

# CLUSTER TILTING AND COMPLEXITY

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**ABSTRACT.** We study the notion of positive and negative complexity of pairs of objects in cluster categories. The first main result shows that the maximal complexity occurring is either one, two or infinite, depending on the representation type of the underlying hereditary algebra. In the second result, we study the bounded derived category of a cluster tilted algebra, and show that the maximal complexity occurring is either zero or one whenever the algebra is of finite or tame type.

## 1. INTRODUCTION

Cluster categories associated to finite dimensional hereditary algebras were introduced in [BMRRT]. These 2-Calabi-Yau triangulated categories arise as orbit categories of derived categories, and provide a categorification of the combinatorics of the cluster algebras introduced in [FoZ] by Fomin and Zelevinsky in the acyclic case. They also provide a generalized framework for classical tilting theory, with the cluster tilting objects and their endomorphism rings, the cluster tilted algebras.

Given two objects in a triangulated category defined over a field, their total cohomology is a  $\mathbb{Z}$ -graded vector space over the ground field. It therefore makes sense to study the rate of growth of the dimensions in both the “negative” and the “positive” direction, thus leading to the notion of negative and positive complexity of a pair of objects. In this paper, we study the complexity of pairs of objects in a cluster category, and show that the maximal complexity occurring depends on the representation type of the hereditary algebra we start with:

**Theorem.** *Let  $H$  be a basic finite dimensional hereditary algebra over an algebraically closed field, and let  $\mathcal{C}_H$  be the corresponding cluster category. Then*

$$\sup\{\text{cx}_{\mathcal{C}_H}^*(X, Y) \mid X, Y \in \mathcal{C}_H\} = \begin{cases} 1 & \text{if } H \text{ has finite type,} \\ 2 & \text{if } H \text{ has tame type,} \\ \infty & \text{if } H \text{ has wild type.} \end{cases}$$

We also study the complexity of the derived category of a cluster tilted algebra, and show that in this case, the maximal complexity occurring depends on the representation type of the algebra:

**Theorem.** *If  $\Lambda$  is a cluster tilted algebra of finite or tame representation type, then*

$$\sup\{\text{cx}_{\mathcal{D}^b(\Lambda)}^*(X, Y) \mid X, Y \in \mathcal{D}^b(\Lambda)\} = \begin{cases} 0 & \text{if } \Lambda \text{ is hereditary,} \\ 1 & \text{otherwise} \end{cases}$$

We prove this by showing that a tame cluster tilted algebra has finitely many indecomposable Cohen-Macaulay modules. Finally, we look at some examples showing what can happen for wild cluster tilted algebras.

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2000 *Mathematics Subject Classification.* 16P90, 18E30, 18G15.

*Key words and phrases.* Cluster categories, cluster tilted algebras, complexity.

Both authors were supported by NFR Storforsk grant no. 167130.

## 2. PRELIMINARIES

Throughout this section, we fix a field  $k$  and a triangulated Hom-finite  $k$ -category  $\mathcal{T}$  with suspension functor  $\Sigma$ . Thus for all objects  $X, Y, Z$  in  $\mathcal{T}$ , the set  $\text{Hom}_{\mathcal{T}}(X, Y)$  is a finite dimensional  $k$ -vector space, and the composition

$$\text{Hom}_{\mathcal{T}}(Y, Z) \times \text{Hom}_{\mathcal{T}}(X, Y) \rightarrow \text{Hom}_{\mathcal{T}}(X, Z)$$

is  $k$ -bilinear. Recall that a *Serre functor* on  $\mathcal{T}$  is a triangle equivalence  $\mathcal{T} \xrightarrow{S} \mathcal{T}$ , together with functorial isomorphisms

$$\text{Hom}_{\mathcal{T}}(X, Y) \simeq D \text{Hom}_{\mathcal{T}}(Y, SX)$$

of vector spaces for all objects  $X, Y \in \mathcal{T}$ , where  $D = \text{Hom}_k(-, k)$ . By [BoK], such a functor is unique if it exists. For an integer  $d \in \mathbb{Z}$ , the category  $\mathcal{T}$  is said to be *d-Calabi-Yau* if it admits a Serre functor which is isomorphic as a triangle functor to  $\Sigma^d$ .

A subcategory of  $\mathcal{T}$  is *thick* if it is a full triangulated subcategory closed under direct summands. Now let  $\mathcal{C}$  and  $\mathcal{D}$  be subcategories of  $\mathcal{T}$ . We denote by  $\text{thick}_{\mathcal{T}}^1(\mathcal{C})$  the full subcategory of  $\mathcal{T}$  consisting of all the direct summands of finite direct sums of shifts of objects in  $\mathcal{C}$ . Furthermore, we denote by  $\mathcal{C} * \mathcal{D}$  the full subcategory of  $\mathcal{T}$  consisting of objects  $M$  such that there exists a distinguished triangle

$$C \rightarrow M \rightarrow D \rightarrow \Sigma C$$

in  $\mathcal{T}$ , with  $C \in \mathcal{C}$  and  $D \in \mathcal{D}$ . Now for each  $n \geq 2$ , define inductively  $\text{thick}_{\mathcal{T}}^n(\mathcal{C})$  to be  $\text{thick}_{\mathcal{T}}^1(\text{thick}_{\mathcal{T}}^{n-1}(\mathcal{C}) * \text{thick}_{\mathcal{T}}^1(\mathcal{C}))$ , and denote  $\bigcup_{n=1}^{\infty} \text{thick}_{\mathcal{T}}^n(\mathcal{C})$  by  $\text{thick}_{\mathcal{T}}(\mathcal{C})$ . This is the smallest thick subcategory of  $\mathcal{T}$  containing  $\mathcal{C}$ .

