THE GORENSTEIN DEFECT CATEGORY

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Dedicated to Ragnar-Olaf Buchweitz on the occasion of his sixtieth birthday

Abstract. We consider the homotopy category of complexes of projective modules over a Noetherian ring. Truncation at degree zero induces a fully faithful triangle functor from the totally acyclic complexes to the stable derived category. We show that if the ring is either Artin or commutative Noetherian local, then the functor is dense if and only if the ring is Gorenstein. Motivated by this, we define the *Gorenstein defect category* of the ring, a category which in some sense measures how far the ring is from being Gorenstein.

1. Introduction

Given a ring, one can associate to it certain triangulated categories: derived categories, homotopy categories, and various triangulated subcategories of these, such as bounded derived categories or homotopy categories of acyclic complexes. When the ring is Gorenstein, classical results by Buchweitz (cf. [Buc]) show that some of these triangulated categories are equivalent: the stable category of maximal Cohen-Macaulay modules, the stable derived category of finitely generated modules, and the homotopy category of totally acyclic complexes of finitely generated projective modules. Thus, for Gorenstein rings, these triangulated categories (together with their respective cohomology theories) virtually coincide.

In this paper, we provide a categorical characterization of Gorenstein rings. Let A be a left Noetherian ring, and $\operatorname{proj} A$ the category of finitely generated projective left A-modules. Brutal truncation at degree zero induces a map from the homotopy category $\mathbf{K}_{\mathrm{tac}}(\operatorname{proj} A)$ of totally acyclic complexes to the homotopy category $\mathbf{K}^{-,\mathrm{b}}(\operatorname{proj} A)$ of right bounded eventually acyclic complexes. However, this map is not a functor. We therefore consider instead the stable derived category

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A) \stackrel{\mathrm{def}}{=} \mathbf{K}^{-,\mathrm{b}}(\operatorname{proj} A)/\mathbf{K}^{\mathrm{b}}(\operatorname{proj} A)$$

of A, where $\mathbf{K}^{\mathrm{b}}(\operatorname{proj} A)$ is the homotopy category of bounded complexes. Brutal truncation now induces a triangle functor

$$\beta_{\operatorname{proj} A}\colon \mathbf{K}_{\mathrm{tac}}(\operatorname{proj} A) \longrightarrow \mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A),$$

and we show that this functor is always full and faithful. The main result, Theorem 2.7, shows that the properties of this functor actually characterize Gorenstein rings. Namely, if A is either an Artin ring or a commutative Noetherian local ring, then the functor $\beta_{\operatorname{proj} A}$ is dense if and only if A is Gorenstein. The "if" part here is classical: it is part of [Buc, Theorem 4.4.1].

Motivated by this, we define the *Gorenstein defect category* $\mathbf{D}^{\mathrm{b}}_{\mathrm{G}}(A)$ of A as the Verdier quotient

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{G}}(A) \stackrel{\mathrm{def}}{=} \mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)/\langle \operatorname{Im} \beta_{\operatorname{proj} A}\rangle,$$

where $\langle \operatorname{Im} \beta_{\operatorname{proj} A}\rangle$ is the isomorphism closure of the image of $\beta_{\operatorname{proj} A}$ in $\mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)$: this is a thick subcategory. Then Theorem 2.7 translates to the following: if A is either

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an Artin ring or a commutative Noetherian local ring, then $\mathbf{D}^{\mathrm{b}}_{\mathrm{G}}(A) = 0$ if and only if A is Gorenstein. The dimension of the Gorenstein defect category is therefore in some sense a measure of "how far" the ring is from being Gorenstein.

2. The results

Let $\mathscr{P}$ be an additive category, and $\mathbf{K}\mathscr{P}$ the homotopy category of complexes in $\mathscr{P}$. This is a triangulated category, with suspension $\Sigma\colon \mathbf{K}\mathscr{P} \longrightarrow \mathbf{K}\mathscr{P}$ given by shifting a complex one degree to the left, and changing the sign of its differential. That is, for a complex $C \in \mathbf{K}\mathscr{P}$ with differential d, the complex ΣC has C_{n-1} in degree n, and $-d$ as differential. The (distinguished) triangles in $\mathbf{K}\mathscr{P}$ are the sequences of objects and maps that are isomorphic to sequences of the form

$$C_1 \xrightarrow{f} C_2 \longrightarrow C(f) \longrightarrow \Sigma C_1$$

for some map f and its mapping cone $C(f)$.

