

THE IMAGE OF THE DERIVED CATEGORY IN THE CLUSTER CATEGORY

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ABSTRACT. Cluster categories of hereditary algebras have been introduced as orbit categories of their derived categories. Keller has pointed out that for non-hereditary algebras orbit categories need not be triangulated, and he introduced the notion of triangulated hull to overcome this problem. In the more general setup of algebras of global dimension at most 2, cluster categories are defined to be these triangulated hulls of the orbit categories.

In this paper we study the image of the natural functor from the bounded derived category to the cluster category, that is we investigate how far the orbit category is from being the cluster category.

We show that the cluster combinatorics can be worked with in the orbit category, i.e. that it is not necessary to consider the entire cluster category. On the other hand we show that for wide classes of non-piecewise hereditary algebras the orbit category is never equal to the cluster category.

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1. INTRODUCTION

In [Ami2] the first author introduced cluster categories \mathcal{C}_Λ for finite-dimensional algebras Λ with $\text{gl.dim } \Lambda \leq 2$. They are defined to be triangulated orbit categories in the sense of Keller (that is they are triangulated hulls of orbit categories – see [Kel]). For hereditary algebras Λ , Keller showed that the cluster category \mathcal{C}_Λ and the orbit category (see Subsection 2.1) coincide. However he pointed out that this need not be the case in general. The aim of this paper is to investigate what this difference is.

Our approach on this aim is as follows: By [Ami2], under a technical assumption which we will make throughout this paper (called τ_2 -finiteness – see Definition 2.8), the image of the algebra Λ is cluster tilting in the cluster category. We denote the endomorphism ring of this cluster tilting object by $\tilde{\Lambda}$. It follows that there is a natural functor $\mathcal{C}_\Lambda \rightarrow \text{mod } \tilde{\Lambda}$. We point out that $\tilde{\Lambda}$ carries a natural \mathbb{Z} -grading. Then we show the following:

Theorem 1.1 (see Theorem 3.2). *The orbit category coincides with the cluster category if and only if any $\tilde{\Lambda}$ -module is gradable.*

More precisely, the objects in the orbit category are precisely those objects in the cluster category, for which the corresponding $\tilde{\Lambda}$ -module is gradable.

We then use this theorem to carry over results on gradability of modules to the cluster category setup. In particular we show that objects outside the orbit category come in 1-parameter families (see Theorem 4.2). Applying this, we obtain the following result.

Theorem 1.2 (see Corollary 5.5). *Any cluster tilting object in \mathcal{C}_Λ is the image of some object in \mathcal{D}_Λ .*

This means that to study cluster combinatorics inside the cluster category, it is enough to work in the derived category or the orbit category. These latter categories are more accessible to explicit computations than cluster categories.

We then focus on the question of when the orbit category coincides with the cluster category. The most ambitious hope one could have in this direction is the following:

Conjecture 1.3. *The orbit category coincides with the cluster category for an algebra Λ if and only if Λ is piecewise hereditary.*

Note that this conjecture, combined with Theorem 1.1, is of a similar flavor to the following question.

Question 1.4 (Skowroński [Sko, Question 1]). *Let Λ be a finite dimensional algebra, $T(\Lambda) = \Lambda \ltimes D\Lambda$ the trivial extension of Λ by its k -dual $D\Lambda$, which may be considered as a graded algebra by putting Λ*

in degree 0 and $D\Lambda$ in degree 1. When is the push-down functor from graded $T(\Lambda)$ -modules to ungraded $T(\Lambda)$ -modules dense?

It should be noted that the “if” part of Conjecture 1.3 holds (see [Kel, Theorem in Section 4]) – similarly it is known that the push-down functor of Question 1.4 is dense if Λ is piecewise hereditary. Here we collect evidence for the “only if” part. To this end we show the following two results:

Theorem 1.5 (see Theorem 6.1). *Assume one object in the derived category of Λ satisfies a fractional Calabi-Yau type condition with Calabi-Yau dimension $\neq 1$. Then Conjecture 1.3 holds.*

Theorem 1.6 (see Theorem 7.1). *Assume the quiver of Λ contains an oriented cycle. Then the orbit category is strictly smaller than the cluster category. In particular Conjecture 1.3 holds.*

2. BACKGROUND AND NOTATION

We assume all our algebras to be finite-dimensional associative algebras over an algebraically closed base field k . Moreover, all categories are k -categories.

For an algebra Λ we denote by \mathcal{D}_Λ the bounded derived category of the category of finitely generated right Λ -modules $\text{mod } \Lambda$.

Definition 2.1. Let \mathcal{T} be a triangulated category. A Serre functor of \mathcal{T} is an autoequivalence \mathbb{S} , such that there is a functorial isomorphism

$$\text{Hom}_{\mathcal{T}}(X, Y) \cong D \text{Hom}_{\mathcal{T}}(Y, \mathbb{S}X),$$

where $D = \text{Hom}_k(-, k)$.

We denote by $\mathbb{S}_2 = \mathbb{S} \circ [-2]$ the second desuspension of \mathbb{S} .

Note that if Λ is an algebra of finite global dimension, then \mathcal{D}_Λ has a Serre functor, which is given by $\mathbb{S} = - \otimes_\Lambda^L D\Lambda$.

Definition 2.2. A triangulated category \mathcal{T} is called d -Calabi-Yau if the d -th suspension $[d]$ is a Serre functor on \mathcal{T} .

Definition 2.3. Let \mathcal{T} be a triangulated category. A subcategory $\mathcal{S} \subseteq \mathcal{T}$ is called (2-)cluster tilting if it is functorially finite, and

$$\begin{aligned} \mathcal{S} &= \{T \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(\mathcal{S}, T[1]) = 0\} \\ &= \{T \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(T, \mathcal{S}[1]) = 0\}. \end{aligned}$$

Note that in case \mathcal{T} is 2-Calabi-Yau the two subcategories on the right automatically coincide.

An object $S \in \mathcal{T}$ is called cluster tilting if $\text{add } S$ is a cluster tilting subcategory.

2.1. Orbit categories and generalized cluster categories.

Definition 2.4. Let Λ be an algebra of global dimension at most 2. We denote by $\mathcal{D}_\Lambda/(\mathbb{S}_2)$ the *orbit category* of \mathcal{D}_Λ modulo \mathbb{S}_2 , that is the category with the same objects as \mathcal{D}_Λ , but with morphisms sets

$$\mathrm{Hom}_{\mathcal{D}_\Lambda/\mathbb{S}_2}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}_\Lambda}(\mathbb{S}_2^i X, Y).$$

Theorem 2.5 (Keller [Kel]). *In the setup of Definition 2.4 there is a triangulated category \mathcal{C}_Λ and triangle functor $\pi: \mathcal{D}_\Lambda \rightarrow \mathcal{C}_\Lambda$ such that π factors into*

$$\mathcal{D}_\Lambda \xrightarrow{\text{nat. proj.}} \mathcal{D}_\Lambda/(\mathbb{S}_2) \xrightarrow{\text{fully faithful}} \mathcal{C}_\Lambda,$$

and \mathcal{C}_Λ is minimal with these properties.

This category \mathcal{C}_Λ is defined to be the *cluster category* of Λ in [Ami2]. There are more explicit descriptions of cluster categories in terms of localizations in [Ami2], but we do not need them here.

In the case that Λ is hereditary the cluster category can more easily be understood by the following fact.

Theorem 2.6 (Keller [Kel]). *Assume Λ is hereditary. Then the functor $\mathcal{D}_\Lambda/(\mathbb{S}_2) \rightarrow \mathcal{C}_\Lambda$ of Theorem 2.5 is an equivalence.*

We are now interested in the object $\pi\Lambda$ in \mathcal{C}_Λ . We denote by $\tilde{\Lambda}$ its endomorphism algebra.

Theorem 2.7 ([Ami2, Proposition 4.7]). *Let Λ be an algebra with $\mathrm{gl.dim} \Lambda \leq 2$. Then*

$$\tilde{\Lambda} = \mathrm{End}_{\mathcal{C}_\Lambda}(\pi\Lambda) = \bigoplus_{i \geq 0} \mathrm{Hom}_{\mathcal{D}}(\Lambda, \mathbb{S}_2^{-i}\Lambda) = T_\Lambda \mathrm{Ext}_\Lambda^2(D\Lambda, \Lambda),$$

where T_Λ denotes the tensor algebra over Λ .

In particular $\tilde{\Lambda}$ has a natural positive \mathbb{Z} -grading.

Note that it follows from the right hand side above that $\tilde{\Lambda}$ is generated in degrees 0 and 1, and that minimal generators in degree 1 correspond to a minimal set of relations in Λ (see [ABS, Lemma 2.4]).

Definition 2.8. An algebra Λ is called τ_2 -finite if $\mathrm{gl.dim} \Lambda \leq 2$, and $\tilde{\Lambda}$ is finite-dimensional, that is if $\tilde{\Lambda}$ is non-zero in only finitely many degrees.