Given two objects  $X$  and  $Y$  of  $\mathcal{T}$ , we define the *positive complexity* of the ordered pair  $(X, Y)$  as

$$\text{cx}_{\mathcal{T}}^+(X, Y) \stackrel{\text{def}}{=} \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} : \dim \text{Hom}_{\mathcal{T}}(X, \Sigma^n Y) \leq an^{t-1} \text{ for } n \gg 0\}.$$

Similarly, we define the *negative complexity* as

$$\text{cx}_{\mathcal{T}}^-(X, Y) \stackrel{\text{def}}{=} \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} : \dim \text{Hom}_{\mathcal{T}}(X, \Sigma^{-n} Y) \leq an^{t-1} \text{ for } n \gg 0\}.$$

Whenever we write  $\text{cx}_{\mathcal{T}}^*(X, Y)$  and make a statement, it is to be understood that the statement holds for both the positive and the negative complexity. By definition, the positive complexity is zero if and only if  $\text{Hom}_{\mathcal{T}}(X, \Sigma^n Y) = 0$  for large  $n$ , whereas the negative complexity is zero if and only if  $\text{Hom}_{\mathcal{T}}(X, \Sigma^n Y) = 0$  for small  $n$ . Moreover, given integers  $a, b \in \mathbb{Z}$ , there is an equality  $\text{cx}_{\mathcal{T}}^*(X, Y) = \text{cx}_{\mathcal{T}}^*(\Sigma^a X, \Sigma^b Y)$ . Note also that if  $\mathcal{T}$  is  $d$ -Calabi-Yau for some  $d$ , then  $\text{cx}_{\mathcal{T}}^+(X, Y) = \text{cx}_{\mathcal{T}}^-(Y, X)$ ; in particular the equality  $\text{cx}_{\mathcal{T}}^+(X, X) = \text{cx}_{\mathcal{T}}^-(X, X)$  holds in this case.

The following elementary lemma shows that complexity in some sense behaves nicely on thick subcategories.

**Lemma 2.1.** *Let  $X$  and  $Y$  be objects of  $\mathcal{T}$ . Then  $\text{cx}_{\mathcal{T}}^*(X', Y) \leq \text{cx}_{\mathcal{T}}^*(X, Y)$  for all objects  $X' \in \text{thick}_{\mathcal{T}}(X)$ , and  $\text{cx}_{\mathcal{T}}^*(X, Y') \leq \text{cx}_{\mathcal{T}}^*(X, Y)$  for all objects  $Y' \in \text{thick}_{\mathcal{T}}(Y)$ . In particular, the inequality  $\text{cx}_{\mathcal{T}}^*(X', X'') \leq \text{cx}_{\mathcal{T}}^*(X, X)$  holds for all objects  $X', X'' \in \text{thick}_{\mathcal{T}}(X)$ .*

*Proof.* We prove only the first inequality, by induction on the number  $n$  such that  $X'$  belongs to  $\text{thick}_{\mathcal{T}}^n(X)$ . If  $\text{cx}_{\mathcal{T}}^*(X, Y) = \infty$ , then the inequality obviously holds. Hence we may assume that  $\text{cx}_{\mathcal{T}}^*(X, Y)$  is finite, say  $\text{cx}_{\mathcal{T}}^*(X, Y) = c$ . If  $n = 1$ , then  $X'$  is a direct summand of finite direct sums of shifts of  $X$ , hence the inequality holds in this case. Next, suppose  $n > 1$  and that  $X'$  belongs to  $\text{thick}_{\mathcal{T}}^{n-1}(X) * \text{thick}_{\mathcal{T}}^1(X)$ . Then there exists a triangle

$$X_1 \rightarrow X' \rightarrow X_2 \rightarrow \Sigma X_1$$

in which  $X_1 \in \text{thick}_{\mathcal{T}}^{n-1}(X)$  and  $X_2 \in \text{thick}_{\mathcal{T}}^1(X)$ . This triangle induces an exact sequence

$$\text{Hom}_{\mathcal{T}}(X_2, \Sigma^n Y) \rightarrow \text{Hom}_{\mathcal{T}}(X', \Sigma^n Y) \rightarrow \text{Hom}_{\mathcal{T}}(X_1, \Sigma^n Y)$$

of vector spaces for every  $n \in \mathbb{Z}$ . By induction, both  $\text{cx}_{\mathcal{T}}^*(X_1, Y)$  and  $\text{cx}_{\mathcal{T}}^*(X_2, Y)$  are at most  $c$ . Therefore, there exist real numbers  $a_1$  and  $a_2$  such that

$$\begin{aligned} \dim \text{Hom}_{\mathcal{T}}(X_1, \Sigma^n Y) &\leq a_1 |n|^{c-1} \\ \dim \text{Hom}_{\mathcal{T}}(X_2, \Sigma^n Y) &\leq a_2 |n|^{c-1} \end{aligned}$$

for  $|n| \gg 0$ . This gives

$$\begin{aligned} \dim \text{Hom}_{\mathcal{T}}(X', \Sigma^n Y) &\leq \dim \text{Hom}_{\mathcal{T}}(X_1, \Sigma^n Y) + \dim \text{Hom}_{\mathcal{T}}(X_2, \Sigma^n Y) \\ &\leq (a_1 + a_2) |n|^{c-1} \end{aligned}$$

for  $|n| \gg 0$ , showing that  $\text{cx}_{\mathcal{T}}^*(X', Y)$  is at most  $c$ . The result now follows from the definition of  $\text{thick}_{\mathcal{T}}^n(X)$ .  $\square$

