LINEAR WAVE THEORY

PART A

Regular waves

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PART A - REGULAR WAVES

1 INTRODUCTION

These notes give an elementary introduction to linear wave theory. Linear wave theory is the core theory of ocean surface waves used in ocean and coastal engineering and naval architecture. The treatment is kept at a level that should be accessible to first year undergraduate students and does not require more than elementary calculus, probability and statistics.

Part A will cover the linear theory of regular gravity waves on the surface of a fluid, in our case, the surface of water. For gravity waves, gravitation constitutes the restoration force, that is the force that keep the waves going. This applies to waves with wavelengths larger than a few centimeters. For shorter surface waves, capillary forces come into action.

Chapter 2 covers basic wave motion and applies to all kind of waves. In the following chapter we briefly discuss the equations and boundary conditions which lead to water waves. Plane waves are treated in detail and simple superposition is also mentioned. We then proceed to three dimensional waves.

The notes are rather short in the sense that they discuss the equations rather than the applications.

Most of the material covered may be found in standard textbooks on the topic, see the references.

2 BASIC WAVE MOTION

The *sine (*or *cosine*) function defines what is called a regular wave. In order to specify a regular wave we need its amplitude, a, its *wavelength,* λ, its *period*, T, and in order to be fully specified. also its propagation direction and phase at a given location and time. All these concepts will be introduced below.

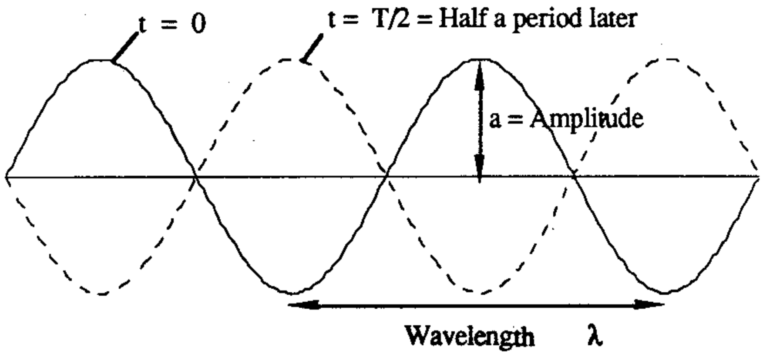


Fig. 1: The sine wave

Consider the function η of the two variables position, *x*, and time, *t*:

$$\eta(x,t) = a\sin(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x)$$

Convince yourself that this function has the following properties:

- For a fixed t_0, $\eta(x,t_0)$ is a sine function of *x*
- For a fixed x_0, $\eta(x_0,t)$ is a sine function of *t*
- $\eta(x,t) = \eta(x+\lambda,t) = \eta(x+n\lambda,t), n = \cdots -1,0,1,\cdots$,which shows that the function repeats itself each time *x* is increased with λ. This explains why $\lambda$is called *wavelength*.
- $\eta(x,t) = \eta(x,t+T) = \eta(x,t+nT), n = \cdots -1,0,1,\cdots$ which shows that the function repeats itself with *period T*.
- $\eta(x+x_0,t+t_0) = \eta(x,t)$ provided $x_0/\lambda = t/T$.

The quantity $2\pi/\lambda$ is called the *wavenumber* and is usually denoted by the letter *k*. Similarly, $2\pi/T$ is written ω (the Greek letter *omega*) and called the *angular frequency*. The unit for *k* is *rad/m* and for ω *rad/s*. Note that $f = 1/T$ is called frequency and measured in Hertz (*Hz*=s^{-1}).

The constant *a* in front of the sine is called the amplitude of the wave. Note that since $-1 \le \sin(\alpha) \le 1$, $|\eta(x,t)| \le a$. That is, $|\eta(x,t)|$ is never larger than the amplitude.

The basic feature of the wave as defined above is that the whole pattern *moves along the x-axis* as the time changes. Consider for simplicity the point *x* = 0, *t* = 0, where η is equal to 0.

If now t starts to increase, the points $x_0(t)$ defined by $x_0(t)/\lambda = t/T$ will have the property that $\eta(x_0(t),t) = 0$ for all t. The point where η is 0, x_0, thus moves with velocity λ/T along the x-axis. The last property stated above shows this in general.

Exercise 2.1: Consider the functions

$$\eta_1 = \sin(\omega t - kx)$$

and

$$\eta_2 = \sin(\omega t + kx)$$

where both k and ω are larger than 0. Show that the first represents a wave moving to the right and the second a wave moving to the left!

An additional angle α in the expression $\eta = a\sin(\omega t + kx + \alpha)$ is called a *phase term.* Show that a phase term does not affect the wavelength, the period or the propagation direction of the wave.

Exercise 2.2: Show that if we allow ω and k to be negative and arbitrary phase terms to be included, all functions

$$a\sin(kx - \omega t)$$
$$a\sin(kx + \omega t + \pi/4)$$
$$a\cos(\omega t - kx) + b\sin(\omega t - kx)$$

may be written $a'\sin(\omega' t - k' x + \alpha')$ for appropriate choices of a', ω', k' and α'.

The argument of the sine, *i.e.* $\omega t - kx + \alpha$, is in general called the *phase*. The phase is often denoted by the letter ϕ (Greek *phi*). Since $\sin(\phi + 2n\pi) = \sin(\phi), n = \cdots -1,0,1,\cdots$ phase differences of any multiple of 2π do not matter at all. The phase of the point (x_1, t_1) will be equal to the phase of the point (x_2, t_2) if

$$\omega t_1 + kx_1 = \omega t_2 + kx_2$$

that is,

$$\frac{x_2 - x_1}{t_2 - t_1} = \frac{\omega}{k} = \frac{\lambda}{T}$$

or,

$$x_2 = x_1 + \frac{\omega}{k}(t_2 - t_1)$$

The point x_2 on the x-axis which moves with velocity ω/k will therefore experience the same phase for all times. Therefore, the velocity $c = \omega/k = \lambda/T$ is called the *phase velocity* associated with the wave.

Let us see what happens if we add two general waves, one travelling to the right and one to the left. We first recall the trigonometric identity

$$\sin(A) + \sin(B) = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$$

Consider

$$\mu(x,t) = \sin(\omega t - kx) + \sin(\omega t + kx) = 2\sin(\omega t)\cos(kx)$$

This function is a product of a sine and a cosine; the first with *x* as argument and the second with *t*. Figure 2 shows a plot of the functions for different *t*'s.

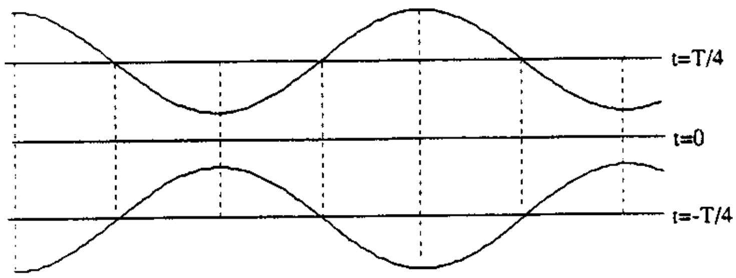


Fig. 2: The standing wave

Note that the function is *always* 0 for $kx = n\pi, n = \cdots -1,0,1,\cdots$. In this case we therefore do not have a travelling wave. However, since we still have a periodic behaviour both in *x* and *t*, it is customary to call this case a *standing wave*.

Exercise 2.3: The phase velocity for light waves is equal to 300 000 km/s. The periods for FM-band broadcasting range from (1/88)·10-6 s to (1/108)·10-6 s. What are the wavelengths of such waves?

Review questions:

1. What is a regular wave?
2. How is the period and the wavelength defined?
3. What is the amplitude of the wave?
4. How do we define the wavenumber and the angular frequency?
5. Why do we call the velocity of the wave the phase velocity, and how can we derive the phase velocity?
6. How can we make up a standing wave?

3. THE EQUATIONS FOR SURFACE WAVES

In this section we shall see how waves may occur on the surface of water in nature or in a manmade water tank. Unfortunately, it is rather difficult to derive these equations and we shall therefore not give a complete derivation, but assume some familiarity with fluid mechanics.

The water motion is governed by the laws of mechanics. These laws are all conservation laws which tell you that something is conserved. The most familiar one is mass conservation which says that *mass cannot be created or disappear.*

We shall first consider waves in a channel with parallel walls and horizontal bottom. We shall also assume that the waves travels along the channel and that there are no variation in the water motion across the channel.

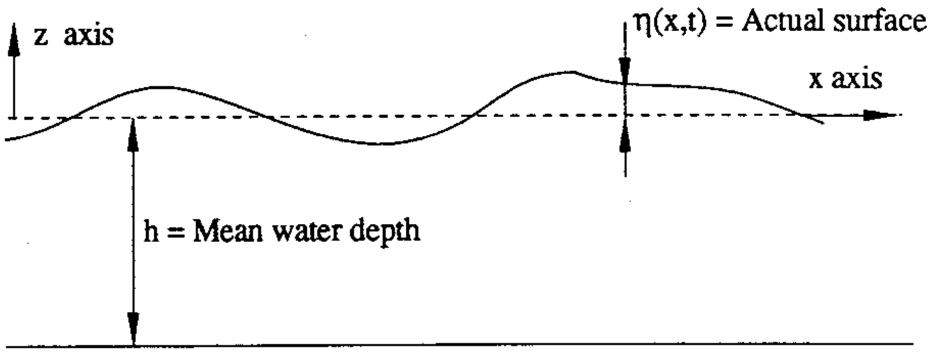


Fig. 3.1: Waves along a channel

Since everyone have seen waves on the surface of the water, waves which propagate along the channel, is is obvious that such waves *exist.* The problem is then to derive how the wavelength and the period of the waves may be expressed in terms of, say the water depth, the acceleration of gravity etc.

The waves on the surface set the rest of the water into motion, and at each point, (x,z), the fluid has a velocity

$$\mathbf{v}(x,z,t) = u(x,z,t)\mathbf{i} + w(x,z,t)\mathbf{k}$$

where z denotes the vertical coordinate measured upwards from the mean water level. We have now introduced unit vectors **i,** pointing along the x-axis and **k**, pointing along the z-axis (This **k** should not be confused with the wavenumber vector which we are going to use later).