In this paper we will focus on the case that Λ is τ_2 -finite. In this case the object $\pi\Lambda$ is cluster tilting in \mathcal{C}_Λ by the following result.

Theorem 2.9. *Let Λ be τ_2 -finite.*

- [Iya, Theorem 1.23] *The subcategory $\mathrm{add}\{\mathbb{S}_2^n \Lambda \mid n \in \mathbb{Z}\}$ is cluster tilting in \mathcal{D}_Λ .*
- [Ami2, Theorem 4.10] *The cluster category \mathcal{C}_Λ is a Hom-finite 2-Calabi-Yau category, and $\pi\Lambda$ is a cluster tilting object.*

2.2. Graded algebras and coverings. Let $A = \bigoplus_{p \in \mathbb{Z}} A_p$ be a finite dimensional \mathbb{Z} -graded algebra. We denote by $\text{mod}_{\text{gr}} A$ the category of finitely generated graded A -modules. For a graded A -module $M = \bigoplus_{p \in \mathbb{Z}} M_p$ and $q \in \mathbb{Z}$, we denote by $M\langle q \rangle$ the graded module $\bigoplus_{p \in \mathbb{Z}} M_{p+q}$ (that is the degree p part of $M\langle q \rangle$ is M_{p+q}).

We denote by $\text{proj}_{\text{gr}} A$ the category of graded projective A -modules $\text{add}\{A\langle p \rangle \mid p \in \mathbb{Z}\}$. A category with \mathbb{Z} -action is called \mathbb{Z} -covering of A if it is equivalent to $\text{proj}_{\text{gr}} A$ as category with \mathbb{Z} -action. Then by a classical result (see [GrM]) we have an equivalence between modules over this category and $\text{mod}_{\text{gr}} A$.

Example 2.10. Let Λ be a τ_2 -finite algebra of global dimension 2. Then the category $\text{add}\{\mathbb{S}_2^n \Lambda \mid n \in \mathbb{Z}\}$ is a \mathbb{Z} -covering of $\tilde{\Lambda}$.

3. CONNECTION TO GRADABLE MODULES

Throughout this section we assume Λ to be τ_2 -finite. Then, by Theorem 2.9 the cluster category is 2-Calabi-Yau. Hence, by the arguments of [BMR, Theorem 2.2], for any cluster tilting object T the functor

$$\begin{aligned} \mathbb{M}_T: \mathcal{C}_\Lambda &\longrightarrow \text{mod End}_{\mathcal{C}}(T) \\ X &\longmapsto \text{Hom}_{\mathcal{C}}(T, X) \end{aligned}$$

induces an equivalence $\mathcal{C}_\Lambda/(T[1]) \xrightarrow{\sim} \text{mod End}_{\mathcal{C}}(T)$. In particular we have

$$\mathbb{M} := \mathbb{M}_{\pi\Lambda}: \mathcal{C}_\Lambda/(\pi\Lambda[1]) \xrightarrow{\sim} \text{mod } \tilde{\Lambda}. \quad (3.1)$$

Similarly, since $\text{add}\{\mathbb{S}_2^n \Lambda \mid n \in \mathbb{Z}\}$ is a cluster tilting subcategory of \mathcal{D}_Λ and a \mathbb{Z} -covering of $\tilde{\Lambda}$, the functor

$$\begin{aligned} \mathbb{M}_{\text{gr}}: \mathcal{D}_\Lambda &\longrightarrow \text{mod}\{\mathbb{S}_2^n \Lambda \mid n \in \mathbb{Z}\} \xrightarrow{\sim} \text{mod}_{\text{gr}} \tilde{\Lambda} \\ X &\longmapsto \text{Hom}_{\mathcal{D}}(-, X) \longmapsto \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(\mathbb{S}_2^n \Lambda, X) \end{aligned}$$

induces an equivalence

$$\mathcal{D}_\Lambda/(\mathbb{S}_2^n \Lambda[1] \mid n \in \mathbb{Z}) \xrightarrow{\sim} \text{mod}_{\text{gr}} \tilde{\Lambda}. \quad (3.2)$$

It is immediate from the definitions of the functors that we have the following commutative square.

$$\begin{array}{ccc} \mathcal{D}_\Lambda & \xrightarrow{\mathbb{M}_{\text{gr}}} & \text{mod}_{\text{gr}} \tilde{\Lambda} \\ \pi \downarrow & & \downarrow \text{forgetful} \\ \mathcal{C}_\Lambda & \xrightarrow{\mathbb{M}} & \text{mod } \tilde{\Lambda} \end{array} \quad (3.3)$$

Remark 3.1. Since $\text{add}\{\mathbb{S}_2^n\Lambda \mid n \in \mathbb{Z}\}$ is cluster tilting, for any $X \in \mathcal{D}_\Lambda$, there is a triangle

$$T_1 \longrightarrow T_0 \longrightarrow X \longrightarrow T_1[1]$$

with $T_0, T_1 \in \text{add}\{\mathbb{S}_2^n\Lambda \mid n \in \mathbb{Z}\}$. Under the functor \mathbf{M}_{gr} this triangle corresponds to a graded projective presentation

$$\mathbf{M}_{\text{gr}}T_1 \longrightarrow \mathbf{M}_{\text{gr}}T_0 \longrightarrow \mathbf{M}_{\text{gr}}X \longrightarrow 0.$$

Similarly, under \mathbf{M} , triangles involving two terms from $\text{add } \pi\Lambda$ in \mathcal{C}_Λ correspond to projective presentations in $\text{mod } \tilde{\Lambda}$.

In this language, the vertical functors of Diagram (3.3) correspond to applying π to triangles, and forgetting the grading on projective presentations, respectively.

We denote the (full, isomorphism-closed) subcategory of $\text{mod } \tilde{\Lambda}$ consisting of modules in the image of the forgetful functor $\text{mod}_{\text{gr}} \tilde{\Lambda} \rightarrow \text{mod } \tilde{\Lambda}$ by $\text{mod}_{\text{gr.able}} \tilde{\Lambda}$, and call its objects *gradable* modules.

Theorem 3.2. *Let Λ be a τ_2 -finite algebra. Then for $X \in \mathcal{C}_\Lambda$ the following are equivalent:*

- (1) X is in the image of π , that is there is $Y \in \mathcal{D}_\Lambda$ with $X \cong \pi Y$,
and
- (2) $\mathbf{M}X \in \text{mod}_{\text{gr.able}} \tilde{\Lambda}$.

Proof. By the observations above we know that (1) \implies (2).

For the implication (2) \implies (1) note that both conditions (1) and (2) are satisfied for some X if and only if they are satisfied for all indecomposable direct summands of X . Hence we may assume X to be indecomposable.

If $X \in \text{add } \pi\Lambda[1]$ then clearly X is in the image of π , so the implication holds. If $X \notin \text{add } \pi\Lambda[1]$ it suffices to show that $\mathbf{M}X \cong \mathbf{M}\pi Y$ for some $Y \in \mathcal{D}_\Lambda$ (since \mathbf{M} preserves isomorphism classes of objects without direct summands in $\text{add } \pi\Lambda[1]$ by the equivalence in (3.1)). By Property (2), the equivalence in (3.2), and the commutative diagram (3.3), we know that there is an indecomposable $Y \in \mathcal{D}_\Lambda$ such that $\mathbf{M}X$ is obtained from $\mathbf{M}_{\text{gr}}Y$ by forgetting the grading. But this means precisely that $\mathbf{M}X = \mathbf{M}\pi Y$. \square

Remark 3.3. For Theorem 3.2 (and throughout this paper) it is important that we work with 2-cluster categories. For $n > 2$ the functor from an n -cluster category to the module category of an n -Calabi-Yau tilted algebra does not preserve “almost all” indecomposables.

From Theorem 3.2 we deduce a new proof of the following result of Keller.

Corollary 3.4. *Let Λ be hereditary. Then $\pi: \mathcal{D}_\Lambda \rightarrow \mathcal{C}_\Lambda$ is dense.*

Proof. If Λ is hereditary, then $\text{Ext}_\Lambda^2 = 0$, so $\tilde{\Lambda} = \Lambda$ is concentrated in degree 0 as graded algebra. Thus clearly any module is gradable. Now the claim follows from Theorem 3.2 above. \square

Corollary 3.5. *Let Λ be as above, and $f: P \rightarrow Q$ in \mathcal{C} with $P, Q \in \text{add } \pi\Lambda$. Then $\text{Cone}_\mathcal{C} f \in \text{Im } \pi$ if and only if f is isomorphic to a map in the image of π .*

Proof. The “if” follows from the fact that for $f = \pi g$ we have $\text{Cone}_\mathcal{C} f = \pi(\text{Cone}_\mathcal{D} g)$.