The aim of this paper is to determine the maximal complexity occurring in certain triangulated categories, via maximal orthogonal subcategories. Recall that a subcategory  $\mathcal{C}$  of  $\mathcal{T}$  is *contravariantly finite* in  $\mathcal{T}$  if every object in  $\mathcal{T}$  admits a right  $\mathcal{C}$ -approximation. Thus, for every object  $X \in \mathcal{T}$  there exists a morphism  $C \rightarrow X$  with  $C \in \mathcal{C}$ , such that every morphism  $C' \rightarrow X$  with  $C' \in \mathcal{C}$  factors through  $C$ . The following lemma provides a criterion under which a contravariantly finite subcategory  $\mathcal{C}$  generates  $\mathcal{T}$  (see [Iya] and [KeR, Section 5.5]). Consequently, we see from Lemma 2.1 that the maximal complexity of  $\mathcal{T}$  equals that of  $\mathcal{C}$ .

**Lemma 2.2.** *Let  $\mathcal{C}$  be a contravariantly finite subcategory of  $\mathcal{T}$ , and suppose there exists an integer  $n \geq 1$  such that the following are equivalent for every object  $X \in \mathcal{T}$ :*

- (1)  $X \in \mathcal{C}$ ,
- (2)  $\text{Hom}_{\mathcal{T}}(C, \Sigma^i X) = 0$  for  $1 \leq i \leq n$  and all  $C \in \mathcal{C}$ .

*Then  $\text{thick}_{\mathcal{T}}^{n+1}(\mathcal{C}) = \mathcal{T}$ .*

*Proof.* Choose  $n$  triangles

$$\begin{array}{ccccccc} K_1 & \longrightarrow & C_0 & \xrightarrow{f_0} & X & \longrightarrow & \Sigma K_1 \\ & & \vdots & & \vdots & & \vdots \\ & & \vdots & & \vdots & & \vdots \\ K_{n-1} & \longrightarrow & C_{n-2} & \xrightarrow{f_{n-2}} & K_{n-2} & \longrightarrow & \Sigma K_{n-1} \\ & & \vdots & & \vdots & & \vdots \\ K_n & \longrightarrow & C_{n-1} & \xrightarrow{f_{n-1}} & K_{n-1} & \longrightarrow & \Sigma K_n \end{array}$$

in which the morphisms  $f_i$  are right  $\mathcal{C}$ -approximations. Let  $C$  be any object in  $\mathcal{C}$ . The triangles induce exact sequences

$$\cdots \rightarrow \text{Hom}_{\mathcal{T}}(C, \Sigma^j C_i) \xrightarrow{(\Sigma^j f_i)^*} \text{Hom}_{\mathcal{T}}(C, \Sigma^j K_i) \rightarrow \text{Hom}_{\mathcal{T}}(C, \Sigma^{j+1} K_{i+1}) \rightarrow \cdots$$

for  $0 \leq i \leq n-1$  (where we have denoted  $X$  by  $K_0$ ). An induction argument shows that  $\text{Hom}_{\mathcal{T}}(C, \Sigma^i K_n)$  vanishes for  $1 \leq i \leq n$ , hence  $K_n$  belongs to  $\mathcal{C}$ . Then another induction argument shows that  $X$  belongs to  $\text{thick}_{\mathcal{T}}^{n+1}(\mathcal{C})$ .  $\square$

**Corollary 2.3.** *Given the assumptions from the previous lemma, the equality*

$$\sup\{\text{cx}_{\mathcal{T}}^*(X, Y) \mid X, Y \in \mathcal{T}\} = \sup\{\text{cx}_{\mathcal{T}}^*(C, C') \mid C, C' \in \mathcal{C}\}$$

*holds.*

In the next section, we apply the above results to Calabi-Yau triangulated categories admitting subcategories with the properties displayed in the assumption of Lemma 2.2. Recall therefore that, if  $\mathcal{T}$  is  $d$ -Calabi-Yau for some  $d \geq 2$ , then a *cluster tilting subcategory* of  $\mathcal{T}$  is a contravariantly finite subcategory  $\mathcal{C}$  such that the following are equivalent for any object  $X \in \mathcal{T}$ :

- (1)  $X \in \mathcal{C}$ ,
- (2)  $\text{Hom}_{\mathcal{T}}(C, \Sigma^i X) = 0$  for  $1 \leq i \leq d-1$  and all  $C \in \mathcal{C}$ .

Since  $\mathcal{T}$  is  $d$ -Calabi-Yau, property (2) is equivalent to

- (3)  $\text{Hom}_{\mathcal{T}}(X, \Sigma^i C) = 0$  for  $1 \leq i \leq d-1$  and all  $C \in \mathcal{C}$ .

An object  $T \in \mathcal{T}$  is a *cluster tilting object* of  $\mathcal{T}$  if  $\text{add } T$  is a cluster tilting subcategory.

Note that it follows directly from Corollary 2.3 that if  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are cluster tilting subcategories of  $\mathcal{T}$ , then

$$\begin{aligned} \sup\{\text{cx}_{\mathcal{T}}^*(X, Y) \mid X, Y \in \mathcal{T}\} &= \sup\{\text{cx}_{\mathcal{T}}^*(C_1, C'_1) \mid C_1, C'_1 \in \mathcal{C}_1\} \\ &= \sup\{\text{cx}_{\mathcal{T}}^*(C_2, C'_2) \mid C_2, C'_2 \in \mathcal{C}_2\}. \end{aligned}$$

Therefore, in order to determine the maximal complexity of a Calabi-Yau triangulated category, any cluster tilting subcategory will do.