Note that vectors are written with **bold** letters. *u* and *w* are thus the *x*- and the *z*-components of the velocity.

Water is hard to compress, and for our purpose, we will assume that this is impossible, that is, we consider water to be *incompressible*. In an incompressible fluid, the velocity $\mathbf{v} = (u, v, w)$ at each point will satisfy the equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

called the *equation of continuity*.

In our case the *y*-component of the velocity, *v*, is assumed to be zero, that is, we do not assume any variations across the channel.

If, in addition, the fluid is considered to be *irrotational,* the velocity may be expressed in terms of a so-called *velocity potential* Φ such that

$$u = \frac{\partial \Phi}{\partial x}$$
$$v = \frac{\partial \Phi}{\partial y}$$
$$w = \frac{\partial \Phi}{\partial z}$$

The concepts "irrotationality" and "velocity potential" are treated in courses in Fluid Mechanics, and also in about every textbook about water waves. If we introduce the velocity potential in the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

we obtain

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

(In the general case, we should also add a term $\partial^2 \Phi / \partial^2 y$.)

This equation is a very famous partial differential equation called the *Laplace equation.*

It turns out that Laplace equation is all we have to know about the water motion away from the boundaries and the surface.

The bottom of the channel is not permeable to the water, and therefore the vertical water velocity at the bottom *must be zero at all times*:

$$w(x,z=-h,t)=\frac{\partial\Phi}{\partial z}(x,z=-h,t)=0$$

This constitutes a relation which must hold at the boundary, and it is therefore called a *boundary condition.* For the moment we assume that the channel very long so that we do not have to bother about conditions and the far ends.

The conditions at the water surface are harder to obtain. It has been observed that the fluid near the surface remains near the surface during the wave motion as long as the motion is smooth. That is, unless the waves break. The first boundary condition at the free surface consists of stating this property in mathematical terms. Consider a part of the surface at two neighbouring times as indicated in Fig. 3.2.

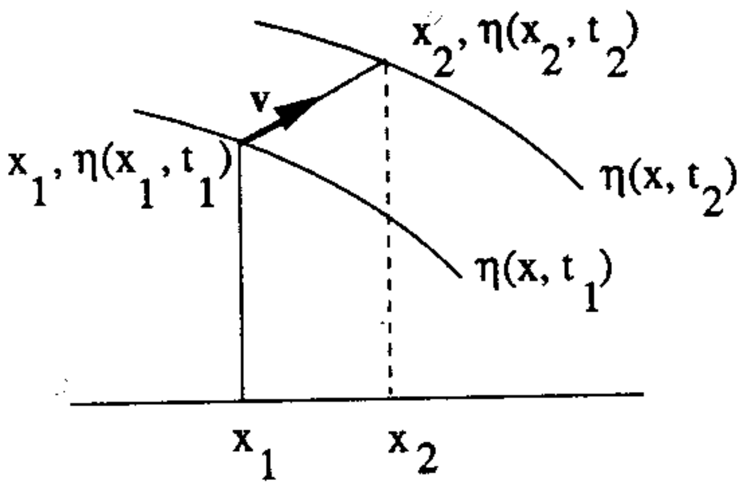


Figure 3.2: Motion of a fluid point on the free surface

The point at $(x_1,\eta(x_1,t_1))$ moves with velocity **v** to $(x_2,\eta(x_2,t_2))$ during the time interval $\Delta t=t_2-t_1$. Thus,

$$\eta(x_2,t_2)=\eta(x_1,t_1)+w\cdot(t_2-t_1),$$
$$x_2=x_1+u\cdot(t_2-t_1).$$

Let us also expand $\eta(x_1,t_1)$ in a Taylor series:

$$\eta(x_2,t_2)=\eta(x_1,t_2)+\frac{\partial\eta}{\partial x}(x_1,t_2)(x_2-x_1)+\cdots$$

If this is introduced in first equation above, we obtain

$$\eta(x_1,t_2)-\eta(x_1,t_1)\frac{\partial\eta}{\partial x}(x_1,t_2)(x_2-x_1)=w\cdot(t_2-t_1)+\cdots$$

Or, if we divide by $t_2-t_1$and let $t_2\to t_1$,

$$\frac{\partial\eta}{\partial t}+u\frac{\partial\eta}{\partial x}=w$$

This is the mathematical formulation of the physical condition that a fluid particle at the surface should remain at the surface at all times. It tells you something about the *motion* of the surface and is therefore called the *kinematic boundary condition.*

The other condition to be satisfied at the surface comes from the fact that the pressure p at the surface must be equal to the atmospheric pressure, which we assume is constant. This condition may be derived from *Bernoulli's Equation* which is also treated in basic courses on Fluid Mechanics (actually, a version of Benoulli's Equation is different from the more familiar form for steady flow). The equation states that for irrotational flow

$$\frac{p}{\rho}+\frac{\partial\Phi}{\partial t}+\frac{1}{2}(u^2+w^2)+gz=C(t)$$

The function $C(t)$ is not important and may be set to an arbitrary convenient constant. If we let $C(t)=p_{\text{atm}}/\rho$ Bernoulli's Equation gives for the free surface:

$$\frac{p}{\rho}+\frac{\partial\Phi}{\partial t}+\frac{1}{2}(u^2+w^2)+g\eta=0$$

This condition, dealing with the *force* on the surface, is usually called the *dynamic boundary condition.*

All together, we have now formulated the mathematical problem which must be solved in order to find the motion of the surface:

1) Within the fluid, Laplace's equation must be satisfied

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

2) At the closed bottom,

$$w(x,z=-h,t)=\frac{\partial\Phi}{\partial z}(x,z=-h,t)=0$$

3) The surface is always made up of the same fluid particles:

$$\frac{\partial\eta}{\partial t}+u\frac{\partial\eta}{\partial x}=w$$

(has to hold at the surface $z = \eta(x,t)$).

4) The pressure in the fluid at the free surface is equal to the atmospheric pressure:

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(u^2 + w^2) + g\eta = 0$$

(has to hold at the surface $z = \eta(x,t)$).

The mathematical problem stated in (1) to (4) is very difficult. No complete solution is known, although we know a lot about *special cases.*

The next section will treat the case where the magnitude of $\eta(x,t)$ is very small compared to the variations in the *x*-direction (For a wave we would say that the amplitude is small compared to the wavelength).

Review questions:

1. What is the equation of continuity?
2. What is the velocity potential?
3. Which equation must the velocity potential satisfy? What is it called?
4. Which condition must hold at the bottom of the channel?
5. What is a boundary condition?
6. What is the physical content of the kinematic boundary condition?
7. How can we state the kinematic boundary condition in mathematical terms?
8. What is the physical content of the dynamic boundary condition?

4. SMALL AMPLITUDE WAVES

The equations stated in the previous section are much too complicated to be solved in full generality. We are going to *linearize* the equation and the boundary condition, and in order to do so, we shall apply a useful technique called *dimensional analysis and scaling.* Actually, in most textbooks, the linearization is treated very briefly.

Assume that the typical *length scale* for variations in the *x*-direction is *L* (for ocean waves, *L* could typically be of order 100m which we write as *O*(100m)). Assume further that the time scale is *T*. (This could be a typical wave period which for ocean waves would be around 8s.) The amplitude is of the order *A,* that is, $\eta = O(A)$. The two *physical parameters* in our problem are

h = the mean water depth

and

$$g = \text{the acceleration of gravity.}$$

(It turns out that water density and viscosity, which did not occur in our equations anyway, are of virtually no significance.)

From the five quantities

$$L,\ T,\ A,\ h \text{ and } g$$

we may form three *dimensionless combinations*:

$$\pi_1 = \frac{A}{L}$$
$$\pi_2 = \frac{L}{h}$$
$$\pi_3 = \frac{gT^2}{L}$$

(There are other possible combinations but these turn out to be the most convenient).

The *small amplitude gravity waves* case is when $\pi_1 << 1$, that is, when $A << L$, and gravity is essential, that is, $\pi_3 = O(1)$.

The appropriate *water velocity scale* follows from the vertical motion of the surface. Thus the scale for |**v**| is *A/T*. Consider the kinematic condition,

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = w.$$

The first term is $O(A/T)$, the right hand side is of the same order, whereas the second term is

$$u\frac{\partial \eta}{\partial x} = O(\frac{A}{T}\frac{A}{L}) = O(\frac{A}{T}) \times O(\frac{A}{L})$$

Since *A/L* is supposed to be much smaller than 1, we may neglect the second term and use the *simplified kinematic condition*

$$\frac{\partial \eta}{\partial t} = w$$

in the present case.