We say that a complex C in $\mathscr{P}$ is *acyclic* if the complex $\operatorname{Hom}_{\mathscr{P}}(P, C)$ of abelian groups is acyclic for all objects $P \in \mathscr{P}$. If in addition $\operatorname{Hom}_{\mathscr{P}}(C, P)$ is acyclic for all $P \in \mathscr{P}$, then C is *totally acyclic*. Moreover, C is *eventually acyclic* if for all objects $P \in \mathscr{P}$, the complex $\operatorname{Hom}_{\mathscr{P}}(P, C)$ is eventually acyclic, i.e. $\mathrm{H}_n(\operatorname{Hom}_{\mathscr{P}}(P, C)) = 0$ for $|n| \gg 0$. Note that we cannot define acyclicity directly for complexes in $\mathscr{P}$, since the category is only assumed to be additive.

We shall be working with the following full subcategories of $\mathbf{K}\mathscr{P}$ (the definitions are up to isomorphism in $\mathbf{K}\mathscr{P}$):

$$\begin{aligned}
\mathbf{K}_{\mathrm{tac}}\mathscr{P} &= \{C \in \mathbf{K}\mathscr{P} \mid C \text{ is totally acyclic}\} \\
\mathbf{K}^{-,\mathrm{b}}\mathscr{P} &= \{C \in \mathbf{K}\mathscr{P} \mid C_n = 0 \text{ for } n \ll 0 \text{ and } C \text{ is eventually acyclic}\} \\
\mathbf{K}^{\mathrm{b}}\mathscr{P} &= \{C \in \mathbf{K}\mathscr{P} \mid C_n = 0 \text{ for } |n| \gg 0\}.
\end{aligned}$$

These are all triangulated subcategories of $\mathbf{K}\mathscr{P}$. For example, when $\mathscr{P}$ is the category of finitely generated left projective modules over a left Noetherian ring A, then $\mathbf{K}^{\mathrm{b}}\mathscr{P}$ is by definition the category of perfect complexes. Moreover, in this setting it is well known that the triangulated categories $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}$ and $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ are equivalent, where $\operatorname{mod} A$ is the category of finitely generated left A-modules.

The category $\mathbf{K}^{\mathrm{b}}\mathscr{P}$ is a thick subcategory of $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}$, that is, a triangulated subcategory closed under direct summands. The Verdier quotient $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}/\mathbf{K}^{\mathrm{b}}\mathscr{P}$ is therefore a well defined triangulated category. Recall that the objects in this quotient are the same as the objects in $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}$. A morphism $C_1 \longrightarrow C_2$ in the quotient is an equivalence class of diagrams of the form

$$\begin{tikzcd}
 & D \arrow[dl, "g"'] \arrow[dr, "f"] & \\
C_1 & & C_2
\end{tikzcd}$$

where f and g are morphisms in $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}$, and g has the property that its mapping cone $C(g)$ belongs to $\mathbf{K}^{\mathrm{b}}\mathscr{P}$. Two such morphisms (g, D, f) and (g', D', f') are equivalent if there exists a third such morphism (g'', D'', f''), and two morphisms $h\colon D'' \longrightarrow D$ and $h'\colon D'' \longrightarrow D'$ in $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}$, such that the diagram

$$\begin{tikzcd}
 & D \arrow[dl, "g"'] \arrow[dr, "f"] & \\
C_1 & D'' \arrow[l, "g''"'] \arrow[r, "f''"] \arrow[u, "h"] \arrow[d, "h'"] & C_2 \\
 & D' \arrow[ul, "g'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

is commutative. For further details, we refer to [Nee, Chapter 2]. The natural triangle functor $\mathbf{K}^{-,\mathrm{b}}\mathscr{P} \longrightarrow \mathbf{K}^{-,\mathrm{b}}\mathscr{P}/\mathbf{K}^{\mathrm{b}}\mathscr{P}$ maps an object to itself, and a morphism $f\colon C_1 \longrightarrow C_2$ to the equivalence class of the diagram

$$\begin{tikzcd} & C_1 \arrow[dl, "1"'] \arrow[dr, "f"] & \\ C_1 & & C_2 \end{tikzcd}$$