### 3. CLUSTER CATEGORIES

Cluster categories associated to finite dimensional hereditary algebras were introduced in [BMRRT] (and for hereditary algebras of Dynkin type  $A_n$  in [CCS]). Let  $k$  be an algebraically closed field and  $H$  a basic finite dimensional hereditary  $k$ -algebra. Let  $\mathcal{D}^b(H)$  be the bounded derived category of finitely generated left  $H$ -modules; this category is triangulated, its suspension functor  $\Sigma$  is just the shift of a complex. Finally, denote by  $\tau$  the Auslander-Reiten translate in  $\mathcal{D}^b(H)$ ; this functor is induced by the usual Auslander-Reiten translate  $D \text{Tr}$  on the non-projective indecomposable  $H$ -modules. It was shown in [Kel] that the orbit category  $\mathcal{D}^b(H)/\tau^{-1}\Sigma$  is triangulated, with suspension functor induced by  $\Sigma$ . This is the *cluster category*  $\mathcal{C}_H$  associated to  $H$ . Its objects coincide with the objects in  $\mathcal{D}^b(H)$ , and the functors  $\Sigma$  and  $\tau$  are equal. Given objects  $X$  and  $Y$  of  $\mathcal{C}_H$ , the morphism space  $\text{Hom}_{\mathcal{C}_H}(X, Y)$  is given by

$$\text{Hom}_{\mathcal{C}_H}(X, Y) \stackrel{\text{def}}{=} \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(H)}(\tau^{-i} \Sigma^i X, Y),$$

which is finite dimensional since  $H$  is hereditary. We shall denote the suspension functor of  $\mathcal{C}_H$  by  $\Sigma$  as well. Moreover, given an  $H$ -module  $M$ , we shall also denote its image in  $\mathcal{C}_H$  by  $M$ . By [BMRRT, Proposition 1.7(b)] the cluster category  $\mathcal{C}_H$  is 2-Calabi-Yau, that is, there is an isomorphism

$$D \text{Hom}_{\mathcal{C}_H}(X, \Sigma Y) \simeq \text{Hom}_{\mathcal{C}_H}(Y, \Sigma X)$$

of vector spaces for all objects  $X$  and  $Y$  in  $\mathcal{C}_H$ .

In order to prove the main result, we need a result on the rate of growth of the sequence  $\{\dim \tau^{-n} H\}_{n=1}^{\infty}$  for a hereditary algebra  $H$ . Recall first that the representation type of a finite dimensional algebra (over an algebraically closed field) is either *finite*, *tame* or *wild*. An algebra is of finite representation type if there are only finitely many non-isomorphic indecomposable modules. Furthermore, an algebra is of tame representation type if there exist infinitely many non-isomorphic indecomposable modules, but they all belong to one-parameter families, and in each dimension there are finitely many such families. Finally, an algebra is of wild representation type if it is not of finite or tame type. In the latter case, the

representation theory of the algebra is at least as complicated as the classification of finite dimensional vector spaces together with two non-commuting endomorphisms.

**Proposition 3.1.** *Let  $H$  be a finite dimensional hereditary algebra of infinite representation type over an algebraically closed field. Define*

$$\gamma(\tau_H^{-1}) := \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} : \dim \tau^{-n}H \leq an^{t-1} \text{ for } n \gg 0\}.$$

*Then the following hold:*

- (1)  $\gamma(\tau_H^{-1}) = 2$  if (and only if)  $H$  is tame.
- (2)  $\gamma(\tau_H^{-1}) = \infty$  if (and only if)  $H$  is wild.

*Proof.* (1) Suppose  $H$  is tame. We may assume that  $H$  is the path algebra of one of the Euclidean quivers  $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ . We prove this case by using the theory of quadratic forms, and refer to [Ri1, Chapter 1] for unexplained notation and terminology.

Let  $r \in \mathbb{Z}^n$  be a minimal positive radical vector. It is shown in [DIR] (see the formulas in the middle of page 11 there, and note that the coxeter transformation coincides with the induced action of  $\tau$  on the Grothendieck groups) that there is a natural number  $m$  such that for any indecomposable  $H$ -module  $X$

$$\exists \partial \in \mathbb{Z} \forall n \in \mathbb{Z} \quad [\tau^{mn}X] = [X] + n\partial r,$$

provided  $\tau^{mn}X \neq 0$ .

This shows that the sequence  $\{\dim \tau^{-n}H\}_{n=1}^{\infty}$  grows linearly.

(2) Suppose  $H$  is wild, and let  $P$  be an indecomposable  $H$ -module. By [Tak, Theorem 2.4], there exists an integer  $m$  such that

$$\lim_{n \rightarrow \infty} \frac{\dim \tau^{-n}P}{\rho^n n^{m-1}}$$

is nonzero, where  $\rho$  is the spectral radius of the Coxeter transformation of  $H$ . Now suppose that  $\gamma(\tau_H^{-1})$  is finite, so that there exist a  $t \geq 0$  and an  $a \in \mathbb{R}$  such that  $\dim \tau^{-n}P \leq an^{t-1}$  for large  $n$ . Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\dim \tau^{-n}P}{\rho^n n^{m-1}} &\leq \lim_{n \rightarrow \infty} \frac{an^{t-1}}{\rho^n n^{m-1}} \\ &= \lim_{n \rightarrow \infty} \frac{an^{t-m}}{\rho^n} \\ &= 0 \end{aligned}$$

since, by [Ri2, Theorem], the spectral radius  $\rho$  satisfies  $\rho > 1$ . This is a contradiction, hence  $\gamma(\tau_H^{-1}) = \infty$ .  $\square$

We now prove the main result. It shows that the maximal complexity in  $\mathcal{C}_H$  (positive and negative) is either one, two or infinite, depending on the representation type of  $H$ .