We then consider the dynamic condition:

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(u^2 + w^2) + g\eta = 0$$

The magnitude of Φ is $O(LA/T)$ since

$$\frac{\partial \Phi}{\partial x} = O(\frac{A}{T})$$

The first term is thus of order

$$O(\frac{AL}{T^2})$$

The second term is of order

$$O\left((A/T)^2\right) = O(\frac{AL}{T^2}) \times O(\frac{A}{L})$$

Thus, the second is negligible compared to the first. Finally, the last term is

$$g\eta = O(\frac{LA}{T^2})$$

since we were considering the case where

$$\pi_3 = \frac{gT^2}{L} = O(1)$$

The last term is thus of the same order as the first term, and we obtain the simplified condition

$$\frac{\partial \Phi}{\partial t} + g\eta = 0$$

Unfortunately, the simplified problem is still too difficult for us since the velocities and the potential should be taken at the free surface,- which we do not know. However,

$$w(x,\eta,t) = w(x,0,t) + \frac{\partial w}{\partial z}(x, z = 0, t) \cdot \eta + O(\eta^2).$$

since

$$\left| \frac{\partial w}{\partial z}(x, z = 0, t) \cdot \eta \right| = O(\frac{A/T}{L} A) = O(\frac{A}{T}) \frac{A}{L}.$$

In accordance with the approximations we have already done, we may neglect the term $\partial w / \partial z \eta$, and simply use the *linearized kinematic condition*

$$\frac{\partial \eta(x,t)}{\partial t} = w(x,0,t).$$

The η–dependence is thus gone. A similar argument also linearizes the dynamic condition:

$$\frac{\partial \Phi(x,0,t)}{\partial t} = -g\eta(x,t).$$

We are now finally ready for attacking the *linearized equations*:

$$\frac{\partial^2 \Phi(x,z,t)}{\partial x^2} + \frac{\partial^2 \Phi(x,z,t)}{\partial z^2} = 0, \quad -h \le z \le \eta, \tag{1}$$

$$\frac{\partial \Phi}{\partial z}(x, z=-h, t) = 0, \tag{2}$$

$$\frac{\partial \eta}{\partial t}(x,t) = w(x,0,t), \tag{3}$$

$$\frac{\partial \Phi}{\partial t}(x,0,t) = -g\eta(x,t). \tag{4}$$

We are primarily looking for solutions that are regular waves so let us first see whether (1) may have such solutions. For a given *z*, we thus assume that Φ has the form

$$\Phi(x,z,t) = A(z)\sin(\omega t - kx + \phi_0)$$

where *k*, ω and ϕ_0 are unknowns and *A* is an amplitude which we assume is dependent of *z* (It is conceivable that *A* should depend on *z*). If this function is inserted into (1), we easily obtain

$$\left[-k^2 A(z) + A''(z)\right]\sin(\omega t - kx + \phi_0) = 0$$

Shall this be fulfilled for all *x* and *t*, we must have that the term in the bracket vanishes completely. This leads to an second order ordinary linear differential equation for *A* which has the general solution

$$A(z) = C_1 \cosh(kz + C_2)$$

Equation (2) requires

$$\frac{\partial \Phi}{\partial z}(x, z=-h, t) = \frac{dA}{dz}(z=-h)\sin(\omega t - kx + \phi_0) = 0,$$

which means that $A'(z=-h) = 0$. But $A'(z) = kC_1 \sinh(kz + C_2)$ which vanishes at $z = -h$ if $C_2 = kh$. Thus, a possible solution which satisfies both (1) and (2) is

$$\Phi(x,z,t) = C_1 \cosh(k(z+h))\sin(\omega t - kx + \phi_0).$$

It remains to be seen whether (3) and (4) can be satisfied. Equation (4) actually gives an expression for η since

$$\eta(x,t) = -\frac{1}{g}\frac{\partial}{\partial t}\Phi(x,z=0,t)$$
$$= -\frac{\omega}{g}C_1\cosh(k(z+h))\cos(\omega t - kx + \phi_0)$$

But (3) must also hold, that is,

$$\frac{\partial\eta}{\partial t}(x,t) = \frac{\omega^2}{g}C_1\cosh(kh)\sin(\omega t - kx + \phi_0)$$

must be equal to

$$w(x,z=0,t) = \frac{\partial\Phi}{\partial z}(x,z=0,t) = kC_1\sinh(kh)\sin(\omega t - kx + \phi_0)$$

Shall this last condition be true for all *x*- and *t*-s, we must have

$$\frac{\omega^2}{g}\cosh(kh) = k\sinh(kh)$$

or

$$\omega^2 = gk\tanh(hk)$$

This is an equation which says that ω and k can not be chosen at will. For a given $k \neq 0$, only the two frequencies, ω and -ω which satisfies the equation are allowed. The equation is called a *dispersion relation.* The dispersion relation tells us how the frequency and the wavenumber are connected.

If we now let $\phi_0 = -\pi/2$ and set

$$\frac{\omega}{g}C_1\cosh(kh) = a$$

we recover the familiar running regular wave for η:

$$\eta(x,t) = a\sin(\omega t - kx).$$

For the potential Φ we obtain:

$$\Phi(x,z,t) = C_1\cosh(k(z+h))\sin(\omega t - kx - \pi/2)$$
$$= \left[-\frac{ag}{\omega}\frac{1}{\cosh(kh)}\right]\cosh(k(z+h))\left(-\cos(\omega t - kx)\right)$$
$$= \frac{ag}{\omega}\frac{\cosh(k(z+h))}{\cosh(kh)}\cos(\omega t - kx)$$

The equations for η, Φ and the dispersion relation represent the core of linear wave theory.

Review questions A4:

1. Which two physical parameters are of importance for channel waves?
2. What are the three convenient dimensionless combinations which can be formed from the length, time and amplitude scales and the physical parameters? What are the sizes of these dimensionless combinations for small amplitude waves?
3. Find the order of magnitude of the following quantities: $\eta, w, \partial\eta/\partial t, \Phi, \partial\Phi/\partial t$, and, $u^2 + w^2$?
4. Derive the *simplified (linearized)* kinematic and dynamic boundary conditions for small amplitude waves.
5. Show that the surface conditions may be simplified further such that w and $\partial\Phi/\partial t$ are evaluated at $z = 0$.
6. Recall the basic steps in the derivation of the regular wave solution.
7. What is the dispersion relation?

5 THE DISPERSION RELATION

The dispersion relation says that waves with a given frequency must have a certain wavelength. For the wave $\eta(x,t) = a\sin(\omega t - kx)$ the wavenumber k and ω must be connected by the dispersion relation $\omega^2 = gk\tanh(hk)$. Note that for a given k , there are two possibilities for ω, namely $\omega = +(gk\tanh(hk))^{1/2}$ and $\omega = -(gk\tanh(hk))^{1/2}$. This corresponds to waves going to the right and to the left, respectively. The mathematical function $y = \tanh(x)$ is shown in Fig. 5.1.

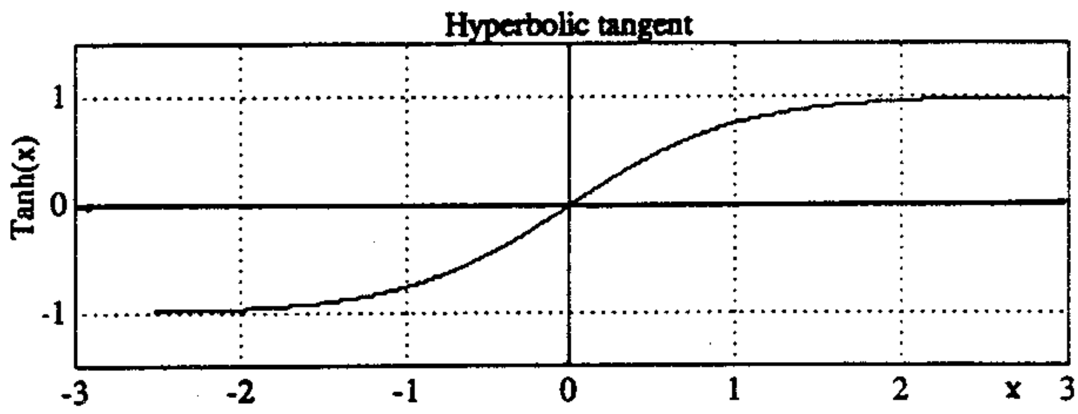


Fig. 5.1: The hyperbolic tangent, $y = \tanh(x)$.

For small values of the argument

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{(1 + x + O(x^2)) - (1 - x + O(x^2))}{(1 + O(x)) + (1 + O(x))} = x + O(x^2)$$

Moreover,

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \xrightarrow[x\to\infty]{} 1.$$

We recall that h is the water depth, and $k = 2\pi/\lambda$ where λ is the wavelength. Thus, $kh = 2\pi h/\lambda$. If kh is small, then $h << \lambda$, that is, the water depth is much smaller than the wavelength. This corresponds to *shallow water*. Conversely, if kh is large, this corresponds to deep water. Let us consider the dispersion relation in these particular cases.

Shallow water:

Now $kh << 1$and $\tanh(kh)$ may be replaced by kh. Thus, $\omega^2 = gk \cdot kh$, or

$$\omega = \pm(gh)^{1/2} k.$$

Deep water:

In this case, we set $\tanh(kh) = 1$ and

$$\omega = \pm\sqrt{gk}$$

Note there is a wide range of water depths which are *neither shallow nor deep* for a given wavelength. The *rules of thumb* are:

- Use the *deep water* expression when $h > \lambda/2$.
- Use the *shallow water* expression when $h < \lambda/20$

Exercise 5.1: When tanh(x) > 0.99, we may replace tanh(x) by 1 for practical calculations. Find, by means of your calculator, a value x_0 such that tanh(x) > 0.99 when $x_0 < x$. Verify that we may use $\omega^2 = gk$ when $h > \lambda/2$. Find the maximum relative error in ω if we use the shallow water expression when $h < \lambda/20$.

Exercise 5.2: Determine the wavelength of a wave with period 10s when the water depth is
a) 2000m , b) 1m. (Answ. (a) 156m, (b) 31.3m)

Exercise 5.3 Show that in deep water, the wavelength, λ, in metres of a wave with period T seconds is $1.56T^2$.

The *phase velocity*, c_p , of a regular wave was defined as

$$c_p = \frac{\omega}{k} = \frac{L}{T}$$

We recall that this was, *e.g.* the speed of the top (crest) of the wave as it moves along. From the dispersion relation we obtain the following expression for the phase velocity

$$c_p = \frac{g}{\omega}\tanh(kh).$$

Let us see what this amount to in deep and shallow water. In shallow water, we obtain the somewhat surprising answer that the phase velocity is independent of both ω and k:

$$c_p = \frac{\omega}{k} \approx \frac{\sqrt{gh}\cdot k}{k} = \sqrt{gh}$$

However, the velocity is now dependent of the depth, *h. Waves* for which the phase velocity is constant are called non-dispersive (Light waves in vacuum and regular sound waves in air are also non-dispersive).

For deep water we obtain

$$c_p = \frac{\omega}{k} \approx \frac{\omega}{\omega^2 / g} = \frac{g}{\omega} = \frac{\sqrt{gk}}{k} = \sqrt{\frac{g}{k}}\,.$$

In deep water, the speed increases with increasing wave period and wavelength. The graph below is copied from the book of K.F. Bowden. Note that as long $h < \lambda / 10$, the waves move with constant speed.