Assume conversely that $\text{Cone}_\mathcal{C} f \in \text{Im } \pi$. Then, by Theorem 3.2, $\mathbf{M}(\text{Cone}_\mathcal{C} f) \in \text{mod}_{\text{gr.able}} \tilde{\Lambda}$. So $\mathbf{M}(\text{Cone}_\mathcal{C} f)$ comes from a graded $\tilde{\Lambda}$ -module $\mathbf{M}_{\text{gr}} X$. There is a triangle

$$\widehat{P} \xrightarrow{\widehat{f}} \widehat{Q} \longrightarrow X \longrightarrow \widehat{P}[1]$$

in \mathcal{D}_Λ , with $\widehat{P}, \widehat{Q} \in \text{add}\{\mathbb{S}_2^n \Lambda \mid n \in \mathbb{Z}\}$, since this subcategory is cluster tilting. We may moreover assume \widehat{f} to be a radical map.

Applying π to this triangle we see that $\text{Cone}_\mathcal{C} f = \text{Cone}_\mathcal{C} \pi \widehat{f}$, so f is isomorphic to the direct sum of $\pi \widehat{f}$ and an isomorphism. \square

Proposition 3.6. *Let Λ be as above. Let P and Q be indecomposable projective Λ -modules. Then the following are equivalent:*

- (1) $\{\text{Cone}_\mathcal{C} f \mid f: \pi P \rightarrow \pi Q\} \subseteq \text{Im } \pi$, and
- (2) for any $i < j$ and any $g: \mathbb{S}_2^i P \rightarrow Q$ and $g': \mathbb{S}_2^j P \rightarrow Q$ with $g \neq 0$ there are $r: \mathbb{S}_2^j P \rightarrow \mathbb{S}_2^i P$ and $s: Q \rightarrow \mathbb{S}_2^{i-j} Q$ such that

$$g' = g \circ r + \mathbb{S}_2^{j-i}(s \circ g).$$

Proof. Assume first (1), and let g and g' as in (2). Thus we have $\text{Cone}_\mathcal{C}(\pi g + \pi g') \in \text{Im } \pi$. (Note that πP is canonically isomorphic to $\pi \mathbb{S}_2^\ell P$ for all ℓ .) By Corollary 3.5 we have

$$\begin{array}{ccc} \pi P & \xrightarrow{\pi g + \pi g'} & \pi Q \\ r \downarrow & & \downarrow s \\ \pi P & \xrightarrow{\pi h} & \pi Q \end{array}$$

for some $h: \mathbb{S}_2^\ell P \rightarrow Q$, and $r \in \text{Aut}(\pi P)$, $s \in \text{Aut}(\pi Q)$. Passing to $\tilde{\Lambda}$ -modules and decomposing the maps into their homogeneous parts we obtain

$$\begin{array}{ccc} \mathbf{M}\pi P & \xrightarrow{\mathbf{M}_{\text{gr}} g + \mathbf{M}_{\text{gr}} g'} & \mathbf{M}\pi Q \\ \sum \mathbf{M}_{\text{gr}} r_i \downarrow & & \downarrow \sum \mathbf{M}_{\text{gr}} s_i \\ \mathbf{M}\pi P & \xrightarrow{\mathbf{M}_{\text{gr}} h} & \mathbf{M}\pi Q \end{array}$$

with $r_i: \mathbb{S}_2^i P \rightarrow P$ and $s_i: \mathbb{S}_2^i Q \rightarrow Q$. Since r and s are automorphisms so are r_0 and s_0 , and we may assume $r_0 = 1 = s_0$.

Since there are no morphisms in negative degree, taking the non zero morphisms of the smallest degree of the maps $(\sum \mathbf{M}_{\text{gr}} s_i)(\mathbf{M}_{\text{gr}} g + \mathbf{M}_{\text{gr}} g')$ and $(\mathbf{M}_{\text{gr}} h)(\sum \mathbf{M}_{\text{gr}} r_i)$, one gets that $\ell = i$ and $h = g$. Now looking at the morphism of degree j one gets $g' = g \circ \mathbb{S}_2^i(r_{j-i}) - s_{j-i} \circ \mathbb{S}_2^{j-i}(g)$, which is the factorization property of part (2) of the corollary.

Now assume (2) holds, and assume there is a map $f: \pi P \rightarrow \pi Q$ such that $\text{Cone}_{\mathcal{E}} f \notin \text{Im } \pi$. Then f is not isomorphic to a map in the image of π . We will consider maps which are isomorphic to f , and their decompositions into homogeneous parts.

Let $f' \cong f$. We may write $\mathbf{M}f' = \sum_{i \in \mathbb{Z}} \mathbf{M}_{\text{gr}} f'_i$, where $f'_i: \mathbb{S}_2^i P \rightarrow Q$. By assumption at least two of these homogeneous parts do not vanish, and thus we may define

$$d(f') = (d(f')_1, d(f')_2), \quad \text{with } d(f')_1 = \min\{i \in \mathbb{Z} \mid f'_i \neq 0\}, \text{ and} \\ d(f')_2 = \min\{i > d(f')_1 \mid f'_i \neq 0\}.$$

Note that since Λ is τ_2 -finite, nonzero maps $\pi P \rightarrow \pi Q$ can only exist in finitely many degrees. In particular the set

$$D = \{d(f') \mid f' \cong f\}$$

is finite. Ordering pairs of integers lexicographically (that is by $(a_1, a_2) \leq (b_1, b_2)$ if $a_1 < b_1$ or $(a_1 = b_1 \text{ and } a_2 \leq b_2)$), we may assume that $d(f)$ is maximal in D .

By (2) we have $r: \mathbb{S}_2^{d(f)_2} P \rightarrow \mathbb{S}_2^{d(f)_1} P$ and $s: Q \rightarrow \mathbb{S}_2^{d(f)_1 - d(f)_2} Q$ such that $\mathbf{M}_{\text{gr}} f_j = (\mathbf{M}_{\text{gr}} r)(\mathbf{M}_{\text{gr}} f_i) + (\mathbf{M}_{\text{gr}} f_i)(\mathbf{M}_{\text{gr}} s)$. Now $\mathbf{M}f \cong (1 - \mathbf{M}_{\text{gr}} s)(\mathbf{M}f)(1 - \mathbf{M}_{\text{gr}} r)$. Looking at this degree wise we have

$$\begin{aligned} \mathbf{M}f &\cong (1 - \mathbf{M}_{\text{gr}} s)(\mathbf{M}_{\text{gr}} f_{d(f)_1} + \mathbf{M}_{\text{gr}} f_{d(f)_2} + [\text{terms of degree } > d(f)_2])(1 - \mathbf{M}_{\text{gr}} r) \\ &= \mathbf{M}_{\text{gr}} f_{d(f)_1} + \mathbf{M}_{\text{gr}} f_{d(f)_2} - (\mathbf{M}_{\text{gr}} f_{d(f)_1})(\mathbf{M}_{\text{gr}} r) - (\mathbf{M}_{\text{gr}} s)(\mathbf{M}_{\text{gr}} f_{d(f)_1}) \\ &\quad + [\text{terms of degree } > d(f)_2] \\ &= \mathbf{M}_{\text{gr}} f_{d(f)_1} + [\text{terms of degree } > d(f)_2] \end{aligned}$$

contradicting our assumption that $d(f)$ is maximal in D . Hence (1) must hold. \square

Example 3.7. Let Λ be that algebra given by the quiver

$$\begin{array}{ccc} & 2 & \\ \alpha \nearrow & & \searrow \beta \\ 1 & \xleftarrow{\gamma} & 3 \end{array}$$

subject to the relation $r = \beta\alpha$.

Then the quiver of $\tilde{\Lambda}$ is

$$\begin{array}{ccc} & 2 & \\ \alpha \nearrow & & \searrow \beta \\ 1 & \xleftarrow[r]{r} & 3 \\ & \xleftarrow{\gamma} & \end{array}$$

where γ (and α and β) is of degree 0, and r is of degree 1. Since r cannot be expressed as a sum up multiples of γ , the second assertion of Proposition 3.6 is not fulfilled for P_3 and P_1 . Thus Proposition 3.6 shows that π is not dense for this algebra, and more precisely that there is an object of the form $\text{Cone } f$ not in the image of π for some $f: \pi P_3 \rightarrow \pi P_1$.

4. GRADABLE MODULES AND PARAMETER FAMILIES

In this section we show that a module over a graded algebra is either gradable, or belongs to a one-parameter family of modules which are almost all non-isomorphic. In particular it will follow that modules which represent an open orbit in the representation variety are always gradable.

Definition 4.1. Let R be a \mathbb{Z} -graded k -algebra. For $\alpha \in k^\times$ we denote by σ_α the algebra-automorphism given on homogeneous elements by

$$r \mapsto \alpha^{\deg r} r.$$

For a (non-graded) R -module M we denote by M_α the module twisted by the automorphism σ_α . That is, $M_\alpha = M$ as k -vector spaces, but with the new module multiplication given by

$$m \cdot_\alpha r = \alpha^{\deg r} mr$$

for homogeneous $r \in R$.