**Theorem 3.2.** *Let  $H$  be a basic finite dimensional hereditary algebra over an algebraically closed field, and let  $\mathcal{C}_H$  be the corresponding cluster category. Then*

$$\sup\{\text{cx}_{\mathcal{C}_H}^*(X, Y) \mid X, Y \in \mathcal{C}_H\} = \begin{cases} 1 & \text{if } H \text{ has finite type,} \\ 2 & \text{if } H \text{ has tame type,} \\ \infty & \text{if } H \text{ has wild type.} \end{cases}$$

*Proof.* Consider the subcategory  $\text{add } H$  of  $\mathcal{C}_H$ . It is contravariantly finite since it contains only finitely many non-isomorphic indecomposable objects. Moreover, by [BMRRT, Theorem 3.3(b)], the following are equivalent for any object  $X \in \mathcal{C}_H$ :

- (1)  $X \in \text{add } H$ ,
- (2)  $\text{Hom}_{\mathcal{C}_H}(H, \Sigma X) = 0$ .

Thus the object  $H$  is cluster tilting in  $\mathcal{C}_H$ , and so from Corollary 2.3 we see that

$$\sup\{\mathrm{cx}_{\mathcal{C}_H}^*(X, Y) \mid X, Y \in \mathcal{C}_H\} = \mathrm{cx}_{\mathcal{C}_H}^*(H, H).$$

Since  $\mathcal{C}_H$  is Calabi-Yau, the maximal positive complexity equals the maximal negative complexity. It therefore suffices to prove the result for negative complexity.

By definition, the negative complexity  $\mathrm{cx}_{\mathcal{C}_H}^-(H, H)$  equals the rate of growth of the dimensions of the vector spaces  $\mathrm{Hom}_{\mathcal{C}_H}(H, \Sigma^{-n}H)$  as  $n$  grows. Since  $\tau = \Sigma$  on  $\mathcal{C}_H$ , we obtain isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}_H}(H, \Sigma^{-n}H) &\simeq \mathrm{Hom}_{\mathcal{C}_H}(H, \tau^{-n}H) \\ &\simeq \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(H)}(\tau^{-i}\Sigma^i H, \tau^{-n}H) \\ &\simeq \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(H)}(H, \tau^{i-n}\Sigma^{-i}H) \end{aligned}$$

of vector spaces. If  $H$  is of finite representation type, then  $\dim \mathrm{Hom}_{\mathcal{C}_H}(H, \tau^{-n}H)$  is bounded as  $n \rightarrow \infty$ . Hence the result follows in this case.

Suppose  $H$  is of infinite representation type. Given integers  $i$  and  $j$ , the stalk complex  $\tau^i \Sigma^j H$  in  $\mathcal{D}^b(H)$  is nonzero in degree  $j - 1$  when  $i \geq 1$ , and in degree  $j$  when  $i \leq 0$ . Thus when  $n$  is positive, the only nonzero term in the above direct sum appears when  $i = 0$ , that is, the term  $\mathrm{Hom}_{\mathcal{D}^b(H)}(H, \tau^{-n}H)$ . Therefore, for such  $n$ , we obtain the isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}_H}(H, \Sigma^{-n}H) &\simeq \mathrm{Hom}_{\mathcal{D}^b(H)}(H, \tau^{-n}H) \\ &\simeq \mathrm{Hom}_H(H, \tau^{-n}H) \\ &\simeq \tau^{-n}H. \end{aligned}$$

Consequently, the negative complexity  $\mathrm{cx}_{\mathcal{C}_H}^-(H, H)$  equals the rate of growth of the sequence  $\{\dim \tau^{-n}H\}_{n=1}^\infty$ . The result now follows from Proposition 3.1.  $\square$

#### 4. CLUSTER TILTED ALGEBRAS

Let  $H$  be a basic finite dimensional hereditary algebra over some algebraically closed field, and  $T$  a cluster tilting object in the cluster category  $\mathcal{C}_H$ . The corresponding *cluster tilted algebra* is the endomorphism ring  $\mathrm{End}_{\mathcal{C}_H}(T)$ , itself a finite dimensional algebra. By [BMR], the functor

$$\mathrm{Hom}(T, -): \mathcal{C}_H/(\tau T) \rightarrow \mathrm{mod}(\mathrm{End}_{\mathcal{C}_H}(T))$$

is an equivalence, hence one might suspect from Theorem 3.2 that tame cluster tilted algebras have complexity two. However, this is *not* the case: we show in this section that their complexity is at most one.

In order to show this, we first recall some facts on Gorenstein algebras. Let  $\Gamma$  be such an algebra, and denote by  $\mathrm{CM}(\Gamma)$  the category of Cohen-Macaulay  $\Gamma$ -modules, i.e.

$$\mathrm{CM}(\Gamma) = \{M \in \mathrm{mod} \Gamma \mid \mathrm{Ext}_\Gamma^i(M, \Gamma) = 0 \text{ for all } i > 0\}.$$

It follows from general cotilting theory that this is a Frobenius exact category, in which the projective injective objects are the projective  $\Gamma$ -modules, and the injective envelopes are the left add  $\Gamma$ -approximations. Therefore the stable category  $\underline{\mathrm{CM}}(\Gamma)$ , which is obtained by factoring out all morphisms which factor through projective  $\Gamma$ -modules, is a triangulated category. Its shift functor is given by cokernels of left add  $\Gamma$ -approximations, the inverse shift is the usual syzygy functor. Now let  $\mathcal{D}^b(\Gamma)$  be the bounded derived category of finitely generated  $\Gamma$ -modules. Furthermore, let  $\mathcal{D}^{\mathrm{perf}}(\Gamma)$  be the thick subcategory of  $\mathcal{D}^b(\Gamma)$  consisting of objects isomorphic to bounded complexes of finitely generated projective  $\Gamma$ -modules. It follows from

work by Buchweitz, Happel and Rickard (cf. [Buc], [Hap], [Ric]) that  $\underline{\text{CM}}(\Gamma)$  and the quotient category  $\mathcal{D}^b(\Gamma)/\mathcal{D}^{\text{perf}}(\Gamma)$  are equivalent as triangulated categories.