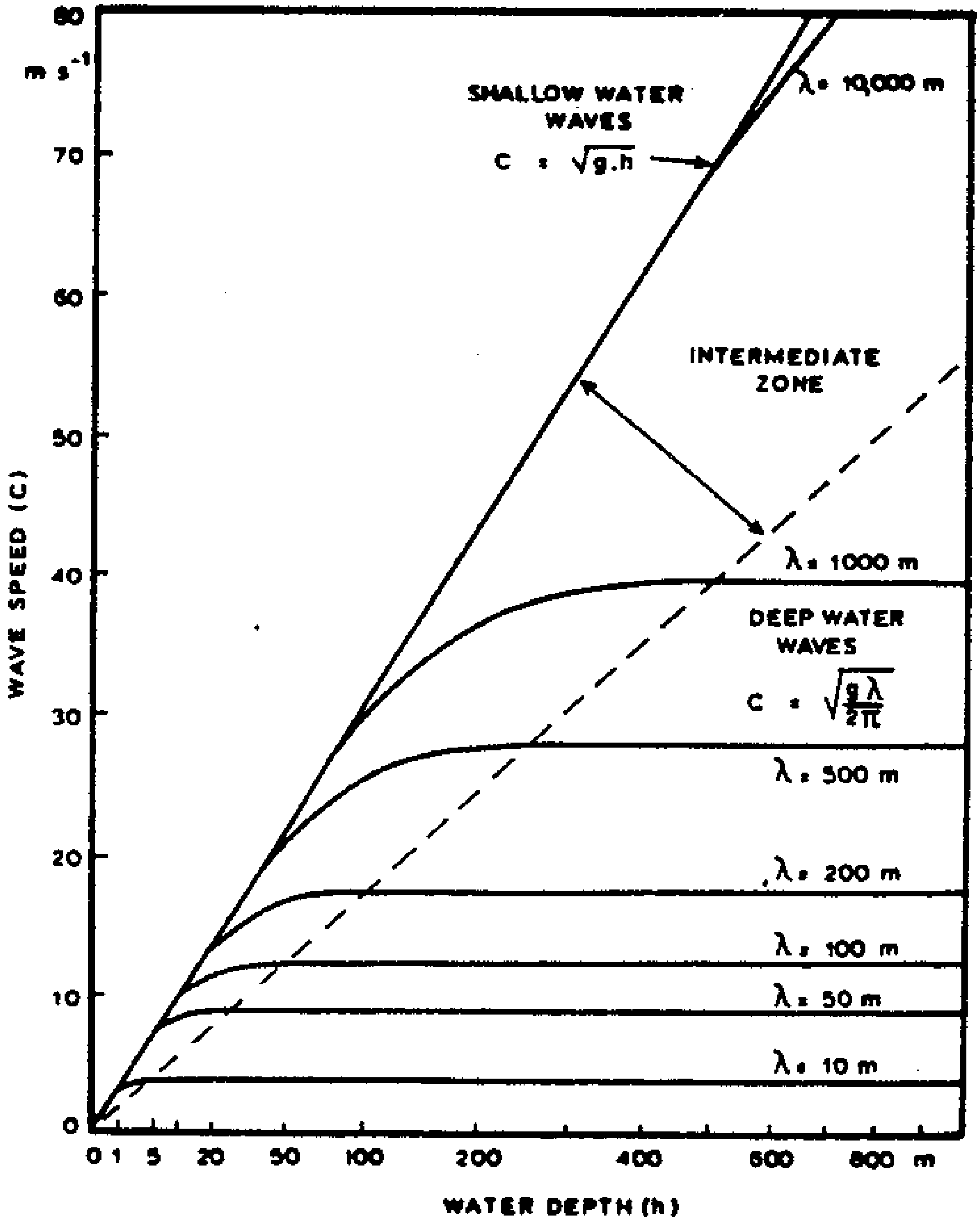


Fig. 5.2: The phase velocity of regular water waves (From Bowden, 1983)

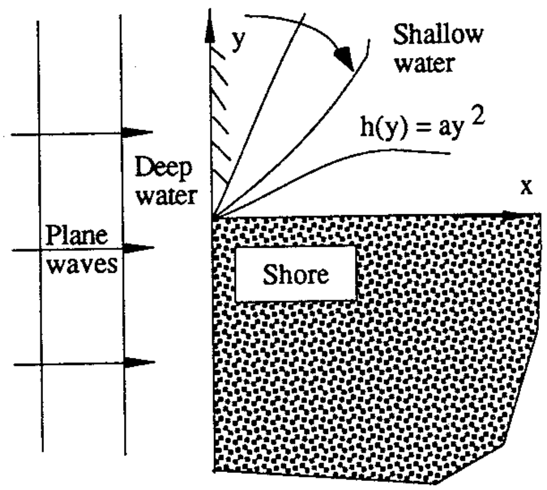
Exercise 5.4: Find the phase velocity for the waves in Exercise 5.2. (Answ (a) 15.6m/s, (b) 3.13m/s.)

Exercise 5.5: In 1966 Cartwright, Snodgrass, Munk and others observed ocean swell generated from storms offshore New Zealand up into Alaska. Estimate how many days the waves have been travelling across the Pacific if their period is 25 s. (Answ. ≈ 7-8 days. In Chapter 9 we shall learn that one should actually use only half the phase velocity in such calculations.)

Exercise 5.6: The same team as in Exercise 5.5 observed that the frequency of the swell when it was observed far from the source (the generating storm) tended to be linearly increasing with time. Assume that all waves were generated at the same instant of time and show that the slope of the observed curve "frequency vs time" by a distant observer, gives us a way to estimate the distance to the storm.

Exercise 5.7: Determine the speed of the tidal wave (period 12.4 h) as it enters a channel 5 m deep. (Answ. 7 m/s).

Exercise 5.7: Plane waves (that is, waves with long and parallel crests) $(\eta(x,t) = a\sin(\omega t - kx))$ are



approaching from the left as seen from above on the graph. The negative y-axis consists of a vertical wall and for y > 0, the water depth is given by the function $h(y) = ay^2$. Sketch the wave crests for x > 0 and y > 0 as long as y satisfies $ay^2 < \lambda / 25$. *Hint:* Remember that the phase velocity also is the velocity of the wave crests!

If we want to find the wavenumber *k* corresponding to a certain ω, we are faced with a so-called *transcendental equation*. It is in general impossible to turn the dispersion relation around and express *k* as a function ω. Of course, for shallow and deep water, the approximate solutions are fully adequate. However, it is very simple to solve it as closely as we want numerically, *i.e.* on a computer. The FORTRAN function below solves *k* to a relative accuracy better than 10^{-6} for all ω.

```
      REAL FUNCTION WAVEN(OMEGA,DEPTH)
C     PURPOSE  :
C         The real function WAVEN computes the wavenumber from
C         the finite depth dispersion relation
C         OMEGA  = sqrt( g*WAVEN*tanh(WAVEN*DEPTH) )
C         with a relative accuracy better than 1.E-6 .
C
C     VARIABLES:
C         OMEGA : Angular frequency , rad/s .
C         DEPTH : Water depth , m , > 0.
C
C**********************************************************************
C
      G = 9.81
      WAVEN = .0
      IF ( OMEGA .EQ. .0 ) RETURN
      WLIN = OMEGA**2/G
      DKL   = WLIN*ABS(DEPTH)
      IF ( DKL .GT. 3.) THEN
        WAVEN = WLIN/TANH(DKL)
      ELSE
        IF ( DKL .LT. .01 ) THEN
          WAVEN = SQRT(WLIN/(ABS(DEPTH)*(1-DKL/3)))
        ELSE
C
C-----     NEWTON ITERATION
          W = MAX(WLIN,SQRT(WLIN/ABS(DEPTH)) )
          DO 10 LOOP = 1,20
            W0 = W
            DK = W*DEPTH
            FX = WLIN - W*TANH(DK)
            FXDER = -TANH(DK) - DK/COSH(DK)**2
            W = W - FX/FXDER
            IF ( ABS((W0-W)/W) .LT. 1.E-5 ) GOTO 20
 10       CONTINUE
 20       WAVEN = W
        END IF
      END IF
C
      RETURN
      END
```

Fig. 5.3: A FORTRAN function for solving for *k* as a function of ω and the depth *h*.

Review questions:

1) Write down the dispersion relation for small amplitude waves and explain all the terms.

2) How does the function *y* = tanh(*x*) behave for small and large values of the argument?

3) Derive simplified forms of the dispersion relation for shallow and deep water, respectively. What is deep and shallow water in this context?

4) What is the general expression for the phase velocity, and what are the corresponding expressions for deep and shallow water?

5) Explain why the dispersion relation has two solutions of ω for each wavenumber.

6 FURTHER PROPERTIES OF THE WAVES

Apart from the surface elevation η and the velocity potential, there are several other quantities of interest. In the present section we shall look more closely into the velocity of the water due to the waves, the track of the fluid particles and the pressure variations due to the wave motion on the surface.