Theorem 4.2. *Let R be a finitely generated \mathbb{Z} -graded k -algebra, and M a finite-dimensional R -module. Then exactly one of the following happens:*

- (1) *The modules M_α with $\alpha \in k^\times$ are all isomorphic, and M is gradable.*
- (2) *For any $\alpha \in k^\times$ there are only finitely many $\beta \in k^\times$ such that $M_\alpha \cong M_\beta$, and M is not gradable.*

Proof. By assumption R is generated (as k -algebra) by a finite set $\{r_i\}$ of homogeneous elements. We set $g = \max\{|\deg r_i|\}$.

Assume $M_\alpha \cong M_\beta$, and $(\frac{\alpha}{\beta})^i \neq 1$ for $i \leq g \cdot \dim M$. Let $\psi: M_\alpha \rightarrow M_\beta$ be an isomorphism. Then ψ can be considered as automorphism of the k -vector space M . Let

$$M = \bigoplus_{\lambda \in k^\times} M(\lambda)$$

be the generalized eigenspace decomposition of M with respect to ψ . Let $r \in R$ be homogeneous. For $m \in M$ we have

$$\begin{aligned} \left(\psi - \left(\frac{\beta}{\alpha} \right)^{\deg r} \lambda \right) (mr) &= \psi(mr) - \left(\frac{\beta}{\alpha} \right)^{\deg r} \lambda m \cdot r \\ &= \left(\frac{1}{\alpha} \right)^{\deg r} \psi(m \cdot_{\alpha} r) - \left(\frac{\beta}{\alpha} \right)^{\deg r} \lambda mr \\ &= \left(\frac{1}{\alpha} \right)^{\deg r} \psi(m) \cdot_{\beta} r - \left(\frac{\beta}{\alpha} \right)^{\deg r} \lambda mr \\ &= \left(\frac{\beta}{\alpha} \right)^{\deg r} (\psi - \lambda)(m)r \end{aligned}$$

A straightforward induction shows that this implies that for all $n \in \mathbb{N}$ we have

$$\left(\psi - \left(\frac{\beta}{\alpha} \right)^{\deg r} \lambda \right)^n (mr) = \left(\frac{\beta}{\alpha} \right)^{n \deg r} (\psi - \lambda)^n (m)r.$$

It follows that for $m \in M(\lambda)$ we have $mr \in M\left(\left(\frac{\beta}{\alpha}\right)^{\deg r} \lambda\right)$. If we denote by $\langle \frac{\beta}{\alpha} \rangle$ the cyclic subgroup of k^\times generated by $\frac{\beta}{\alpha}$ we obtain a direct sum decomposition of M into the summands

$$\bigoplus_{\sigma \in \langle \frac{\beta}{\alpha} \rangle} M(\sigma\lambda)$$

where λ runs over representatives of the cosets of $\langle \frac{\beta}{\alpha} \rangle$ in k^\times . Moreover these summands are $\langle \frac{\beta}{\alpha} \rangle$ -gradable. Since the order of $\langle \frac{\beta}{\alpha} \rangle$ is bigger than $g \cdot \dim M$ these $\langle \frac{\beta}{\alpha} \rangle$ -gradings can be lifted to \mathbb{Z} -gradings. Thus M is gradable.

Conversely, if M is gradable then it is immediate that

$$\begin{aligned} M &\longrightarrow M_{\alpha} \\ m &\longmapsto \alpha^{\deg m} m \end{aligned}$$

gives an isomorphism of R -modules for any $\alpha \in k^\times$. \square

Definition 4.3. Let R be a finitely generated k -algebra and M a finite-dimensional R -module. We say that M has an open orbit if the orbit of M (under the natural $\mathrm{GL}(\dim M, k)$ -action) in the variety of $(\dim M)$ -dimensional R -modules is open.

With this definition we have the following immediate consequence of Theorem 4.2.

Corollary 4.4. Let R be a finitely generated \mathbb{Z} -graded k -algebra, and M a finite-dimensional R -module, which has an open orbit. Then M is gradable.

Proof. The M_α form a line in the representation variety. Since M has an open orbit they are almost all isomorphic to M . Hence we are in the first case of Theorem 4.2. \square

Corollary 4.5. *Let R be a finitely generated \mathbb{Z} -graded k -algebra, and M a finite-dimensional R -module, such that $\text{Ext}_R^1(M, M) = 0$. Then M is gradable.*

Proof. By [Voi, § 3.5], $\text{Ext}_R^1(M, M) = 0$ implies that M has an open orbit. Hence the claim follows from Corollary 4.4. \square

5. APPLICATION OF RESULTS ON GRADABLE MODULES

In this section we use results on gradable modules by Gordon and Green [GoG] and from the previous section to the situation of the graded algebra $\tilde{\Lambda}$, for a τ_2 -finite algebra Λ (see Subsection 2.1). By Theorem 3.2 this yields results on the image of the derived category in the cluster category.

5.1. Auslander-Reiten components. We recall the following result of Gordon and Green.

Theorem 5.1 ([GoG, Theorem 4.2]). *Let R be graded finite-dimensional k -algebra. A component of the Auslander-Reiten quiver of (ungraded) R -modules either only contains gradable modules, or does not contain any gradable modules.*

Applying this to our setup we obtain the following:

Theorem 5.2. *Let Λ be a τ_2 -finite algebra. Then the image of the derived category in the cluster category is a union of Auslander-Reiten components.*

For the proof we need the following observation.

Observation 5.3. *Let \mathcal{A} be an Auslander-Reiten component of \mathcal{C}_Λ . If \mathcal{A} does not contain any summand of $\pi\Lambda[1]$, then $\mathbb{M}\mathcal{A}$ is an Auslander-Reiten component of $\text{mod } \tilde{\Lambda}$. If \mathcal{A} contains a summand of $\pi\Lambda[1]$ then $\mathbb{M}\mathcal{A}$ is a union of Auslander-Reiten components of $\text{mod } \tilde{\Lambda}$, and all these components contain a projective or an injective $\tilde{\Lambda}$ -module.*

Proof of Theorem 5.2. By Observation 5.3 we have to consider two cases:

If \mathcal{A} is an Auslander-Reiten component of \mathcal{C}_Λ not containing any summand of $\pi\Lambda[1]$ then $\mathbb{M}\mathcal{A}$ is an Auslander-Reiten component of $\text{mod } \tilde{\Lambda}$, and by Theorems 3.2 and 5.1 either all or no objects in \mathcal{A} lie in the image of π .

Now consider a component \mathcal{A} of the Auslander-Reiten quiver of \mathcal{C}_Λ which contains a summand of $\pi\Lambda[1]$. By Observation 5.3 each of the Auslander-Reiten components of $\mathbb{M}\mathcal{A}$ contains at least one projective

or injective, hence gradable, module. So, by Theorem 5.1, all objects in these components are gradable. Hence, by Theorem 3.2, all objects in \mathcal{A} lie in the image of π . \square

5.2. Rigid objects. We call an object in a triangulated or exact category *rigid* if $\text{Ext}^1(X, X) = 0$.

Theorem 5.4. *Let Λ be a τ_2 -finite algebra. Let $X \in \mathcal{C}_\Lambda$ be rigid. Then X is in the image of $\pi: \mathcal{D}_\Lambda \rightarrow \mathcal{C}_\Lambda$.*

Proof. By [KöZ, Theorem 4.9], if X is rigid then so is $\text{M}X$. By Corollary 4.5 any rigid module in $\text{mod } \tilde{\Lambda}$ is gradable, so in particular $\text{M}X$ is gradable. Hence, by Theorem 3.2, X lies in the image of π . \square

Corollary 5.5. *Any cluster tilting object in \mathcal{C}_Λ is the image of some object in \mathcal{D}_Λ .*

6. FRACTIONAL CALABI-YAU TYPE SITUATIONS

In this section, we study the situation that there is an object in \mathcal{D}_Λ satisfying a fractional Calabi-Yau type condition, that is an X such that $X[a] \cong \mathbb{S}^b X$ for certain $a, b \in \mathbb{Z}$. We then show the following:

Theorem 6.1. *Let Λ be a connected τ_2 -finite algebra. Assume that there is some indecomposable object $X \in \mathcal{D}_\Lambda$ such that $X[a] \cong \mathbb{S}^b X$ for some $a, b \in \mathbb{Z}$ with $a \neq b$.*

Then the functor $\pi: \mathcal{D}_\Lambda \rightarrow \mathcal{C}_\Lambda$ is dense if and only if Λ is piecewise hereditary.

Remark 6.2. \bullet If there is a functorial isomorphism $X[a] \cong \mathbb{S}^b X$ for all $X \in D^b(\text{mod } \Lambda)$ then Λ is called *fractionally Calabi-Yau* of dimension $\frac{a}{b}$.