The following lemma shows that if  $\text{CM}(\Gamma)$  is of finite type, i.e. contains only finitely many non-isomorphic indecomposable objects, then the maximal complexity occurring in  $\mathcal{D}^b(\Gamma)$  is either one or zero.

**Lemma 4.1.** *Let  $\Gamma$  be a finite dimensional Gorenstein algebra such that the category  $\text{CM}(\Gamma)$  of Cohen-Macaulay  $\Gamma$ -modules has finitely many non-isomorphic indecomposable objects. Then*

$$\sup\{\text{cx}_{\mathcal{D}^b(\Gamma)}^+(X, Y) \mid X, Y \in \mathcal{D}^b(\Gamma)\} = \begin{cases} 0 & \text{if } \Gamma \text{ has finite global dimension,} \\ 1 & \text{otherwise} \end{cases}$$

*Proof.* Let  $X$  and  $Y$  be complexes in  $\mathcal{D}^b(\Gamma)$ . As mentioned above, the categories  $\underline{\text{CM}}(\Gamma)$  and  $\mathcal{D}^b(\Gamma)/\mathcal{D}^{\text{perf}}(\Gamma)$  are equivalent, and so there is a dense functor  $\mathcal{D}^b(\Gamma) \rightarrow \underline{\text{CM}}(\Gamma)$ . If we denote by  $\bar{X}$  and  $\bar{Y}$  the images of  $X$  and  $Y$  in  $\underline{\text{CM}}(\Gamma)$ , then it follows from [Buc, Corollary 6.3.4] that  $\text{cx}_{\mathcal{D}^b(\Gamma)}^+(X, Y) = \text{cx}_{\underline{\text{CM}}(\Gamma)}^+(\bar{X}, \bar{Y})$ . Since  $\text{CM}(\Gamma)$  is of finite type, we see that  $\text{cx}_{\underline{\text{CM}}(\Gamma)}^+(\bar{X}, \bar{Y})$  is at most one.  $\square$

Recall from [KeR] that a cluster tilted algebra is Gorenstein of dimension one, that is, its injective dimension as a left and right module over itself is one. The following result shows that if such an algebra  $\Lambda$  is tame, then  $\text{CM}(\Lambda)$  is of finite type.

**Theorem 4.2.** *If  $\Lambda$  is a tame cluster tilted algebra, then the category  $\text{CM}(\Lambda)$  of Cohen-Macaulay  $\Lambda$ -modules has finitely many non-isomorphic indecomposable objects.*

*Proof.* We may assume  $\Lambda$  to be connected. By definition, there is a connected hereditary algebra  $H$  and a cluster tilting object  $T \in \mathcal{C}_H$  such that  $\Lambda = \text{End}_{\mathcal{C}_H}(T)$ . Moreover, by a theorem of Krause (cf. [Kra, Corollary 3.4]), the algebra  $H$  is also tame. Therefore, at least two of the indecomposable direct summands of  $T$  lie in the non-regular component (this is the component of  $\mathcal{C}_H$  that comes from the preprojective and preinjective components of  $\text{mod } H$ ). Let  $T_i$  be one such summand lying “as far to the right as possible”, that is, there is no path in this component from  $T_i$  to any other summands of  $T$ . We may assume that  $\tau^-T_i$  comes from a projective  $H$ -module. Now for any  $X \in \text{mod } H$  we have

$$\begin{aligned} X \in \text{CM}(\Lambda) &\iff \text{Ext}_{\Lambda}^1(X, \Lambda) = 0 \\ &\iff \underline{\text{Hom}}_{\Lambda}(\tau^- \Lambda, X) = 0 \\ &\implies \text{Hom}_{\mathcal{C}_H}(\tau^-T_i, X)/(\text{maps factoring through } \tau T) = 0 \\ &\iff \text{Hom}_H(\tau^-T_i, X)/(\text{maps factoring through } \tau T) = 0, \end{aligned}$$

where the implication in the third line holds because of the following: there is an isomorphism

$$\underline{\text{Hom}}_{\Lambda}(\tau^- \Lambda, X) \simeq \text{Hom}_{\mathcal{C}_H}(\tau^- \Lambda, X)/(\text{maps factoring through } T \text{ or } \tau T),$$

since the functor

$$\text{Hom}(T, -): \mathcal{C}_H/(\tau T) \rightarrow \text{mod } \Lambda$$

is an equivalence identifying  $\text{add } T$  with projective  $\Lambda$ -modules, and

$$\begin{aligned} &\text{Hom}_{\mathcal{C}_H}(\tau^- \Lambda, X)/(\text{maps factoring through } T \text{ or } \tau T) \\ &= \text{Hom}_{\mathcal{C}_H}(\tau^- \Lambda, X)/(\text{maps factoring through } \tau T), \end{aligned}$$

since  $\text{Hom}_{\mathcal{C}_H}(\tau^-T, T) = 0$ .

