 1953, pp. 41–55.

[8] Sergio Cecotti, Andrew Neitzke, and Cumrun Vafa, *R-Twisting and $4d/2d$ Correspondences*, arXiv:1006.3435v2 [hep-th].

[9] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke, *Framed BPS states*, arXiv:1006.0146 [hep-th], 123 pages, 52 figures.

[10] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 265–297.

[11] Ch. Geiss, B. Leclerc, and J. Schröer, *Auslander algebras and initial seeds for cluster algebras*, J. London Math. Soc. (2) **75** (2007), 718–740.

[12] Christof Geiß, Bernard Leclerc, and Jan Schröer, *Semicanonical bases and preprojective algebras*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 2, 193–253.

[13] D. Happel, *On the derived category of a finite-dimensional algebra*, Comment. Math. Helv. **62** (1987), no. 3, 339–389.

[14] ———, *Triangulated categories in the representation theory of finite dimensional algebras*, Cambridge University Press, 1988, LMS Lecture Note Series 119.

[15] A. Heller, *Stable homotopy categories*, Bull. Amer. Math. Soc. **74** (1968), 28–63.

[16] Osamu Iyama, *Cluster tilting for higher Auslander algebras*, arXiv:0809.4897 [math.RT], to appear in Adv. Math.

[17] ———, *Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories*, Adv. Math. **210** (2007), no. 1, 22–50.

[18] Osamu Iyama and Steffen Oppermann, *n-representation-finite algebras and n-APR tilting*, arXiv:0909.0593 [math.RT], 33 pages, to appear in Trans. AMS.

[19] ———, *Stable categories of higher preprojective algebras*, arXiv:0912.3412 [math.RT], 37 pages.

[20] Osamu Iyama and Yuji Yoshino, *Mutations in triangulated categories and rigid Cohen-Macaulay modules*, Inv. Math. **172** (2008), 117–168.

[21] B. Keller, *On triangulated orbit categories*, Doc. Math. **10** (2005), 551–581 (electronic).

[22] B. Keller and D. Vossieck, *Sous les catégories dérivées*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 6, 225–228.

[23] Bernhard Keller, *Quantum dilogarithm identities and cluster algebras*, 32 pages, arXiv:1102.4148v2 [math.RT]

[24] Maxim Kontsevich and Yan Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants*, arXiv:1006.2706 [math.AG].

[25] ———, *Stability structures, Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435 [math.AG].

[26] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.

[27] S. Oppermann and H. Thomas, *Higher dimensional cluster combinatorics and representation theory*, 40 pages, 2010.

[28] Dieter Puppe, *On the formal structure of stable homotopy theory.*, Colloq. algebr. Topology, Aarhus 1962, 65-71 (1962)., 1962.

[29] C. M. Ringel, *The self-injective cluster-tilted algebras*, Arch. Math. (Basel) **91** (2008), no. 3, 218–225.

[30] Jean-Louis Verdier, *Catégories dérivées, état 0*, SGA 4.5, Lec. Notes in Math., vol. 569, Springer–Verlag, 1977, pp. 262–308 (French).

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