In Theorem 2.2, we establish a fully faithful triangle functor from $\mathbf{K}_{\mathrm{tac}}\mathscr{P}$ to the quotient $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}/\mathbf{K}^{\mathrm{b}}\mathscr{P}$. To avoid too many technicalities in the proof, we first prove the following lemma. It allows us to complete certain morphisms and homotopies of truncated complexes. Given an integer n and a complex

$$C\colon \quad \cdots \xrightarrow{d_{n+3}} C_{n+2} \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \xrightarrow{d_{n-2}} \cdots$$

in $\mathscr{P}$, we denote its *brutal truncation*

$$C\colon \quad \cdots \xrightarrow{d_{n+3}} C_{n+2} \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \to 0 \to 0 \to \cdots$$

at degree n by $\beta_{\geq n}(C)$. Note that when $C \in \mathbf{K}_{\mathrm{tac}}\mathscr{P}$, then $\beta_{\geq n}(C) \in \mathbf{K}^{-,\mathrm{b}}\mathscr{P}$.

Lemma 2.1. *Let $\mathscr{P}$ be an a additive category, and C, D two complexes in $\mathscr{P}$ with C totally acyclic. Furthermore, let n be an integer, and $f\colon \beta_{\geq n}(C) \longrightarrow \beta_{\geq n}(D)$ a chain map.*

(a) *The map f can be extended to a chain map $\widehat{f}\colon C \longrightarrow D$ (with $\beta_{\geq n}(\widehat{f}) = f$).*
(b) *Let $\pi\colon C \longrightarrow \beta_{\geq n}(C)$ be the natural chain map, and consider the composite chain map $f\circ\pi\colon C \longrightarrow \beta_{\geq n}(D)$. If $f\circ\pi$ is nullhomotopic through a homotopy h, then h can be extended to a homotopy $\widehat{h}$ making $\widehat{f}$ nullhomotopic.*

Proof. (a) The chain map f is given by the solid part of the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r, "d^C_{n+3}"] & C_{n+2} \arrow[r, "d^C_{n+2}"] \arrow[d, "f_{n+2}"] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d, dashed, "f_{n-1}"] & C_{n-2} \arrow[r, "d^C_{n-2}"] & \cdots \\ \cdots \arrow[r, "d^D_{n+3}"] & D_{n+2} \arrow[r, "d^D_{n+2}"] & D_{n+1} \arrow[r, "d^D_{n+1}"] & D_n \arrow[r, "d^D_n"] & D_{n-1} \arrow[r, "d^D_{n-1}"] & D_{n-2} \arrow[r, "d^D_{n-2}"] & \cdots \end{tikzcd}$$

and it suffices to find a map f_{n-1} as indicated, making the square to its left commutative. The composition $d^D_n \circ f_n \circ d^C_{n+1}$ is zero, and by assumption the sequence

$$\mathrm{Hom}_{\mathscr{P}}(C_{n-1}, D_{n-1}) \xrightarrow{(d^C_n)^*} \mathrm{Hom}_{\mathscr{P}}(C_n, D_{n-1}) \xrightarrow{(d^C_{n+1})^*} \mathrm{Hom}_{\mathscr{P}}(C_{n+1}, D_{n-1})$$

is exact. Therefore, there exists a map $f_{n-1}\colon C_{n-1} \longrightarrow D_{n-1}$ such that $f_{n-1} \circ d^C_n = d^D_n \circ f_n$.

(b) The homotopy h is given by maps $h_{n-1}, h_n, h_{n+1}, \ldots$ as in the diagram

$$\begin{tikzcd} \cdots \arrow[r, "d^C_{n+3}"] & C_{n+2} \arrow[r, "d^C_{n+2}"] \arrow[d, "f_{n+2}"] \arrow[dl, "h_{n+2}"'] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"] \arrow[dl, "h_{n+1}"'] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] \arrow[dl, "h_n"'] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d, "f_{n-1}"] \arrow[dl, "h_{n-1}"'] & C_{n-2} \arrow[r, "d^C_{n-2}"] \arrow[d, "f_{n-2}"] \arrow[dl, dashed, "h_{n-2}"'] & \cdots \arrow[dl, dashed, "h_{n-3}"'] \\ \cdots \arrow[r, "d^D_{n+3}"] & D_{n+2} \arrow[r, "d^D_{n+2}"] & D_{n+1} \arrow[r, "d^D_{n+1}"] & D_n \arrow[r, "d^D_n"] & D_{n-1} \arrow[r, "d^D_{n-1}"] & D_{n-2} \arrow[r, "d^D_{n-2}"] & \cdots \end{tikzcd}$$

with $f_i = d^D_{i+1} \circ h_n + h_{i-1} \circ d^C_i$ for all $i \geq n$. We must find maps $h_{n-2}, h_{n-3}, \ldots$ completing the homotopy.