- \bullet We have $b \neq 0$ and $\frac{a}{b} \geq 0$. Since Λ is τ_2 -finite we also have $\frac{a}{b} < 2$.
- \bullet The condition $X[a] \cong \mathbb{S}^b X$ means that in the cluster category we have $\tau^a \pi X \cong \pi X[a] \cong \mathbb{S}^b \pi X \cong \tau^{2b} \pi X$. Thus πX lies in a tube in the cluster category.
- \bullet If Λ is piecewise hereditary, then there can only be indecomposable objects X with $X[a] \cong \mathbb{S}^b X$ for $a \leq b$ (and for $a < b$ the algebra Λ is of Dynkin type, while for $a = b$ it is of Euclidean or tubular type).
- \bullet For algebras which are not piecewise hereditary there may be indecomposable objects X satisfying $X[a] \cong \mathbb{S}^b X$ for $a > b$ or $a < b$. For instance for the algebra given by the quiver

$$\begin{array}{ccc}
 & 2 & \\
 \alpha \nearrow & & \searrow \beta \\
 1 & \xleftarrow{\gamma} & 3
 \end{array}$$

subject to the relation $\beta\alpha$. Then the projective module P_2 corresponding to vertex 2 satisfies $P_2 \cong \mathbb{S}P_2$, while the simple module S_1 corresponding to vertex 1 satisfies $S_1[3] \cong \mathbb{S}^2S_1$.

Outline of proof of Theorem 6.1. If Λ is piecewise hereditary then the functor $\pi: \mathcal{D}_\Lambda \rightarrow \mathcal{C}_\Lambda$ is dense (see [Kel, Theorem in Section 4] for a formal proof).

So it remains to show that if Λ is not piecewise hereditary then π is not dense. By Theorem 3.2 it suffices to find a non-gradable $\tilde{\Lambda}$ -module. For this, by a result of Dowbor and Skowroński (see Theorem 6.13), it suffices to construct a periodic $\tilde{\Lambda}$ -module without finite dimensional summands.

To construct such a module we proceed as follows: We first show that the Auslander-Reiten component of X is of the form $\mathbb{Z}A_\infty$ (see Proposition 6.5 and Lemma 6.7). Then we apply the functor \mathbf{M}_{gr} to the objects in this component, and obtain the desired periodic module as a certain limit in Subsection 6.2.

6.1. The shape of the Auslander-Reiten component. In this subsection we study the shape of the Auslander-Reiten component of X in case the assumptions of Theorem 6.1 are satisfied. We denote this component by \mathcal{A}_X .

We begin with two immediate observations.

Observation 6.3 ([Rie, Struktursatz 1.5]). *The Auslander-Reiten component \mathcal{A}_X is of the form $\mathbb{Z}Q/G$ for some quiver Q and a group G of automorphisms of $\mathbb{Z}Q$. (Actually this is true for any Auslander-Reiten component.)*

Observation 6.4. *For any $Y \in \mathcal{A}_X$ we have $Y[a] \cong \mathbb{S}^b Y$.*

Proposition 6.5. *The Auslander-Reiten component \mathcal{A}_X is of the form $\mathbb{Z}Q$, with Q Dynkin or of type A_∞ .*

For the proof we need the following observation.

Lemma 6.6. *Let Q be a connected quiver. In the mesh category of $\mathbb{Z}Q$ we have*

$$\text{Hom}(Y, \tau^{-i}Y) = 0 \quad \forall Y \forall i \gg 0$$

if and only if Q is either a Dynkin quiver or of type A_∞ .

Proof. Assume the mesh category satisfies $\text{Hom}(Y, \tau^{-i}Y) = 0$ for any Y and i sufficiently large. Fix Y_0 and let σ be the function assigning to any vertex i of Q the sum of $\dim \text{Hom}(Y_0, Y)$, where Y runs over the τ orbit corresponding to i . Then σ is a subadditive function on Q in the sense of [HPR], and, by [HPR, Theorem in Section 1], Q is a Dynkin or Euclidean diagram, or of one of the types A_∞ , A_∞^∞ , or D_∞ . Comparing to the preprojective component of the corresponding path algebras, we see that Q cannot be an Euclidean diagram. A

straight forward calculation shows that for $Q = A_\infty^\infty$ or $Q = D_\infty$ we have $\text{Hom}(Y, \tau^{-i}Y) \neq 0$ for arbitrarily large i . Thus Q is a Dynkin diagram or of type A_∞ .

Conversely it is easy to see that for Q Dynkin or of type A_∞ we have the vanishing property of the lemma. \square

Proof of Proposition 6.5. By Observation 6.4 we have that \mathcal{A}_X is of the form $\mathbb{Z}Q/G$ for some quiver Q and some group G of automorphisms of $\mathbb{Z}Q$.

Note that the assumption $Y[a] \cong \mathbb{S}^b Y$ can be reformulated as $\tau^b Y \cong Y[a-b]$. Recall that we also assumed $a-b \neq 0$. Since $\text{gl.dim } \Lambda < \infty$, for any $Y \in \mathcal{A}_X$ we have $\text{Hom}(Y, Y[i]) = 0$ for almost all i , and hence $\text{Hom}_{\mathcal{D}_\Lambda}(Y, \tau^{-i}Y) = 0$ for $i \gg 0$.

It follows that the same is true in the mesh category of $\mathbb{Z}Q$, and hence, by Lemma 6.6, that Q is Dynkin or of type A_∞ .

Finally note that $\text{Hom}_{\mathcal{D}_\Lambda}(Y, \tau^{-i}Y) = 0$ for $i \gg 0$ also implies that $G = 1$. \square

Lemma 6.7. *If $\mathcal{A}_X = \mathbb{Z}Q$ with Q Dynkin, then Λ is piecewise hereditary of type Q .*

Proof. One can calculate the dimension of the morphisms between objects in \mathcal{A}_X , up to morphisms in the infinite radical, by using mesh relations. Thus these dimensions coincide with the dimensions of morphism spaces in \mathcal{D}_{kQ} .

It follows that, for any given object of \mathcal{A}_X , there are only finitely many objects such that there are morphisms between the two outside the infinite radical. It follows that some power of the radical, and thus the infinite radical, vanishes. Hence one sees that there are no morphisms from \mathcal{A}_X to any other component or vice versa. So, since Λ is connected, \mathcal{A}_X is the entire Auslander-Reiten quiver of \mathcal{D}_Λ . It follows that $\mathcal{D}_\Lambda = \mathcal{D}_{kQ}$ (either note that any complete slice is a tilting object, or use [Ami1, Theorem 7.1]). \square

6.2. A limit of the Auslander-Reiten component. In this subsection we focus on the case that the Auslander-Reiten component \mathcal{A}_X is of type A_∞ . We label the indecomposable objects in \mathcal{A}_X by ${}_i X_j$ with $i \leq j$ as indicated in the following picture.

$$\begin{array}{ccccccccc}
 & \vdots & \\
 \cdots & {}_{-3}X_{-1} & \longleftarrow + & {}_{-2}X_0 & \longleftarrow + & {}_{-1}X_1 & \longleftarrow + & {}_0X_2 & \longleftarrow + & {}_1X_3 & \cdots \\
 & \searrow & & \nearrow & \searrow & & \nearrow & \searrow & & \nearrow & \\
 \cdots & & {}_{-2}X_{-1} & \longleftarrow + & {}_{-1}X_0 & \longleftarrow + & {}_0X_1 & \longleftarrow + & {}_1X_2 & & \cdots \\
 & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & & \searrow & \\
 \cdots & {}_{-2}X_{-2} & \longleftarrow + & {}_{-1}X_{-1} & \longleftarrow + & {}_0X_0 & \longleftarrow + & {}_1X_1 & \longleftarrow + & {}_2X_2 & \cdots
 \end{array}$$

In particular $\tau^\ell {}_i X_j = {}_{i-\ell} X_{j-\ell}$, so $\mathbb{S}^\ell {}_i X_j = {}_{i-\ell} X_{j-\ell}[\ell]$.

Proposition 6.8. *Let $X \in \mathcal{D}_\Lambda$ such that $X[a] = \mathbb{S}^b X$ for some $a \neq b$, and such that the Auslander-Reiten component \mathcal{A}_X of X is of type $\mathbb{Z}A_\infty$. With the labels as above we have the following.*

(1) *For any $T \in \mathcal{D}_\Lambda$ there are i_0 and j_0 such that*

$$\mathrm{Hom}_{\mathcal{D}_\Lambda}(T, {}_i X_j) \cong \mathrm{Hom}_{\mathcal{D}_\Lambda}(T, {}_{i_0} X_{j_0}) \quad \forall i \leq i_0, j \geq j_0.$$

(2) *For any $T \in \mathcal{D}_\Lambda$ there are i_0 and j_0 such that*

$$\mathrm{Hom}_{\mathcal{D}_\Lambda}({}_i X_j, T) \cong \mathrm{Hom}_{\mathcal{D}_\Lambda}({}_{i_0} X_{j_0}, T) \quad \forall i \leq i_0, j \geq j_0.$$

Proof. We only prove (1), (2) is dual.