6.1 The velocity field

We recall that the water velocity $\mathbf{v}(x,z,t)$ for the one-dimensional waves we are considering has two components, $\mathbf{v}=(u,w)$, and

$$u(x,z,t)=\frac{\partial\Phi}{\partial x}(x,z,t)\,,\;\; w(x,z,t)=\frac{\partial\Phi}{\partial z}(x,z,t)\,.$$

The velocity potential for the regular wave was derived in Chapter 3:

$$\Phi(x,z,t)=\frac{ag}{\omega}\frac{\cosh(k(z+h))}{\cosh(kh)}\cos(\omega t-kx)\,.$$

Let us for simplicity first consider *deep water*. For large values of *kh* we may conveniently write the cosh-factor as follows:

$$\frac{\cosh(k(z+h))}{\cosh(kh)}=\frac{e^{kz}e^{kh}+e^{-kz}e^{-kh}}{e^{kz}+e^{-kz}}=e^{kz}\,\frac{1+e^{-2(z+h)k}}{1+e^{-2kz}}$$

When z is near the surface and $h\to\infty$, this expression tends to e^{kz}. Note that z gets increasingly negative as we move down into the water, which means that the factor e^{kz} gets *smaller and smaller.*

For deep water we may thus write

$$\Phi(x,z,t) = \frac{ag}{\omega} e^{kz} \cos(\omega t - kx),$$

from which it follows that

$$u(x,z,t) = \frac{ag}{\omega} k e^{kz} \sin(\omega t - kx) = a\omega e^{kz} \sin(\omega t - kx),$$

and

$$w(x,z,t) = \frac{ag}{\omega} k e^{kz} \cos(\omega t - kx) = a\omega e^{kz} \cos(\omega t - kx).$$

Thus, for a given depth z, both u and w represent running waves with the same amplitude. The waves differ in phase by $\pi/2$, however. The amplitude decreases from ωa at the surface to e^{kz} times the surface amplitude at the depth z. This decrease is rather fast: for $z = -\lambda/2$,

$$e^{kz} = e^{\frac{2\pi}{\lambda}(-\frac{\lambda}{2})} = e^{-\pi} \approx 0.043$$

At a depth equal to half the wavelength, the velocity amplitude is only about 4% of its surface value!

For an arbitrary depth the relations are easily seen to be

$$u(x,z,t) = a\omega \frac{\cosh(k(h+z))}{\sinh(kh)} \sin(\omega t - kx),$$
$$w(x,z,t) = a\omega \frac{\sinh(k(h+z))}{\sinh(kh)} \cos(\omega t - kx),$$

where we have used the dispersion relation for a slight simplification.

By taking the derivative of the velocities with respect to time, we obtain the *fluid accelerations* :

$$\frac{\partial u}{\partial t}(x,z,t) = a\omega^2 \frac{\cosh(k(h+z))}{\sinh(kh)} \cos(\omega t - kx),$$
$$\frac{\partial w}{\partial t}(x,z,t) = -a\omega^2 \frac{\sinh(k(h+z))}{\sinh(kh)} \sin(\omega t - kx).$$

Figure 6.1 shows the velocity and acceleration vectors compared to surface elevation. Note that the velocity is directed in the propagation direction of the wave at the wave crest.

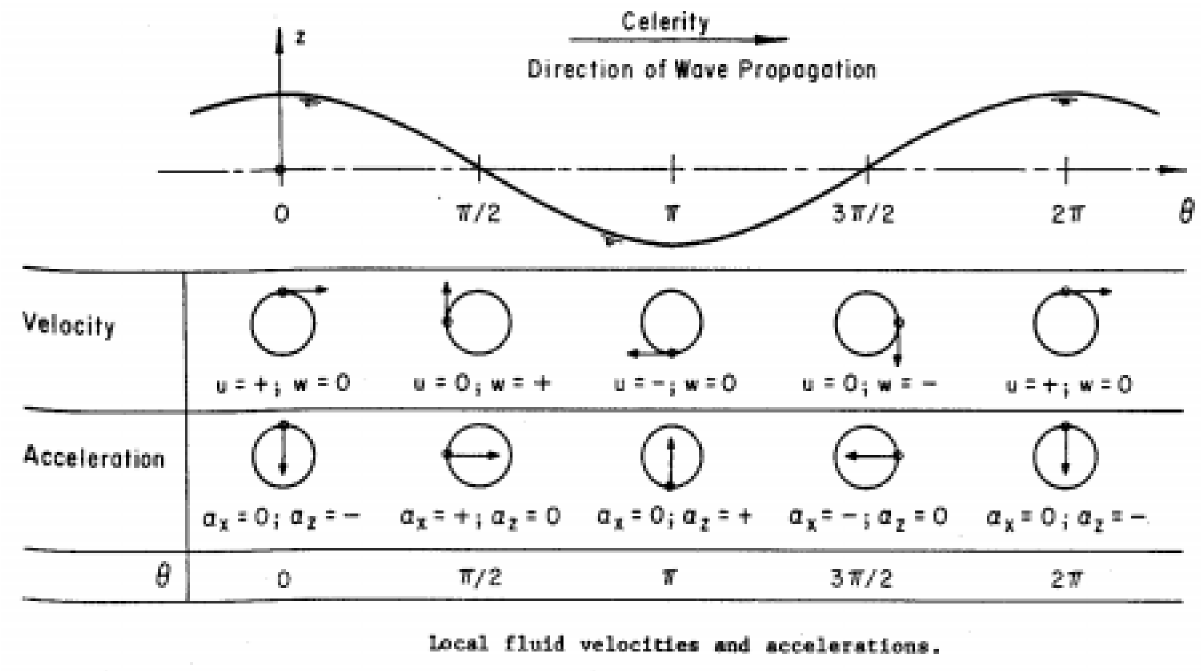
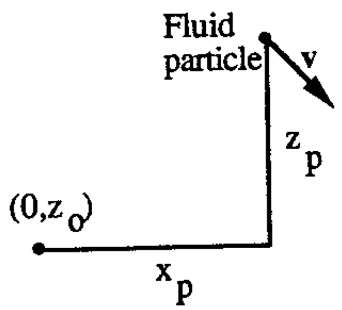


Fig. 6.1: Surface elevation along with velocity and acceleration vectors (From Shore Protection Manual, Vol. 1 p. 2-14)

Exercise 6.1: What is the maximum water particle velocity for a wave of length 200 m and an amplitude equal to 3 m in deep water? (Answ: 1.7 m/s)

6.2 The trajectories of the fluid particles



Consider the fluid near the point $(x=0, z=z_0)$, and let (x_p, z_p) describe the position of a nearby fluid particle at $(0+x_p, z_0+z_p)$. The motion of the fluid particle is given by the differential equations

$$\dot{x}_p = u(x_p, z_0+z_p, t),$$
$$\dot{z}_p = w(x_p, z_0+z_p, t).$$

By expanding *u* and *w* in a Taylor expansion, *e.g.*

$$u(x_p, z_0+z_p, t) = u(0, z_0, t) + \frac{\partial u}{\partial x} x_p + \frac{\partial u}{\partial z} z_p + \cdots,$$

we see by using the expressions for *u* and *w* that the first term dominates (assuming that x_p and z_p are of the order of the wave amplitude). Thus to a first approximation we may set

$$\dot{x}_p = u(0, z_0, t) = A\sin(\omega t),$$
$$\dot{z}_p = w(0, z_0, t) = B\cos(\omega t).$$

Here we have introduced

$$A = \omega a \frac{\cosh(k(h+z_0))}{\sinh(kh)},$$
$$B = \omega a \frac{\sinh(k(h+z_0))}{\sinh(kh)}.$$

If the two equations are integrated with respect to *t*,

$$x_p = -\frac{1}{\omega} A\cos(\omega t),$$
$$z_p = \frac{1}{\omega} B\sin(\omega t).$$

Thus,

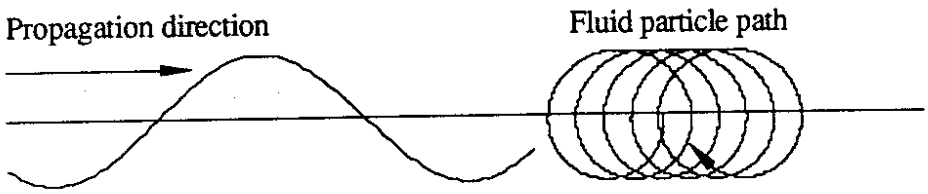
$$\frac{x_p^2}{(A/\omega)^2} + \frac{z_p^2}{(B/\omega)^2} = 1.$$

We recall that this is the equation of an ellipsis, and the fluid particles thus move in *elliptical orbits.* In particular, for deep water we have

$$A = B = \omega a e^{kz}$$

and the fluid particles move in *circles* of radius ae^{kz}.

This is an approximate result. If we look more closely into the equations for *u* and *w*, we see that the *u* velocity on the top of the orbit is slightly larger than the velocity at the bottom of the orbit. The net result is therefore a *slight displacement* along the wave direction.



Stokes drift (exaggerated)

This net motion is called *Stokes drift.*

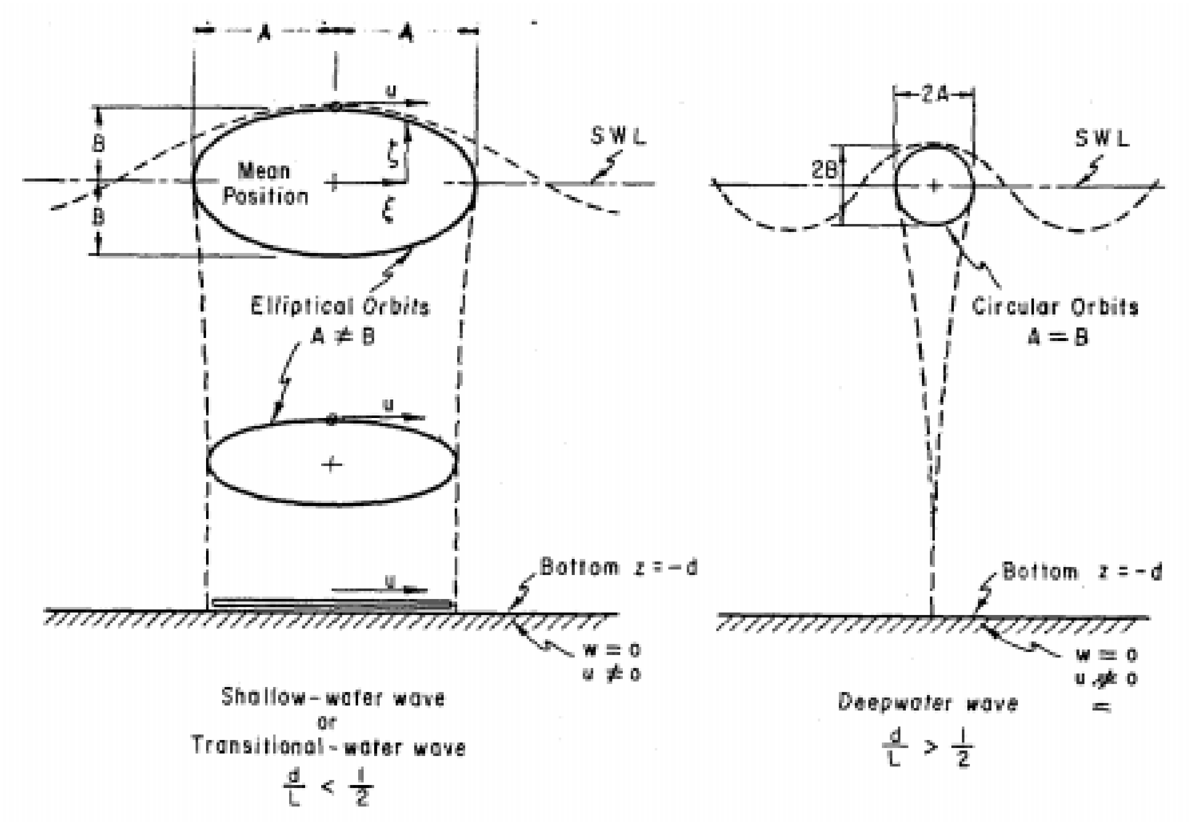


Fig. 6.2: Water particle displacement from the mean location for shallow water and deep water waves. (From the Shore Protection Manual Vol. 1 p. -2-17)

Exercise 6.2: Granted that the fluid particles move in *circles with constant speed* (in deep water), give a direct and simple argument that the velocity has to be ωa for particles at the surface.