To find the map h_{n-2}, consider the map $f_{n-1} - d_n^D \circ h_{n-1}$ in $\operatorname{Hom}_{\mathscr{P}}(C_{n-1}, D_{n-1})$, for which we obtain

$$(f_{n-1} - d_n^D \circ h_{n-1}) \circ d_n^C = d_n^D \circ f_n - d_n^D \circ (f_n - d_{n+1}^D \circ h_n) = 0.$$

Since the sequence

$$\operatorname{Hom}_{\mathscr{P}}(C_{n-2}, D_{n-1}) \xrightarrow{(d_{n-1}^C)^*} \operatorname{Hom}_{\mathscr{P}}(C_{n-1}, D_{n-1}) \xrightarrow{(d_n^C)^*} \operatorname{Hom}_{\mathscr{P}}(C_n, D_{n-1})$$

is exact, there exists a map $h_{n-2} \colon C_{n-2} \longrightarrow D_{n-1}$ such that

$$f_{n-1} - d_n^D \circ h_{n-1} = h_{n-2} \circ d_{n-1}^C.$$

Iterating this procedure, we obtain the maps $h_{n-3}, h_{n-4}, \dots$ giving $\widehat{h}$. □

We are now ready to prove Theorem 2.2. It establishes a fully faithful triangle functor from $\mathbf{K}_{\mathrm{tac}}\mathscr{P}$ to the quotient $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}/\mathbf{K}^{\mathrm{b}}\mathscr{P}$, mapping a complex C to its brutal truncation $\beta_{\geq 0}(C)$ at degree zero. Note that brutal truncation does not define a functor between homotopy categories.

Theorem 2.2. *For an additive category $\mathscr{P}$, brutal truncation at degree zero induces a fully faithful triangle functor*

$$\beta_{\mathscr{P}} \colon \mathbf{K}_{\mathrm{tac}}\mathscr{P} \longrightarrow \mathbf{K}^{-,\mathrm{b}}\mathscr{P}/\mathbf{K}^{\mathrm{b}}\mathscr{P}.$$

Proof. To simplify notation, we denote $\beta_{\mathscr{P}}$ by just β. The first issue to address is well-definedness. Let $f \colon C \longrightarrow D$ be a map of complexes in $\mathbf{K}_{\mathrm{tac}}\mathscr{P}$, as indicated in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "d_3^C"] & C_2 \arrow[r, "d_2^C"] \arrow[d, "f_2"] \arrow[dl, dashed, "h_2"] & C_1 \arrow[r, "d_1^C"] \arrow[d, "f_1"] \arrow[dl, dashed, "h_1"] & C_0 \arrow[r, "d_0^C"] \arrow[d, "f_0"] \arrow[dl, dashed, "h_0"] & C_{-1} \arrow[r, "d_{-1}^C"] \arrow[d, "f_{-1}"] \arrow[dl, dashed, "h_{-1}"] & \cdots \arrow[dl, dashed, "h_{-2}"] \\
\cdots \arrow[r, "d_3^D"] & D_2 \arrow[r, "d_2^D"] & D_1 \arrow[r, "d_1^D"] & D_0 \arrow[r, "d_0^D"] & D_{-1} \arrow[r, "d_{-1}^D"] & \cdots
\end{tikzcd}$$

It suffices to show that if f vanishes in $\mathbf{K}_{\mathrm{tac}}\mathscr{P}$, that is, if there is a homotopy h as indicated by the dashed arrows above, then $\beta_{\geq 0}(f)$ vanishes in $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}/\mathbf{K}^{\mathrm{b}}\mathscr{P}$. Using this homotopy h, we see that the map

$$\begin{tikzcd}
\cdots \arrow[r, "d_3^C"] & C_2 \arrow[r, "d_2^C"] \arrow[d, "f_2"] & C_1 \arrow[r, "d_1^C"] \arrow[d, "f_1"] & C_0 \arrow[r] \arrow[d, "f_0 - h_{-1} \circ d_0^C"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r, "d_3^D"] & D_2 \arrow[r, "d_2^D"] & D_1 \arrow[r, "d_1^D"] & D_0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is nullhomotopic in $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}$. Consequently, the map $\beta_{\geq 0}(f)$ is homotopic to the map

$$\begin{tikzcd}
\cdots \arrow[r, "d_3^C"] & C_2 \arrow[r, "d_2^C"] \arrow[d, "0"] & C_1 \arrow[r, "d_1^C"] \arrow[d, "0"] & C_0 \arrow[r] \arrow[d, "h_{-1} \circ d_0^C"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r, "d_3^D"] & D_2 \arrow[r, "d_2^D"] & D_1 \arrow[r, "d_1^D"] & D_0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

in $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}$. Clearly, this map factors through the stalk complex with C_{-1} in degree zero. Therefore $\beta(f)$, which equals the image of $\beta_{\geq 0}(f)$ in the quotient $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}/\mathbf{K}^{\mathrm{b}}\mathscr{P}$, vanishes. This shows that the functor β is well-defined.