By Observation 6.4 we have $[a] = \mathbb{S}^b$ on objects of the component \mathcal{A}_X . Hence, for i and $\ell \in \mathbb{Z}$, we have

$${}_{i+\ell b} X_{i+\ell b} = \mathbb{S}^{-\ell b} {}_i X_i[\ell b] = {}_i X_i[\ell(b-a)].$$

In particular, for given i the space $\mathrm{Hom}_{\mathcal{D}_\Lambda}(T, {}_{i+\ell b} X_{i+\ell b})$ is non-zero for only finitely many ℓ . It follows that $\mathrm{Hom}_{\mathcal{D}_\Lambda}(T, {}_i X_i) \neq 0$ for only finitely many i .

Thus we can choose i_0 and j_0 such that

$$\mathrm{Hom}_{\mathcal{D}_\Lambda}(T[-1] \oplus T \oplus T[1], {}_\ell X_\ell) = 0 \quad \forall \ell \notin \{i_0, \dots, j_0\}.$$

To complete the proof, it suffices to show that for $i \leq i_0$ and $j \geq j_0$ the maps ${}_{i-1} X_j \rightarrow {}_i X_j$ and ${}_i X_j \rightarrow {}_i X_{j+1}$ induce isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}_\Lambda}(T, {}_{i-1} X_j) &\longrightarrow \mathrm{Hom}_{\mathcal{D}_\Lambda}(T, {}_i X_j) \text{ and} \\ \mathrm{Hom}_{\mathcal{D}_\Lambda}(T, {}_i X_j) &\longrightarrow \mathrm{Hom}_{\mathcal{D}_\Lambda}(T, {}_i X_{j+1}), \text{ respectively.} \end{aligned}$$

This follows from our choice of i_0 and j_0 , and the fact that the cones of the two maps are ${}_{i-1} X_{i-1}[1]$ and ${}_{j+1} X_{j+1}$, respectively. \square

Construction 6.9. In the setup of Proposition 6.8, let $\mathbf{M}_{\mathrm{gr}} \mathcal{A}_X$ be the graded $\tilde{\Lambda}$ -module given by

$$(\mathbf{M}_{\mathrm{gr}} \mathcal{A}_X)_\ell = \mathrm{Hom}_{\mathcal{D}_\Lambda}(\mathbb{S}_2^\ell \Lambda, {}_i X_j) \quad \text{for } i \ll 0, j \gg 0.$$

(This is well-defined by Proposition 6.8.) Note that $\mathbf{M}_{\mathrm{gr}} \mathcal{A}_X$ does not have finite-dimension, but its graded pieces do.

Lemma 6.10. *The graded $\tilde{\Lambda}$ -module $\mathbf{M}_{\mathrm{gr}} \mathcal{A}_X$ does not have any non-zero finite-dimensional direct summands.*

In the proof we will use the following observation:

Observation 6.11. *Let Γ be a \mathbb{Z} -graded algebra, which is generated over Γ_0 by Γ_1 . For a graded Γ -module M , and a finite interval $I \subset \mathbb{Z}$, we denote by M_I the module coinciding with M in degrees in I , and vanishing in degrees outside I . Then*

- (1) If $M \in \text{mod}_{\text{gr}} \Gamma$ is concentrated in degrees $\ell_{\min}, \dots, \ell_{\max}$, and $N \in \text{mod}_{\text{gr}} \Gamma$, then

$$\begin{aligned} \text{Hom}_{\text{mod}_{\text{gr}} \Gamma}(M, N) &= \{(f_i)_{i \in \mathbb{Z}} \in \prod_{i \in \mathbb{Z}} \text{Hom}_{\Gamma_0}(M_i, N_i) \mid \forall g \in \Gamma_1 \\ &\quad \forall m \in M_i: f_i(m)g = f_{i+1}(mg)\} \\ &= \text{Hom}_{\text{mod}_{\text{gr}} \Gamma}(M, N_{[\ell_{\min}, \ell_{\max}+1]}), \end{aligned}$$

where the latter equality holds because the conditions are empty unless $M_i \neq 0$.

- (2) Similarly, for $N \in \text{mod}_{\text{gr}} \Gamma$ concentrated in degrees $\ell_{\min}, \dots, \ell_{\max}$ and $M \in \text{mod}_{\text{gr}} \Gamma$ arbitrary we have

$$\text{Hom}_{\text{mod}_{\text{gr}} \Gamma}(M, N) = \text{Hom}_{\text{mod}_{\text{gr}} \Gamma}(M_{[\ell_{\min}-1, \ell_{\max}]}, N).$$

Proof of Lemma 6.10. Assume H is a finite-dimensional direct summand of $\mathbf{M}_{\text{gr}} \mathcal{A}_X$. Then H is concentrated in finitely many degrees, say ℓ_{\min} to ℓ_{\max} . By Proposition 6.8 and the construction of $\mathbf{M}_{\text{gr}} \mathcal{A}_X$ there are i_0 and j_0 such that

$$(\mathbf{M}_{\text{gr}}(iX_j))_{\ell} = (\mathbf{M}_{\text{gr}} \mathcal{A}_X)_{\ell} \quad \forall i \leq i_0, j \geq j_0, \ell \in \{\ell_{\min} - 1, \dots, \ell_{\max} + 1\}.$$

Since $\tilde{\Lambda}$ is generated in degrees 0 and 1 it follows from Observation 6.11 above that for all $i \leq i_0$ and $j \geq j_0$

$$\begin{aligned} \text{Hom}_{\text{mod}_{\text{gr}} \tilde{\Lambda}}(H, \mathbf{M}_{\text{gr}} \mathcal{A}_X) &= \text{Hom}_{\text{mod}_{\text{gr}} \tilde{\Lambda}}(H, \mathbf{M}_{\text{gr}}(iX_j)), \text{ and} \\ \text{Hom}_{\text{mod}_{\text{gr}} \tilde{\Lambda}}(\mathbf{M}_{\text{gr}} \mathcal{A}_X, H) &= \text{Hom}_{\text{mod}_{\text{gr}} \tilde{\Lambda}}(\mathbf{M}_{\text{gr}}(iX_j), H). \end{aligned}$$

It follows that H is a direct summand of $\mathbf{M}_{\text{gr}}(iX_j)$ for any $i \leq i_0$ and $j \geq j_0$. However the graded $\tilde{\Lambda}$ -modules $\mathbf{M}_{\text{gr}}(iX_j)$ are indecomposable (or zero), and pairwise non-isomorphic. Hence $H = 0$. \square

Lemma 6.12. *We have $(\mathbf{M}_{\text{gr}} \mathcal{A}_X) \langle a - b \rangle \cong \mathbf{M}_{\text{gr}} \mathcal{A}_X$.*

Proof. We have

$$\begin{aligned} (\mathbf{M}_{\text{gr}}(iX_j)) \langle a - b \rangle &= \bigoplus_{\ell \in \mathbb{Z}} \text{Hom}_{\mathcal{D}_{\Lambda}}(\mathbb{S}_2^{\ell+a-b} \Lambda, iX_j) \\ &= \bigoplus_{\ell \in \mathbb{Z}} \text{Hom}_{\mathcal{D}_{\Lambda}}(\mathbb{S}_2^{\ell} \Lambda, \mathbb{S}^{b-a} iX_j[2a - 2b]) \\ &= \bigoplus_{\ell \in \mathbb{Z}} \text{Hom}_{\mathcal{D}_{\Lambda}}(\mathbb{S}_2^{\ell} \Lambda, \mathbb{S}^{2b-a} iX_j[a - 2b]) \\ &= \bigoplus_{\ell \in \mathbb{Z}} \text{Hom}_{\mathcal{D}_{\Lambda}}(\mathbb{S}_2^{\ell} \Lambda, i_{-2b+a} X_{j-2b+a}) \\ &= \mathbf{M}_{\text{gr}}(i_{-2b+a} X_{j-2b+a}). \end{aligned}$$

Now the claim follows from the construction of $\mathbf{M}_{\text{gr}} \mathcal{A}_X$ (Construction 6.9). \square

We now only have to recall the following result of Dowbor and Skowroński before we can give a proof for the main result of this section.

Theorem 6.13 ([DoS]). *Let R be a finite-dimensional graded algebra. Assume there is a periodic graded R -module which has finite-dimensional graded pieces, and does not have any finite-dimensional direct summands. Then there is a finite-dimensional R -module which is not gradable.*

Proof of Theorem 6.1. If Λ is piecewise hereditary then the functor $\pi: \mathcal{D}_\Lambda \rightarrow \mathcal{C}_\Lambda$ is dense (see [Kel, Theorem in Section 4]).

Assume conversely that Λ is not piecewise hereditary. By Proposition 6.5 and Lemma 6.7 we know that the Auslander-Reiten component \mathcal{A}_X of X is of type $\mathbb{Z}A_\infty$.