Exercise 6.3: We shall prove later that the sum of two small amplitude waves also is a solution. *I.e.* if (η_1, Φ_1) and (η_2, Φ_2) are solutions, so are $\eta_1 + \eta_2$ and $\Phi_1 + \Phi_2$. Derive the velocities and the particle trajectories for the standing wave

$$\eta = a\sin(\omega t - kx) + a\sin(\omega t + kx)$$

6.3 The varying pressure from the waves

In general the pressure in the water is equal to the atmospheric pressure + the hydrostatic pressure (due to the weight of the water above) and a *dynamic part due to the wave motion.*

If we return to our form of the Bernoulli Equation, we can recall that

$$\frac{p}{\rho}+\frac{\partial \Phi}{\partial t}+\frac{1}{2}(u^2+v^2)+gz=\frac{p_{\text{atm}}}{\rho}.$$

If the wave amplitude is small, we may also here, like we did when we derived the linearized equations, neglect the term $(u^2+v^2)/2$ and we therefore obtain the following simple expression:

$$p(x,z,t)=-\rho\frac{\partial \Phi}{\partial t}(x,z,t)-\rho gz+p_{\text{atm}}.$$

The time varying part is usually called the *dynamic pressure* and is for the regular small amplitude wave equal to

$$p(x,z,t)=-\rho\frac{\partial \Phi}{\partial t}(x,z,t)=\rho ag\frac{\cosh(k(z+h))}{\cosh(kh)}\sin(\omega t-kx).$$

Exercise 6.4: What are the pressure variations at the bottom of the sea (h = 100m) for a wave with amplitude 1m if the wavelength is 10m, 100m and 1000m? Express the answer in fractions of the atmospheric pressure. Show that a simplified relation is valid as $h/\lambda \to 0$.

6.4 Summary

In this section we have considered a regular wave which we have expressed as

$$\eta(x,t)=a\sin(\omega t-kx).$$

The wavenumber k is equal to $2\pi/\lambda$ where λ is the wavelength and the angular frequency, ω is equal to $2\pi/T$ where T is the period. The dispersion relation connects the wavelength and the period

$$\omega^2=gk\tanh(kh).$$

If the water depth h is larger than about half the wavelength, the water is deep (as the waves are considered) and we may use the simplified relation $\omega^2=gk$. If we period and the wavelength, we obtain

$$\lambda=\frac{g}{2\pi}T^2,$$

that is,

$$\lambda[m]=1.56T^2,$$

where T is measured in seconds. Thus, a 10s wave in deep water has a wavelength of 156m.

On the contrary if $h<\lambda/25$, the water is shallow as far as the waves are concerned. Then,

$$\omega = \sqrt{gh}\, k,$$

and we obtain

$$\lambda[m] = 3.13 \cdot \sqrt{h} \cdot T,$$

where h is measured in meters and T in seconds. In shallow water the wavelength is thus proportional to the wave period.

In deep water the fluid particles move in circles with constant speed. At the surface, the radius of the circle is equal to the amplitude of the wave. Moreover, the water particle makes one complete turn per wave period. Hence the particle speed at the surface is $2\pi a / T$. The radius of the circle diminishes as e^{kz} as we move downward. When $z = \lambda / 2$, the radius is only 4% of its surface value, and when $z = \lambda$ only 0.18%! What are the corresponding velocities?

In very shallow water, the fluid moves almost horizontally with an amplitude

$$a \frac{\cosh(k(z_p + h))}{\sinh(kh)} \approx \frac{a}{\sinh(kh)} \approx \frac{a}{kh} = \frac{a}{h} \cdot \frac{\lambda}{2\pi}.$$

The table on the next page summarizes the Linear Wave Theory. Its is similar to a table in the *SHORE PROTECTION MANUAL*. Compare both tables and convince yourself that the results are similar. Note that the *SHORE PROTECTION MANUAL* uses $H = 2a,\ L = \lambda,$ and that both tables use *d* instead of *h* for water depth.

Linear Theory of Regular Waves Review

Wave property	**SHALLOW WATER** ($d / \lambda < 1/20$)	**INTERMEDIATE WATER** ($1/20 < d / \lambda < 1/2$)	**DEEP WATER** ($d / \lambda > 1/2$)
Velocity potential ($\boldsymbol{u} = \nabla \phi$)	$\phi = \frac{ag}{\omega} \frac{\cosh k(z+d)}{\cosh kd} \cos(\omega t - kx)$	$\phi = \frac{ag}{\omega} \frac{\cosh k(z+d)}{\cosh kd} \cos(\omega t - kx)$	$\phi = \frac{ag}{\omega} e^{kz} \cos(\omega t - kx)$
Dispersion relation	$\omega^2 = g k^2 d$	$\omega^2 = gk \tanh kd$	$\omega^2 = gk$
Wave length - wave period relation	$\lambda = T\sqrt{gd}$	$\lambda = \frac{g}{2\pi} T^2 \tanh \frac{2\pi d}{\lambda}$	$\lambda = \frac{g}{2\pi} T^2 \quad (\approx 1.56\ T^2)$
Wave profile	$\eta = a \sin(\omega t - kx)$	$\eta = a \sin(\omega t - kx)$	$\eta = a \sin(\omega t - kx)$
Dynamic pressure	$p_d = \rho g a \sin(\omega t - kx)$	$p_d = \rho g a \frac{\cosh k(z+d)}{\cosh kd} \sin(\omega t - kx)$	$p_d = \rho g a\, e^{kz} \sin(\omega t - kx)$
Horizontal particle velocity	$u = \frac{\omega a}{kd} \sin(\omega t - kx)$	$u = \omega a \frac{\cosh k(z+d)}{\sinh kd} \sin(\omega t - kx)$	$u = \omega a e^{kz} \sin(\omega t - kx)$
Vertical particle velocity	$w = \omega a \frac{z+d}{d} \cos(\omega t - kx)$	$w = \omega a \frac{\sinh k(z+d)}{\sinh kd} \cos(\omega t - kx)$	$w = \omega a e^{kz} \cos(\omega t - kx)$
Horizontal particle acceleration	$\dot{u} = \frac{\omega^2 a}{kd} \cos(\omega t - kx)$	$\dot{u} = \omega^2 a \frac{\cosh k(z+d)}{\sinh kd} \cos(\omega t - kx)$	$\dot{u} = \omega^2 a\, e^{kz} \cos(\omega t - kx)$
Vertical particle acceleration	$\dot{w} = -\omega^2 a \frac{z+d}{d} \sin(\omega t - kx)$	$\dot{w} = -\omega^2 a \frac{\sinh k(z+d)}{\sinh kd} \sin(\omega t - kx)$	$\dot{w} = -\omega^2 a\, e^{kz} \sin(\omega t - kx)$
Group velocity	$c_g = c$	$c_g = \frac{1}{2} c \left(1 + \frac{2kd}{\sinh 2kd}\right)$	$c_g = \frac{1}{2} c$

$\omega = 2\pi / T,\ k = 2\pi / \lambda$

T = wave period
λ = wave length
a = wave amplitude
g = acceleration of gravity
$c = \lambda / T$ = phase speed

t = time

x = direction of propagation

z = vertical co-ordinate positive upward, origin at still water level

d = water depth

p_d = dynamic pressure
$p_d - \rho g z + p_o$ = total pressure in the water ($-\rho g z$ = hydrostatic pressure, p_o = atmospheric pressure).
$E = \frac{1}{2} \rho g a^2$ = wave energy (per unit surface area)
$P = E c_g$ = wave energy flux (per unit width along the wave crest)

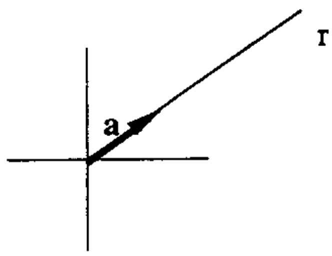
7 PLANE WAVES

So far our waves have been waves in a channel with the spatial coordinates x and z and the time coordinate t. From now we are going to consider waves on a two dimensional ocean, waves that may travel in any direction. By a *plane wave* is meant a wave with infinitely long crests (maxima) and constant elevation along lines orthogonal to the travel direction (think of corrugated iron).

The general *regular plane wave* may be written

$$\eta(\mathbf{x},t) = a\sin(\omega t - \mathbf{kx} + \alpha)$$

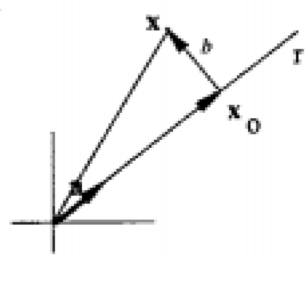
where **x** is the *position vector* consisting of the coordinates (x,y) and **k** is called the *wavenumber vector* with coordinates which we usually write (k_x, k_y) . We use **kx** for the scalar product of **k** and **x.** $k_x x + k_y y$. The wavenumber vector has magnitude, k, equal to the wavenumber and *direction* equal to the propagation direction of the wave. Let us verify this for a wave $\sin(\omega t - \mathbf{kx})$.