It is easy to check that β is a triangle functor. The natural isomorphism $\beta \circ \Sigma \longrightarrow \Sigma \circ \beta$ is given by

$$\begin{tikzcd}
 & & \overset{\deg 2}{C_1} \arrow[r] \arrow[d, equal] & \overset{\deg 1}{C_0} \arrow[r] \arrow[d, equal] & \overset{\deg 0}{C_{-1}} \arrow[r] \arrow[d] & \overset{\deg -1}{0} \arrow[r] \arrow[d] & \cdots \\
\beta(\Sigma C)\colon \arrow[d] & \cdots \arrow[r] & C_1 \arrow[r] & C_0 \arrow[r] & C_{-1} \arrow[r] & 0 \arrow[r] & \cdots \\
\Sigma(\beta(C))\colon & \cdots \arrow[r] & C_1 \arrow[r] & C_0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

This is indeed an isomorphism in $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}/\mathbf{K}^{\mathrm{b}}\mathscr{P}$, since its mapping cone in $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}$ is isomorphic to the stalk complex with C_{-1} in degree one, which belongs to $\mathbf{K}^{\mathrm{b}}\mathscr{P}$. Using a similar isomorphism, one checks that β commutes with mapping cones.

Next, we prove that β is faithful. Let $f\colon C \longrightarrow D$ be a morphism in $\mathbf{K}_{\mathrm{tac}}\mathscr{P}$ such that $\beta(f) = 0$. We may think of f as a morphism of complexes. Then the condition $\beta(f) = 0$ means that, up to homotopy, the brutal truncation $\beta_{\geq 0}(f)$ factors through a bounded complex $C' \in \mathbf{K}^{\mathrm{b}}\mathscr{P}$. Choose a positive integer n such that $C'_i = 0$ for $i \geq n$. By truncating at degree n, we see that the induced map $\beta_{\geq n}(f) \circ \pi\colon C \longrightarrow \beta_{\geq n}(D)$ of complexes is nullhomotopic. It then follows from Lemma 2.1(b) that f itself is nullhomotopic, and this shows that the functor β is faithful.

It remains to show that β is full. Let $\psi\colon \beta(C) \longrightarrow \beta(D)$ be a morphism in $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}/\mathbf{K}^{\mathrm{b}}\mathscr{P}$ between two complexes in the image of β. Then ψ is represented by a diagram

$$\begin{tikzcd}
 & C' \arrow[dl, "g"'] \arrow[dr, "f"] & \\
\beta_{\geq 0}(C) & & \beta_{\geq 0}(D)
\end{tikzcd}$$

of complexes and maps in $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}$, where the mapping cone $C(g)$ of the map g belongs to $\mathbf{K}^{\mathrm{b}}\mathscr{P}$. Up to homotopy, in sufficiently high degrees, the complex C' then coincides with $\beta_{\geq 0}(C)$, and then also with C. Therefore, for some positive integer n, there is an equality $\beta_{\geq n}(C) = \beta_{\geq n}(C')$, and the truncation $\beta_{\geq n}(g)$ is the identity. Furthermore, the truncation $\beta_{\geq n}(f)$ is a morphism $\beta_{\geq n}(f)\colon \beta_{\geq n}(C) \longrightarrow \beta_{\geq n}(D)$. By Lemma 2.1(a), it admits and extension $\widehat{f}\colon C \longrightarrow D$ of complexes in $\mathbf{K}_{\mathrm{tac}}\mathscr{P}$: we shall prove that $\psi = \beta(\widehat{f})$.