By Lemma 6.10 the graded $\tilde{\Lambda}$ -module $M_{\text{gr}} \mathcal{A}_X$ has no finite-dimensional summands. By Lemma 6.12 it is periodic. It follows from Theorem 6.13 above that the push-down functor $\text{mod}_{\text{gr}} \tilde{\Lambda} \rightarrow \text{mod } \tilde{\Lambda}$ is not dense. Now the claim follows from Theorem 3.2. \square

Example 6.14. Let Λ be the algebra given by the quiver

$$\begin{array}{ccccccc} 1 & \longrightarrow & 2 & \longrightarrow & \cdots & \longrightarrow & n \\ \downarrow & & \downarrow & & & & \downarrow \\ \tilde{1} & \longrightarrow & \tilde{2} & \longrightarrow & \cdots & \longrightarrow & \tilde{n} \end{array}$$

with relations making all small squares commutative. Then the following are equivalent.

- (1) Λ is piecewise hereditary,
- (2) The functor $\mathcal{D}_\Lambda \rightarrow \mathcal{C}_\Lambda$ is dense, and
- (3) $n \leq 5$.

Proof. (1) \iff (3): By [Lad] the algebra Λ is equivalent to $kA_{2n}/\text{Rad}^3 kA_{2n}$, where A_{2n} denotes a linearly oriented quiver of type A_{2n} . By [HaS] these algebras are piecewise hereditary if and only if $2n \leq 11$, that is $n \leq 5$.

(1) \iff (2): Note that the algebra is isomorphic to the tensor product $kA_2 \otimes kA_n$. The Serre functor acts diagonally on modules which are tensor products of modules for A_2 and A_n . Thus

$$\mathbb{S}^{3n+3} S_1 = \mathbb{S}^{3n+3} (S_1^{A_2} \otimes S_1^{A_n}) = \mathbb{S}^{3n+3} S_1^{A_2} \otimes \mathbb{S}^{3n+3} S_1^{A_n},$$

where $S_1^{A_2}$ and $S_1^{A_n}$ denote the simple projective modules for kA_2 and kA_n , respectively. Since $\mathbb{S}^3 S_1^{A_2} = S_1[1]$ and $\mathbb{S}^{n+1} S_1^{A_n} = S_1[n-1]$ we obtain

$$\mathbb{S}^{3n+3} S_1 = S_1^{A_2}[n+1] \otimes S_1^{A_n}[3n-3] = S_1[4n-2].$$

Now note that for $n \neq 5$ we have $4n-2 \neq 3n+3$. Thus, by Theorem 6.1, for $n \neq 5$ we have the equivalence (2) \iff (3).

Finally note that for $n = 5$ we know that (1) holds, so (2) also holds by [Kel]. \square

Remark 6.15. Using the similar arguments as in the proof of Theorem 6.1, one can show that, provided there is an indecomposable object $X \in \mathcal{D}_\Lambda$ satisfying $X[a] \cong \mathbb{S}^b X$ for some $a \neq b$, the answer to Skowroński's question (see 1.4) is that all $T(\Lambda)$ modules are gradable precisely if Λ is piecewise hereditary. (In that case one uses Happel's equivalence $\mathcal{D}_\Lambda \rightarrow \underline{\text{mod}}_{\text{gr}} T(\Lambda)$.)

7. ORIENTED CYCLES

In this section we show that for a τ_2 -finite algebra whose quiver contains an oriented cycle the functor $\pi: \mathcal{C}_\Lambda \rightarrow \mathcal{D}_\Lambda$ is never dense. Since such algebras cannot be piecewise hereditary, this means we prove Conjecture 1.3 in this case.

We show the following theorem.

Theorem 7.1. *Let Λ be a τ_2 -finite algebra such that the quiver of Λ contains an oriented cycle. Then the functor $\pi: \mathcal{D}_\Lambda \rightarrow \mathcal{C}_\Lambda$ is not dense.*

As in the setup of Section 6, we use Dowbor's and Skowroński's result (Theorem 6.13) to show that there exists a non-gradable $\tilde{\Lambda}$ -module. We start by preparing a technical result on the indecomposability of certain zigzag-shaped complexes in Subsection 7.1. Then we complete the proof of Theorem 7.1 in Subsection 7.2.

7.1. Indecomposability of zigzags.

Lemma 7.2. *Let \mathcal{A} be an additive k -category. Assume we are given the following objects and morphisms*

$$\begin{array}{ccccccccccc} & & A_{-1} & & A_0 & & A_1 & & A_2 & & \\ & & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \\ \dots & & g_{-1} & & f_{-1} & g_0 & & f_0 & g_1 & & f_1 & g_2 & & f_2 & \dots \\ & & \searrow & \swarrow & \searrow & \\ & & B_{-2} & & B_{-1} & & B_0 & & B_1 & & B_2 & & \end{array}$$

such that for all i

- A_i and B_i have local endomorphism ring,
- $f_i \notin \text{End}_{\mathcal{A}}(B_i) \cdot g_{i+1} \cdot \text{Hom}_{\mathcal{A}}(A_i, A_{i+1})$
 $\quad + \text{Hom}_{\mathcal{A}}(B_{i-1}, B_i) \cdot g_i \cdot \text{End}_{\mathcal{A}}(A_i)$, and
- $g_i \notin \text{Hom}_{\mathcal{A}}(B_{i+1}, B_i) \cdot f_i \cdot \text{End}_{\mathcal{A}}(A_i)$
 $\quad + \text{End}_{\mathcal{A}}(B_{i-1}) \cdot f_{i-1} \cdot \text{Hom}_{\mathcal{A}}(A_i, A_{i-1})$.

(Note that the last two requirements essentially mean that no morphism factors through its neighbors.) Assume moreover that the A_i are pairwise non-isomorphic, and the B_i are pairwise non-isomorphic.

Then the complex

$$\coprod A_i \xrightarrow{\begin{pmatrix} \ddots & & & & & \\ & \ddots & & & & \\ & & f_{i+1} & & & \\ & & g_{i+1} & f_i & & \\ & & & \ddots & \ddots & \end{pmatrix}} \coprod B_i$$

is indecomposable.

Proof. Since the A_i are pairwise non-isomorphic and have local endomorphism rings any idempotent endomorphism of $\coprod A_i$ is of the form $\pi_I + (r_{ij})$ for some $I \subseteq \mathbb{Z}$, π_I the projection to the summands A_i with $i \in I$, and radical morphisms $r_{ij}: A_i \rightarrow A_j$. Similarly an idempotent of $\coprod B_i$ is of the form $\pi_J + (s_{ij})$.

Thus an idempotent of the complex of the lemma is a commutative diagram as below.

$$\begin{array}{ccc} \coprod A_i & \xrightarrow{(\cdot \cdot)} & \coprod B_i \\ \pi_I + (r_{ij}) \downarrow & & \downarrow \pi_J + (s_{ij}) \\ \coprod A_i & \xrightarrow{(\cdot \cdot)} & \coprod B_i \end{array}$$

Assume for some i we have $i \in I$ but $i \notin J$. Composing the above diagram with the injection of A_i and the projection to B_i we obtain the following diagram (omitting summands that do not contribute anything).

$$\begin{array}{ccc} A_i & \xrightarrow{\begin{pmatrix} g_i \\ f_i \end{pmatrix}} & B_{i-1} \oplus B_i \\ \begin{pmatrix} 1+r_{ii} \\ r_{i,i+1} \end{pmatrix} \downarrow & & \downarrow (s_{i-1,i}, s_{ii}) \\ A_i \oplus A_{i+1} & \xrightarrow{(f_i, g_{i+1})} & B_i \end{array}$$

Thus

$$\begin{aligned} (s_{i-1,i}, s_{ii}) \begin{pmatrix} g_i \\ f_i \end{pmatrix} &= (f_i, g_{i+1}) \begin{pmatrix} 1+r_{ii} \\ r_{i,i+1} \end{pmatrix} \\ \iff s_{i-1,i}g_i + s_{ii}f_i &= f_i + f_i r_{ii} + g_{i+1} r_{i,i+1} \\ \iff f_i + f_i r_{ii} - s_{ii}f_i &= s_{i-1,i}g_i - g_{i+1} r_{i,i+1} \end{aligned}$$

Note that $1 \otimes r_{ii} - s_{ii} \otimes 1 \in \text{Rad}(\text{End}_{\mathcal{A}}(B_i) \otimes \text{End}_{\mathcal{A}}(A_i))$, so $1 \otimes 1 + 1 \otimes r_{ii} - s_{ii} \otimes 1$ is invertible. Thus

$f_i \in \text{End}_{\mathcal{A}}(B_i) \cdot g_{i+1} \cdot \text{Hom}_{\mathcal{A}}(A_i, A_{i+1}) + \text{Hom}_{\mathcal{A}}(B_{i-1}, B_i) \cdot g_i \cdot \text{End}_{\mathcal{A}}(A_i)$ contradicting the second point of the assumption.

Similarly the cases

- $i \in I$ and $i - 1 \notin J$,
- $i \notin I$ and $i \in J$, and
- $i \notin I$ and $i - 1 \in J$

lead to contradictions.

It follows that $I = J = \emptyset$ or $I = J = \mathbb{Z}$, so the idempotent is trivial. \square

For the rest of this section the following piece of notation will be useful.