For  $X$  preprojective, the space

$$\mathrm{Hom}_H(\tau^-T_i, X)/(\text{maps factoring through } \tau T)$$

is just  $\mathrm{Hom}_H(\tau^-T_i, X)$ , and this vanishes only for finitely many  $X$ .

We denote by  $R$  the direct sum of the regular summands of  $T$ . For almost all regular and almost all preinjective  $H$ -modules  $X$  we have

$$\begin{aligned} & \mathrm{Hom}_H(\tau^-T_i, X)/(\text{maps factoring through } \tau T) \\ &= \mathrm{Hom}_H(\tau^-T_i, X)/(\text{maps factoring through } \tau R). \end{aligned}$$

If  $X$  lies in a homogeneous tube then the denominator vanishes, and hence the space is non-zero.

For any indecomposable regular  $X$  the dimension  $\dim \mathrm{Hom}(\tau R, X)$  is at most the number of indecomposable summands of  $R$ . For all preinjective  $X$  we have  $\dim \mathrm{Hom}(\tau R, X) = \dim \mathrm{Hom}(\tau^{1-\ell}R, X) = \dim \mathrm{Hom}(\tau R, \tau^\ell X)$ , for some  $\ell$  only depending on  $R$ , and hence there is also a common bound for all the  $\dim \mathrm{Hom}(\tau R, X)$  with  $X$  preinjective. Thus for any indecomposable regular or preinjective  $X$  we have

$$\begin{aligned} & \dim \mathrm{Hom}_H(\tau^-T_i, X)/(\text{maps factoring through } \tau R) \\ & \geq \dim \mathrm{Hom}_H(\tau^-T_i, X) - \dim \mathrm{Hom}_H(\tau^-T_i, \tau R) \cdot \underbrace{\dim \mathrm{Hom}_H(\tau R, X)}_{\text{bounded}}, \end{aligned}$$

and hence the space

$$\mathrm{Hom}_H(\tau^-T_i, X)/(\text{maps factoring through } \tau R)$$

can only vanish if  $\dim \mathrm{Hom}_H(\tau^-T_i, X)$  is sufficiently small. However, this only happens for finitely many modules which are preinjective or lie in non-homogeneous regular tubes.  $\square$

Combining Lemma 4.1 and Theorem 4.2, we see that cluster tilted algebras of finite or tame type are of complexity at most one.

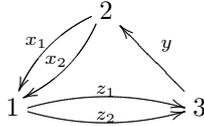
**Theorem 4.3.** *If  $\Lambda$  is a cluster tilted algebra of finite or tame representation type, then*

$$\mathrm{cx}_{\mathcal{D}^b(\Lambda)}^+(X, Y) = \begin{cases} 0 & \text{if } \Lambda \text{ is hereditary,} \\ 1 & \text{otherwise} \end{cases}$$

*Proof.* By [KeR, Proposition 2.1], the algebra  $\Lambda$  is Gorenstein of dimension one, that is, its injective dimension as a left/right module over itself is one. In particular, if  $\Lambda$  has finite global dimension, then it is hereditary. By Theorem 4.2, there are only finitely many isomorphism classes of indecomposable Cohen-Macaulay  $\Lambda$ -modules, hence the result follows from Lemma 4.1.  $\square$

Next, we look at three examples of wild cluster-tilted algebras. These examples show that Theorem 4.3, and hence also Theorem 4.2, does not generalize to wild cluster tilted algebras. For background on mutations of quivers with potentials, see [BIRS] and [DWZ].

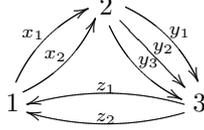
- Examples.** (1) Let  $\Lambda_0 = k[1 \rightrightarrows 2 \longrightarrow 3]$  be the path algebra of the wild quiver  $1 \rightrightarrows 2 \longrightarrow 3$ . This is a hereditary algebra, and therefore  $\mathrm{cx}_{\mathcal{D}^b(\Lambda_0)}^*(X, Y) = 0$  for any  $X, Y \in \mathcal{D}^b(\Lambda_0)$ .
- (2) Let  $\Lambda_1$  be the cluster tilted algebra obtained from  $\Lambda_0$  by mutation at the vertex 2. Then  $\Lambda_1$  is the path algebra of the quiver



subject to the relations given by the cyclic derivatives of the potential  $x_1z_1y + x_2z_2y$ , namely the relations  $\{x_1z_1 + x_2z_2, yx_1, yx_2, z_1y, z_2y\}$ . Then

$$\sup\{\text{cx}_{\mathcal{D}^b(\Lambda_1)}^+(X, Y) \mid X, Y \in \mathcal{D}^b(\Lambda_1)\} = 1.$$

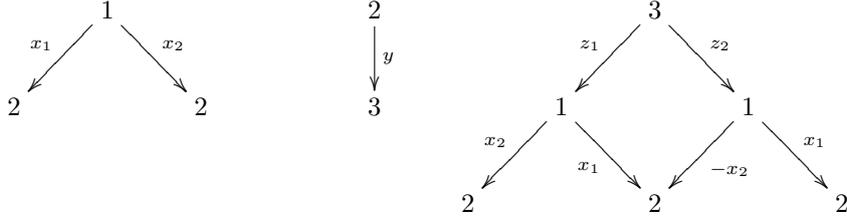
- (3) Let  $\Lambda_2$  be the cluster tilted algebra obtained from  $\Lambda_1$  by mutation at the vertex 1. Then  $\Lambda_2$  is the path algebra of the quiver



subject to the relations given by the cyclic derivatives of the potential  $x_1y_1z_2 + x_2y_2z_1 + x_1y_3z_1 - x_2y_3z_2$ , namely the relations  $\{x_1y_3 + x_2y_2, x_1y_1 - x_2y_3, y_1z_2 + y_3z_1, y_2z_1 - y_3z_2, z_1x_1, z_2x_2, z_1x_1 - z_2x_2\}$ . Then

$$\sup\{\text{cx}_{\mathcal{D}^b(\Lambda_2)}^+(X, Y) \mid X, Y \in \mathcal{D}^b(\Lambda_2)\} = 2.$$