Consider the solid part of the diagram

$$\begin{tikzcd}
 & C' \arrow[dl, "g"'] \arrow[dr, "f"] & \\
\beta_{\geq 0}(C) & \beta_{\geq -1}(C) \arrow[l, "\pi"'] \arrow[r, "\beta_{\geq 0}(\widehat{f}) \circ \pi"] \arrow[u, dashed, "\theta"] \arrow[d, "\pi"] & \beta_{\geq 0}(D) \\
 & \beta_{\geq 0}(C) \arrow[ul, "1"] \arrow[ur, "\beta_{\geq 0}(\widehat{f})"'] &
\end{tikzcd}$$

of complexes and maps in $\mathbf{K}^{-,\mathrm{b}}\mathscr{P}$, where π is the natural projection. The lower two triangles obviously commute. Furthermore, by Lemma 2.1(a), the identity chain map $1\colon \beta_{\geq n}(C) \longrightarrow \beta_{\geq n}(C')$ admits an extension $\theta\colon \beta_{\geq -1}(C) \longrightarrow C'$, and by

construction the equalities

$$\begin{aligned} \beta_{\geq n}(\pi) &= \beta_{\geq n}(g \circ \theta) \\ \beta_{\geq n}\left(\beta_{\geq 0}(\widehat{f}) \circ \pi\right) &= \beta_{\geq n}(f \circ \theta) \end{aligned}$$

hold. The chain map $\pi - g \circ \theta$ can be viewed as a chain map $C \longrightarrow \beta_{\geq 0}(C)$, and its truncation $\beta_{\geq n}(\pi - g \circ \theta)$ is trivially nullhomotopic. Thus, by Lemma 2.1(b), the map $\pi - g \circ \theta$ itself is nullhomotopic, and this shows that the top left triangle commutes. Similarly, the top right triangle commutes, hence the diagram is commutative. Consequently, the map ψ equals $\beta(\widehat{f})$, and so the functor β is full. □

We shall apply Theorem 2.2 to the case when the additive category $\mathscr{P}$ is the category $\operatorname{proj} A$ of finitely generated projective modules over a left Noetherian ring A. The following result shows that, in this situation, if the functor is dense (i.e. an equivalence in view of Theorem 2.2), then all higher extensions between any module and the ring vanish. Recall first that the Verdier quotient

$$\mathbf{K}^{-,\mathrm{b}}(\operatorname{proj} A)/\mathbf{K}^{\mathrm{b}}(\operatorname{proj} A)$$

is the classical *stable derived category* $\mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)$ of A.

Proposition 2.3. *Let A be a left Noetherian ring and* $\operatorname{proj} A$ *the category of finitely generated projective left A-modules. If the functor*

$$\beta_{\operatorname{proj} A} \colon \mathbf{K}_{\mathrm{tac}}(\operatorname{proj} A) \longrightarrow \mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)$$

is dense, then $\operatorname{Ext}^n_A(M, A) = 0$ *for all $n \gg 0$ and every finitely generated left A-module M.*

Proof. Let M be a finitely generated left A-module, and

$$P \colon \cdots \xrightarrow{d_4} P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0$$

its projective resolution: this is a complex in $\mathbf{K}^{-,\mathrm{b}}(\operatorname{proj} A)$. Since the functor $\beta_{\operatorname{proj} A}$ is dense, the complex P is isomorphic in $\mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)$ to $\beta_{\operatorname{proj} A}(T)$ for some totally acyclic complex $T \in \mathbf{K}_{\mathrm{tac}}(\operatorname{proj} A)$. For some $n \gg 0$, the two complexes $\beta_{\geq n}(P)$ and $\beta_{\geq n}(T)$ coincide up to homotopy, giving

$$\operatorname{Ext}^i_A(M, A) \simeq \mathrm{H}_{-i}\left(\operatorname{Hom}_A(P, A)\right) \simeq \mathrm{H}_{-i}\left(\operatorname{Hom}_A(T, A)\right)$$

for all $i \geq n+1$. Since the complex T is totally acyclic, the group $\mathrm{H}_j\left(\operatorname{Hom}_A(T, A)\right)$ vanishes for all $j \in \mathbb{Z}$, and this proves the result. □

Specializing to the case when the ring is either left Artin or commutative Noetherian local, we obtain the following two corollaries. Recall that a commutative local ring is *Gorenstein* if its injective dimension (as a module over itself) is finite.

Corollary 2.4. *Let A be a left Artin ring, and* $\operatorname{proj} A$ *the category of finitely generated projective left A-modules. If the functor*

$$\beta_{\operatorname{proj} A} \colon \mathbf{K}_{\mathrm{tac}}(\operatorname{proj} A) \longrightarrow \mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)$$

is dense, then the injective dimension of A as a left module is finite.