Set **k** = k·**a** where **a** is a unit vector. Consider a vertical pla through the wave and through the origin, parallel to the unit ve The horizontal position vector to all points along the cut in the plane may be written $\mathbf{x} = r\mathbf{a}$. For all *these points* we have

$$\sin(\omega t - \mathbf{kx}) = \sin(\omega t - (k\mathbf{a})\cdot(r\mathbf{a})) = \sin(\omega t - kr)$$

This is the familiar regular wave moving along the r-axis, that is in the direction of **a.**



If we now consider an arbitrary point **x**, we may write $\mathbf{x} = r\mathbf{a} + \mathbf{b}$ w is orthogonal to **a** and hence to **k.** Then,

$$\eta(\mathbf{x},t) = \sin(\omega t - \mathbf{kx}) = \sin(\omega t - (k\mathbf{a})\cdot(r\mathbf{a} + \mathbf{b}))$$
$$= \sin(\omega t - kr - k\mathbf{ab}) = \sin(\omega t - kr)$$

This shows that the value of $\eta(\mathbf{x},t)$ is the same for all values of **b** , that is, along lines orthogonal to **,**or as stated above, along lines orthogonal to the propagation direction.

With $\mathbf{k} = (k_x, k_y)$, we may also introduce polar coordinates (k, θ) and write

$$k_x = k\cos(\theta)$$
$$k_y = k\sin(\theta)$$

The propagation direction of the wave is thus given by θ. In general we then have

$$\eta(\mathbf{x},t) = a\sin\left(\omega t - k(x\cos\theta - y\sin\theta) + \alpha\right)$$

Check what you get for $\theta = 0°, 90°, 180°, 270°$!

For a general ocean surface extending both in the x- and y-directions, our equations and boundary conditions for the surface waves must also include the y-coordinate. It is easy to see that the new linearized equations are

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0, \quad -h \le z \le \eta$$
$$\frac{\partial\Phi}{\partial z}(x,y,-h,t) = 0$$
$$\frac{\partial\eta}{\partial t} = w(x,y,0,t)$$
$$\frac{\partial\Phi}{\partial t}(x,y,0,t) = -g\eta$$

Check that the plane wave travelling along the x-axis,

$$\eta(x,y,t) = a\sin(\omega t - (k\mathbf{i})\mathbf{x}) = a\sin(\omega t - kx)$$

and the corresponding velocity potential

$$\Phi(x,y,z,t) = \frac{ag}{\omega}\frac{\cosh(k(z+h))}{\cosh(kh)}\cos(\omega t - kx)$$

still satisfy the equations! (This is easy since neither η nor Φ contain y!) Since there is in general nothing special with waves in the x-direction in the above equations, we therefore conclude that the general regular plane wave solution to the equations is the one above with kx replaced by $\mathbf{k}\cdot\mathbf{x}$ for an arbitrary wavenumber vector $\mathbf{k}$.

Exercise 7.2: a) What are the propagation directions and wavenumbers for the following waves

$$\begin{aligned}
&\cos(kx-\omega t),\ \ k>0, \omega>0 \\
&\cos(kx-\omega t),\ \ k>0, \omega>0 \\
&\sin(\omega t-\mathbf{kx}),\ \ \omega<0 \\
&\cos(x-2y-\omega t),\ \ \omega>0 \\
&\sin(\omega t+2x-3y),\ \ \omega>0
\end{aligned}$$

(*Hint*: Always try to write the functions as $\sin(\omega t-\mathbf{kx}+\alpha)$ where $\omega > 0$ since we *know* that $\mathbf{k}$ in this case gives us both the propagation direction and the wavelength.)

b) Examine the function $f(x,y,t,)=\cos(x-t)+\cos(y+t)$ as a function of three variables!

Exercise 7.2: Show that the *u*- and *v*-velocities for the case $\mathbf{k}=k(\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j})$ may be written

$$\begin{aligned}
u(x,y,z,t) &= \omega a\cos\theta\frac{\cosh(k(h+z))}{\sinh(kh)}\sin(\omega t-\mathbf{kx}) \\
v(x,y,z,t) &= \omega a\sin\theta\frac{\cosh(k(h+z))}{\sinh(kh)}\sin(\omega t-\mathbf{kx})
\end{aligned}$$

Review questions A7:

1) Explain the meaning of the *wavenumber vector*.
2) Show that η is constant along lines orthogonal to $\mathbf{k}$.
3) How are the equations for surface waves modified for three dimensional waves?
4) How do we determine the propagation directions of "wave-like" functions?

8 SUPERPOSITION OF PLANE WAVES

Superposition means to “put on top of each other” or “add together”, and in this section we shall learn how to obtain more general solutions of the equations in Sec. 7 by *adding together* plane waves. We recall the linearized equations

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0, \quad -h\le z\le\eta$$
$$\frac{\partial\Phi}{\partial z}(x,y,-h,t)=0$$
$$\frac{\partial\eta}{\partial t}=w(x,y,0,t)$$
$$\frac{\partial\Phi}{\partial t}(x,y,0,t)=-g\eta$$

and assume that (η_1,Φ_1) and (η_2,Φ_2) both are solutions. Since all four equations are linear in η and Φ, we easily see that $(\eta_1+\eta_2,\Phi_1+\Phi_2)$ *will be a solution as well*. Check this by putting $\eta_1+\eta_2$ and $\Phi_1+\Phi_2$ into the equations and use the fact that we have assumed that (η_1,Φ_1) and (η_2,Φ_2) are solutions.

Since it is alright to add two solutions, we also see that it is also possible to add together an arbitrary number:

$$\eta(\mathbf{x},t)=\sum_{n=1}^{N}a_n\sin(\omega_n t-\mathbf{k}_n\mathbf{x}+\alpha_n)$$
$$\Phi(x,y,z,t)=\sum_{n=1}^{N}\frac{a_n g}{\omega_n}\frac{\cosh(k_n(z+h))}{\cosh(k_n h)}\cos(\omega_n t-\mathbf{k}_n\mathbf{x}+\alpha_n)$$

Each term in the sum corresponds to a plane wave defined in terms of

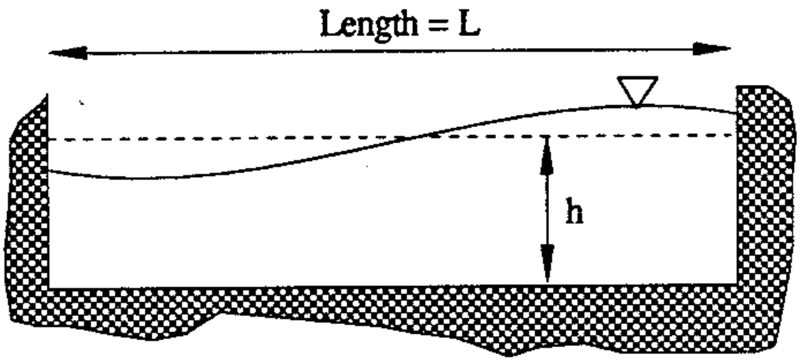
1) Its amplitude: a_n
2) Its wavenumber vector: $\mathbf{k}_n$ (Remember that $k_n=|\mathbf{k}_n|$, $\omega_n^2=gk_n\tanh(hk_n)$)
3) Its phase at $\mathbf{x}$ = 0, t = 0, α_n.

From the velocity potential, we may as before obtain the particle velocities, (u,v,w), the dynamic pressure p_{dyn} and the accelerations.

Exercise 8.1: Examine how the solution made up by two plane waves with the same wavelength and amplitude, but with different directions looks:

$$\eta(\mathbf{x},t)=a\sin(\omega t-\mathbf{k}_1\mathbf{x})+a\sin(\omega t-\mathbf{k}_2\mathbf{x})$$
$$=2a\sin\left(\omega t-\frac{\mathbf{k}_1+\mathbf{k}_2}{2}\mathbf{x}\right)\cos\left(\frac{\mathbf{k}_1-\mathbf{k}_2}{2}\mathbf{x}\right)$$

Exercise 8.2: Determine the period of oscillations of the form indicated below in a basin with vertical walls at both ends.



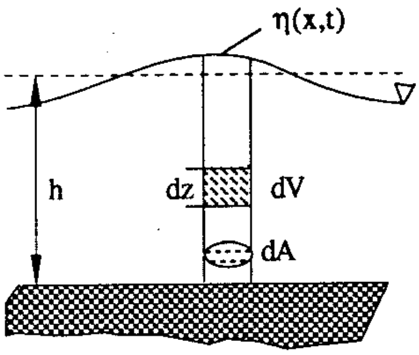
(*Hint*: Recall Exercise 6.3. Which condition must u satisfy at $x = 0$ and $x = L$? Construct the solution by means of the result from Exercise 6.3.)

Review question A8:

1) What is a superposition of waves and why is a superposition of solutions a solution (for small amplitude waves)?

9 ENERGY AND GROUP VELOCITY

When we look at waves breaking on a shore, it is obvious that the waves bring with them a lot of *energy*. The energy content in an infinite plane wave is obviously infinite, so we are more interested in finding the energy per unit area of the surface.



The potential energy contained in a column of water with cross section dA as shown on the graph is

$$dE_p = \int_{z=-h}^{\eta} \rho g z dV = dA \int_{z=-h}^{\eta} \rho g z dz = dA \rho g \frac{\eta^2 - h^2}{2}$$

Since only the excess potential energy is of interest, we subtract the part corresponding to the mean surface and obtain the potential energy per unit area:

$$\frac{dE_p - dE_p(\eta = 0)}{dA} = \frac{1}{2} \rho g \eta^2 .$$

Instead of using the instantaneous value of η, it is more common to use *the average* of η^2 on the right hand side. The average of η^2 for a sinusoidal wave with amplitude *a* is $a^2/2$ (We shall return to this in Part B). For a plane wave with amplitude *a*, the *average potential energy per unit area* is therefore

$$\left\langle \frac{dE_p}{dA} \right\rangle = \frac{\rho g a^2}{4}$$

where the brackets are used for indicating the average value.