*Proof.* The claims for  $\Lambda_0$  are clear. For  $\Lambda_1$ , the indecomposable projectives have the following composition structures:



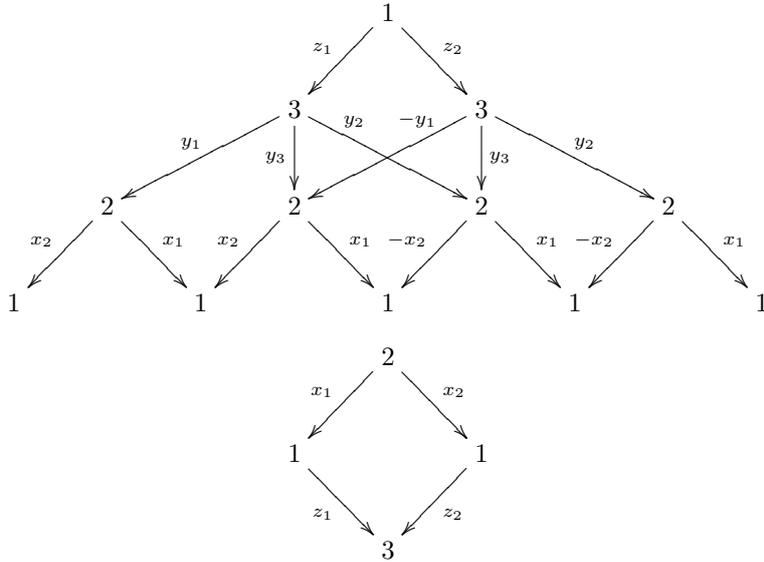
We see that the three simple modules satisfy

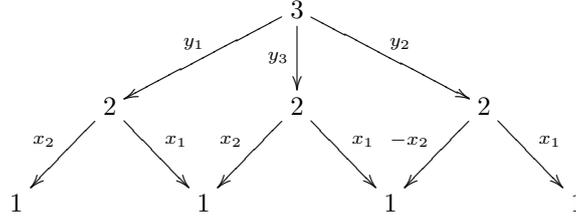
$$\Omega_{\Lambda_1}^1(S_1) = S_2^2 \quad \Omega_{\Lambda_1}^1(S_2) = S_3 \quad \Omega_{\Lambda_1}^2(S_3) = S_2.$$

Consequently every simple  $\Lambda_1$ -module is eventually  $\Omega$ -periodic, and therefore

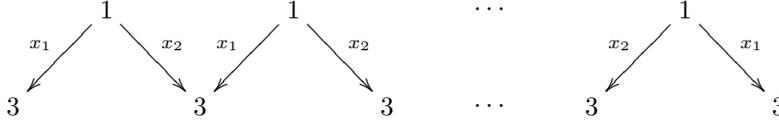
$$\sup\{\text{cx}_{\mathcal{D}^b(\Lambda_1)}^+(X, Y) \mid X, Y \in \mathcal{D}^b(\Lambda_1)\} = 1.$$

For  $\Lambda_2$ , the indecomposable projective modules have the following composition structures:





If we denote by  $M_n$  the module with composition structure



(with  $n$  composition factors  $S_1$  and  $n + 1$  composition factors  $S_3$ ), then one can show by direct calculation that

$$\Omega_{\Lambda_2}^2(M_n) = M_{n+2}.$$

Now note that

$$S_3 = M_0 \quad \text{and} \quad \Omega_{\Lambda_2}^3(S_1) = M_1,$$

so the rate of growth of the dimensions of the syzygies of these simple modules is linear. Finally, note that  $\Omega_{\Lambda_2}^1(S_2)$  is an extension of  $S_1 \oplus S_1$  and  $S_3$ , hence the rate of growth of the dimensions of its syzygies is at most linear. This shows that

$$\sup\{\text{cx}_{\mathcal{D}^b(\Lambda_2)}^+(X, Y) \mid X, Y \in \mathcal{D}^b(\Lambda_2)\} = 2. \quad \square$$

*Remark.* As mentioned, the above example not only shows that Theorem 4.3 does not generalize to wild cluster tilted algebras. It also shows that the same is true for Theorem 4.2, that is, there exist wild cluster tilted algebras with infinitely many non-isomorphic indecomposable Cohen-Macaulay modules. Namely, by Lemma 4.1, the algebra in example (3) has this property.

We conclude this paper with the following more general questions on the complexity of wild cluster tilted algebras:

**Questions.** (1) What numbers occur as

$$\sup\{\text{cx}_{\mathcal{D}^b(\Lambda)}^+(X, Y) \mid X, Y \in \mathcal{D}^b(\Lambda)\}$$

for  $\Lambda$  a cluster tilted algebra of wild type?

(2) Given a wild hereditary algebra  $H$ , do all the numbers in (1) occur as

$$\sup\{\text{cx}_{\mathcal{D}^b(\Lambda)}^+(X, Y) \mid X, Y \in \mathcal{D}^b(\Lambda)\}$$

for some cluster tilted algebra  $\Lambda$  of type  $H$ ?

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