Proof. There are finitely many simple left A-modules $S_1, \ldots, S_t$, and by Proposition 2.3 there exists an integer n such that $\operatorname{Ext}^i_A(\oplus S_j, A) = 0$ for $i \geq n+1$. The injective dimension of A is therefore at most n. □

Corollary 2.5. *Let A be a commutative Noetherian local ring, and* $\operatorname{proj} A$ *the category of finitely generated projective (i.e. free) A-modules. If the functor*

$$\beta_{\operatorname{proj} A} \colon \mathbf{K}_{\mathrm{tac}}(\operatorname{proj} A) \longrightarrow \mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)$$

is dense, then A is Gorenstein.

Proof. Let k be the residue field of A. By Proposition 2.3 there exists an integer n such that $\operatorname{Ext}^i_A(k, A) = 0$ for $i \geq n+1$. The injective dimension of A is therefore at most n. □

The following result shows that, in the situation of Proposition 2.3, every injective module has finite projective dimension.

Proposition 2.6. *Let A be a left Noetherian ring and* $\operatorname{proj} A$ *the category of finitely generated projective left A-modules. If the functor*

$$\beta_{\operatorname{proj} A} \colon \mathbf{K}_{\mathrm{tac}}(\operatorname{proj} A) \longrightarrow \mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)$$

is dense, then the projective dimension of every finitely generated injective left A-module is finite.

Proof. Let I be a finitely generated injective left A-module, and $P_I \in \mathbf{K}^{-,\mathrm{b}}(\operatorname{proj} A)$ a projective resolution of I. For every $n \geq 1$, denote by $\Omega^n_A(I)$ the image of the nth differential in P_I. It suffices to show that the identity on P_I factors through an object in $\mathbf{K}^{\mathrm{b}}(\operatorname{proj} A)$, for this would imply that $\Omega^n_A(I)$ is projective for high n.

Let T be a totally acyclic complex in $\mathbf{K}_{\mathrm{tac}}(\operatorname{proj} A)$, and M the image of its zeroth differential. Then there is a monomorphism $f \colon M \longrightarrow P$ for some $P \in \operatorname{proj} A$ (take for example $P = T_{-1}$). Since I is injective, every map $M \longrightarrow I$ factors through f. Now for every $n \geq 1$, denote by $\Omega^n_A(M)$ the image of the nth differential in T. We claim that every map $g \colon \Omega^n_A(M) \longrightarrow \Omega^n_A(I)$ factors through a projective module. To see this, note that every such map lifts to a chain map $\beta_{\geq n}(T) \longrightarrow \beta_{\geq n}(P_I)$, and by Lemma 2.1(a) this chain map can be extended to a chain map $T \longrightarrow P_I$. This gives a map $g' \colon M \longrightarrow I$, which factors through a projective module by the above. Since $g = \Omega^n_A(g')$, the map g also factors through a projective module, as claimed.

We show next that $\operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)}(\beta_{\operatorname{proj} A}(T), P_I) = 0$. Any morphism ψ in this group is (represented by) a diagram

$$\begin{tikzcd} & C \arrow[dl, "g"'] \arrow[dr, "f"] & \\ \beta_{\geq 0}(T) & & P_I \end{tikzcd}$$

in $\mathbf{K}^{-,\mathrm{b}}(\operatorname{proj} A)$, with the cone of g belonging to $\mathbf{K}^{\mathrm{b}}(\operatorname{proj} A)$. As in the proof of Theorem 2.2, we can assume that (up to homotopy) the two complexes $\beta_{\geq 0}(T)$ and C agree in high degrees. From the above, it then follows that for high n, every map $B^n(C) \longrightarrow B^n(P_I)$ factors through a projective module, where $B^n(D)$ denotes the image of the nth differential in a complex D. This shows that $\psi = 0$.

We can now show that the identity on P_I factors through an object in $\mathbf{K}^{\mathrm{b}}(\operatorname{proj} A)$. Namely, since the functor $\beta_{\operatorname{proj} A}$ is dense, the complex P_I is isomorphic in $\mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)$ to $\beta_{\operatorname{proj} A}(T)$ for some acyclic complex $T \in \mathbf{K}_{\mathrm{tac}}(\operatorname{proj} A)$. From what we showed above, we obtain

$$\operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)}(P_I, P_I) \simeq \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)}(\beta_{\operatorname{proj} A}(T), P_I) = 0,$$

and the result follows. □

Recall that a Noetherian ring (i.e. a ring that is both left and right Noetherian) is *Gorenstein* if its injective dimensions both as a left and as a right module are finite. By a classical result of Zaks (cf. [Zak, Lemma A]), the two injective dimensions then coincide. However, it is an open question whether a Noetherian ring of finite selfinjective dimension on one side is of finite selfinjective dimension on both sides, and therefore Gorenstein.