Definition 7.3. Let Λ be a finite-dimensional algebra. Let P_0 and P_ℓ be indecomposable projective modules. A *sequence of minimal relations* of length ℓ from P_0 to P_ℓ is a sequence

$$\text{top } P_0 = S_0, S_1, \dots, S_\ell = \text{top } P_\ell$$

of simple modules, such that $\text{Ext}_\Lambda^2(S_i, S_{i-1}) \neq 0$ for all $i \in \{1, \dots, \ell\}$. (Note that a 2-extension between simples corresponds to a minimal relation in the opposite direction in the quiver of the algebra; this motivates the name.)

Remark 7.4. For a τ_2 -finite algebra Λ we have the following:

- (1) A sequence of minimal relations of length ℓ corresponds to a sequence of degree 1 arrows in the quiver of $\tilde{\Lambda}$, and thus to a map of degree ℓ .
- (2) By Theorem 2.7 there are no relations in $\tilde{\Lambda}$ having summands which are products of degree 1-arrows. Thus the map corresponding to a minimal sequence of relations is non-zero, and not equal to any other linear combination of paths.
- (3) In particular the length of sequences of minimal relations is bounded above by the maximal degree of $\tilde{\Lambda}$.

Proposition 7.5. *Let Λ be a τ_2 -finite algebra. Assume there are indecomposable projective modules P_1, \dots, P_ℓ and Q_1, \dots, Q_ℓ such that for all $i \in \{1, \dots, \ell\}$ we have*

- *there is a non-zero non-isomorphism $P_i \rightarrow Q_i$,*
- *there is a sequence of minimal relations from Q_{i-1} to P_i (here $Q_0 = Q_\ell$).*

Then the functor $\pi: \mathcal{D}_\Lambda \rightarrow \mathcal{C}_\Lambda$ is not dense.

Proof. By Theorem 3.2 it suffices to show that there is a non-gradable $\tilde{\Lambda}$ -module. For this, by Theorem 6.13, it suffices to find an indecomposable periodic graded $\tilde{\Lambda}$ -module with finite dimensional graded pieces. Finally note that a graded $\tilde{\Lambda}$ -module is indecomposable and periodic if and only if its projective presentation is.

Now apply Lemma 7.2 for $\mathcal{A} = \text{proj}_{\text{gr}} \tilde{\Lambda}$, $A_i = P_{\bar{i}} \langle a_i \rangle$ for $\bar{i} - i \in \ell\mathbb{Z}$ and certain a_i , and $B_i = Q_{\bar{i}} \langle a_i \rangle$. Further we let $f_i: A_i \rightarrow B_i$ be a

map from the first point of the proposition, and $g_i: A_i \rightarrow B_{i-1}$ map corresponding to the sequence of minimal relations in the second point of the proposition.

Since the g_i are all of positive degree, and the f_i are of degree 0, it follows that the a_i are pairwise different. It follows that the A_i and the B_i are pairwise non-isomorphic.

The three points of Lemma 7.2 are easily verified: The second one holds since the f_i are of degree 0 and the g_i are of positive degree. The final one follows from Remark 7.4(2).

Thus, by Lemma 7.2, the complex

$$\coprod A_i \longrightarrow \coprod B_i$$

is indecomposable. It is periodic by construction, and has finite dimensional graded pieces. Thus the claim follows. \square

7.2. Proof of Theorem 7.1.

Lemma 7.6. *Let Λ be a finite-dimensional algebra. Assume there is a sequence of arrows*

$$i_0 \xrightarrow{\alpha_1} i_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} i_n$$

in the quiver of Λ , such that the corresponding map $P_{i_0} \rightarrow P_{i_n}$ vanishes. Then there are $a < b$ such that there is a minimal relation from i_a to i_b . Equivalently $\text{Ext}_{\Lambda}^2(\text{top } P_{i_b}, \text{top } P_{i_a}) \neq 0$.

Proof. Since the map $P_{i_0} \rightarrow P_{i_n}$ vanishes we have

$$\alpha_n \cdots \alpha_1 = \sum_i x_i r_i y_i$$

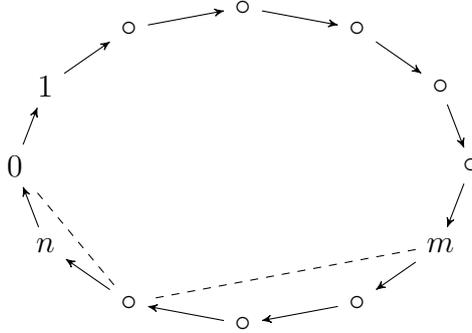
for some x_i and y_i , and minimal relations r_i . With respect to the basis consisting of paths in the quiver we see that at least one r_i has a non-zero scalar multiple of a subpath $\alpha_b \cdots \alpha_{a+1}$ of $\alpha_n \cdots \alpha_1$ as a summand. Thus we have a minimal relation from i_a to i_b . \square

Proof of Theorem 7.1. We may assume that the oriented cycle in the quiver of Λ is

$$0 \rightarrow 1 \rightarrow \cdots \rightarrow n \rightarrow 0.$$

Since Λ is finite-dimensional the map corresponding to some power of this cycle vanishes, so, by Lemma 7.6, there is a minimal relation between two vertices of the cycle. We choose a maximal sequence of minimal relations between vertices of the cycle; note that such a maximal sequence exists by Remark 7.4(3). We may assume the last relation ends in 0, and the first one starts in $m \neq 0$ (if $m = 0$ then there is a cyclic sequence of minimal relations, hence there are arbitrarily long

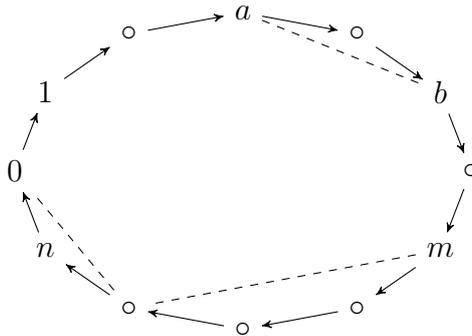
sequences of minimal relations, contradicting Remark 7.4(3)). Thus we obtain a setup as indicated in the following picture



such that no relation among the vertices $0, \dots, m$ starts in 0 or ends in m .

If the map $P_0 \rightarrow P_m$ corresponding to the upper part of the cycle is non-zero, then we are done by Proposition 7.5 (with $\ell = 1$, the map for the first point being the one assumed to be non-zero, and the sequence of relations being the sequence from the lower part of the picture above).

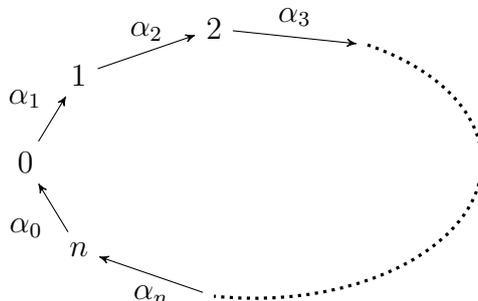
Assume now the map $P_0 \rightarrow P_m$ vanishes. Then there is at least one minimal relation $i \rightarrow j$ with $0 < i < j < m$. Choose a maximal sequence of such relations, such that every one begins where the one before ends. We obtain a setup as indicated in the following picture:



Assume the maps $P_0 \rightarrow P_a$ and $P_b \rightarrow P_m$ corresponding to the remaining paths (those not covered by relations) are both non-zero. Then we are done by Proposition 7.5 (with $\ell = 2$, the maps for the first point being those assumed to be non-zero, and the sequences of relations being the two sequences of relations in the picture above).

If one of the maps $P_0 \rightarrow P_a$ and $P_b \rightarrow P_m$ is non-zero iterate the argument (i.e. find a sequence of relations on it), until all parts of the cycle not covered by relations correspond to non-zero maps. \square

Example 7.7. Let $n \in \mathbb{N}$, and $0 < r_1 < s_1 < r_2 < s_2 < \dots < r_\ell < s_\ell \leq n$ with $\ell \geq 1$. Let Λ be the algebra given by the cyclic quiver



with relations $\{\alpha_{s_i} \cdots \alpha_{r_i} \mid i \in \{1, \dots, \ell\}\}$.

Then Λ is τ_2 -finite, and the functor $\mathcal{D}_\Lambda \rightarrow \mathcal{C}_\Lambda$ is not dense.

Proof. We first check that Λ is τ_2 -finite. Since the relations do not overlap one easily sees that the algebra has global dimension 2. In the quiver of $\tilde{\Lambda}$ the arrows of degree 1 are of the form $s_i \rightarrow r_i - 1$. The only way to come back to higher labels is via the sequence of arrows $\alpha_{s_i} \cdots \alpha_{r_i}$. But this sequence is a relation. Thus $\tilde{\Lambda}$ is finite-dimensional, and hence Λ is τ_2 -finite.

The fact that the functor $\mathcal{D}_\Lambda \rightarrow \mathcal{C}_\Lambda$ is not dense now follows immediately from Theorem 7.1. \square

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