We have now come to the main result. It deals with Artin rings (i.e. rings that are both left and right Artin) and commutative Noetherian local rings. Namely, for such a ring A, the functor $\beta_{\operatorname{proj} A}$ is dense if and only if A is Gorenstein. As mentioned in the introduction, the "if" part of this result is classical: it is part of [Buc, Theorem 4.4.1]. We include a proof for the convenience of the reader.

Theorem 2.7. *Let A be either an Artin ring or a commutative Noetherian local ring, and* $\operatorname{proj} A$ *the category of finitely generated projective left A-modules. Then the functor*

$$\beta_{\operatorname{proj} A}\colon \mathbf{K}_{\mathrm{tac}}(\operatorname{proj} A) \longrightarrow \mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)$$

is dense if and only if A is Gorenstein.

Proof. Suppose the functor $\beta_{\operatorname{proj} A}$ is dense. If A is local, then it is Gorenstein by Corollary 2.5. If A is Artin, then the injective dimension of A as a left module is finite by Corollary 2.4. Moreover, by Proposition 2.6, every finitely generated injective left A-module has finite projective dimension. The duality between finitely generated left and right modules then implies that every finitely generated projective right A-module has finite injective dimension. Therefore A is Gorenstein.

Conversely, suppose that A is Gorenstein, and let C be a complex in $\mathbf{K}^{-,\mathrm{b}}(\operatorname{proj} A)$. Using the same notation as in the previous proof, there is an integer n such that the A-module $B^n(C)$ is maximal Cohen-Macaulay, and such that $\beta_{\geq n}(C)$ is a projective resolution of $B^n(C)$. Since $B^n(C)$ is maximal Cohen-Macaulay, it admits a projective co-resolution C', and splicing $\beta_{\geq n}(C)$ and C' at $B^n(C)$ gives a totally acyclic complex $T \in \mathbf{K}_{\mathrm{tac}}(\operatorname{proj} A)$. Since $\beta_{\geq n}(C) = \beta_{\geq n}(T)$, the complexes C and $\beta_{\operatorname{proj} A}(T)$ are isomorphic in $\mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)$, hence the functor $\beta_{\operatorname{proj} A}$ is dense. □

Motivated by Theorem 2.2 and Theorem 2.7, we now introduce a new triangulated category for any left Noetherian ring A. Since the triangle functor

$$\beta_{\operatorname{proj} A}\colon \mathbf{K}_{\mathrm{tac}}(\operatorname{proj} A) \longrightarrow \mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)$$

is fully faithful by Theorem 2.2, the category $\mathbf{K}_{\mathrm{tac}}(\operatorname{proj} A)$ embeds in $\mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)$ as the image of $\beta_{\operatorname{proj} A}$. The isomorphism closure $\langle \operatorname{Im} \beta_{\operatorname{proj} A} \rangle$ is a thick subcategory of $\mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)$, hence we may form the corresponding Verdier quotient.

Definition. The *Gorenstein defect category* of a left Noetherian ring A is the Verdier quotient

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{G}}(A) \stackrel{\mathrm{def}}{=} \mathbf{D}^{\mathrm{b}}_{\mathrm{st}}(A)/\langle \operatorname{Im} \beta_{\operatorname{proj} A} \rangle,$$

where $\operatorname{proj} A$ is the category of finitely generated projective left A-modules.

In terms of the Gorenstein defect category, Theorem 2.7 takes the following form.

Theorem 2.8. *If A is either an Artin ring or a commutative Noetherian local ring, then $\mathbf{D}^{\mathrm{b}}_{\mathrm{G}}(A) = 0$ if and only if A is Gorenstein.*

Theorem 2.8 suggests that the size of the Gorenstein defect category (of an Artin ring or a commutative Noetherian local ring) measures in some sense "how far" the ring is from being Gorenstein. It would therefore be interesting to find criteria which characterize the rings whose Gorenstein defect categories are n-dimensional, in the sense of Rouquier (cf. [Rou]). An answer to the following question would be a natural start.

Question. What characterizes Artin rings and commutative Noetherian local rings with zero-dimensional Gorenstein defect categories?

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