The kinetic energy is derived similarly by observing that

$$dE_k = \int_{z=-h}^{\eta} \frac{1}{2}\rho\left(u^2 + v^2 + w^2\right) dV.$$

If we (for simplicity) consider deep water and a plane wave $\eta(\mathbf{x},t) = a\sin(\omega t - \mathbf{kx})$, we obtain

$$u^2 + v^2 + w^2 = (\omega a)^2 e^{2kz}$$

and therefore

$$\frac{dE_k}{dA} = \rho\omega^2 a^2 \int_{z=-h}^{\eta} \frac{1}{2} e^{2kz} dz \approx \rho\omega^2 a^2 \int_{-\infty}^{0} \frac{1}{2} e^{2kz} dz = \frac{1}{2}\rho\omega^2 a^2 \frac{1}{2k} = \frac{1}{4}\rho a^2 g.$$

The average kinetic and the potential energies are thus equal (Show that the error introduced by replacing η by 0 is negligible).

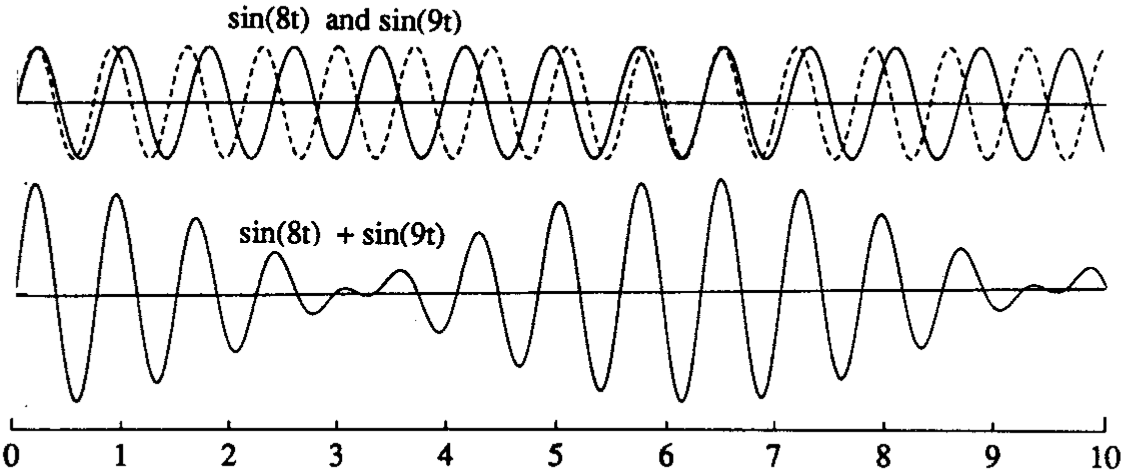
The energy is carried along with the waves, but somewhat surprisingly, the energy is *not travelling* with the phase velocity of the wave. As a matter of fact, in deep water, the transport velocity of the energy is only half the phase velocity!

We shall not give a rigorous proof of this, but follow a more intuitive argument. Consider first the superposition of two plane waves in the same direction, but with slightly different frequencies and wavenumbers:

$$a\sin(\omega_1 t - k_1 x) + a\sin(\omega_2 t - k_2 x)$$

$$= 2a\sin\left(\frac{\omega_1 + \omega_2}{2}t - \frac{k_1 + k_2}{2}x\right)\cos\left(\frac{\omega_1 - \omega_2}{2}t - \frac{k_1 - k_2}{2}x\right).$$

The result is a *product* of two travelling waves. The first wave has frequency about the same frequency and wavenumbers as the two original waves, whereas the second wave has frequency $(\omega_1 - \omega_2)/2$ and wavenumber $(k_1 - k_2)/2$. At a given instant of time, the result will look as shown on the figure.



The result consists of "groups of waves" moving with the phase velocity of the "cosine" wave:

$$c_g = \frac{(\omega_1 - \omega_2)/2}{(k_1 - k_2)/2} = \frac{\omega_2 - \omega_1}{k_2 - k_1}.$$

If we are sitting in a boat in one of the minima for the amplitude (called "knots"), and moving with a velocity equal to c_g, we would not feel any waves at all! And energy will pass us in either direction. From this we conclude that the energy is moving with the same speed as we are, namely *the group velocity,* which in the limit amounts to

$$c_g = \frac{d\omega}{dk}$$

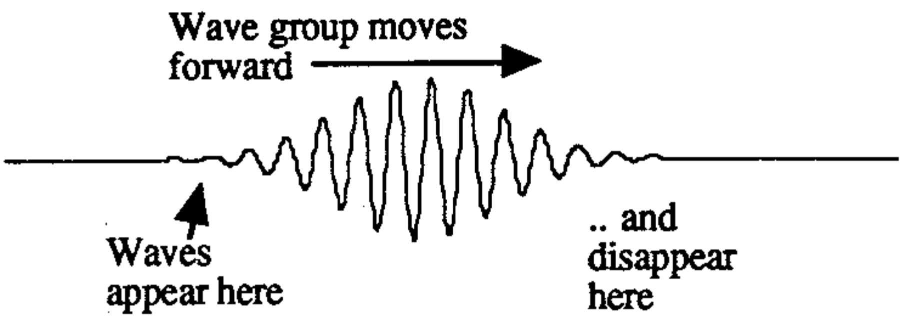
Since we are dealing with water waves fulfilling the dispersion relation, we have

$$c_g = \frac{d\omega}{dk} = \frac{2\omega\, d\omega}{2\omega\, dk} = \frac{(g \cdot \tanh(kh) + gk \cdot h \cdot \cosh^{-2}(kh))dk}{2\omega\, dk} = \frac{g}{2\omega}\left(\tanh(kh) + \frac{kh}{\cosh^2(kh)}\right)$$

In deep water $(h \to \infty)$ we obtain from the expression above, or simply from the corresponding dispersion relation that

$$c_g = \frac{d\omega}{dk} = \frac{g}{2\omega} = \frac{g/\omega}{2} = \frac{c_p}{2}$$

In deep water, *the group velocity is only half the phase velocity*. If we watch wave groups, the individual waves are created at the end of the group, and move forwards until they disappear at the front of the group.

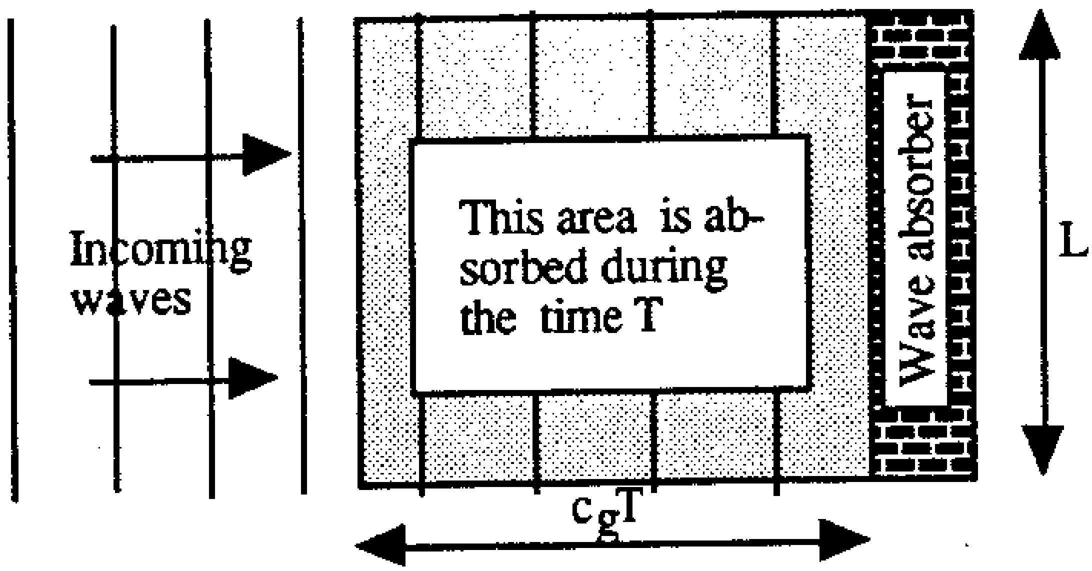


In very shallow water, $\omega = \sqrt{gh}\ k$, and

$$c_g = \frac{d\omega}{dk} = \sqrt{gh} = c_p .$$

In very shallow water, the phase and the group velocities are equal!

For wave power generation, it is the energy coming into the device per time unit rather than the energy content itself which is of interest. In order to derive the available energy, it is convenient to consider an ideal (100%) *wave energy absorber* placed in front of an incoming plane wave as shown on the figure below. We are interested in the absorbed energy per time unit and length unit of the absorber.



During a time interval T, all energy within the dashed area is absorbed by the wave absorber. The energy within the square is

$$\text{Kinetic energy} + \text{potential energy} = 2\frac{1}{4}\rho g a^2 \cdot (c_g T)\cdot L$$

The absorbed energy per time unit and per length unit of the absorber is thus

$$J = \frac{\frac{1}{2}\rho g a^2 \cdot (c_g T)\cdot L}{TL} = \frac{1}{2}\rho g a^2 c_g .$$

In metric units, the unit for J is *Watt per meters (W/m)*.

Consider a typical ocean wave with amplitude a=10m and period T=10s. In deep water,

$$c_g = \frac{g}{2\omega} = g\frac{T}{4\pi} = 8.3 m/s$$

Thus,

$$J = \frac{1}{2}\left(10^3\frac{kg}{m^3}\right)\left(9.81\frac{m}{s^2}\right)\left(1m^2\right)\left(8.3\frac{m}{s}\right) \approx 4\cdot 10^4\frac{kgm^2}{s^2 sm} = 40 kW/m$$

This is considerable! Unfortunately, existing wave power devices are far from being "perfect".

Exercise 9.1: A plane wave is moving along the x-axis in a sea where the depth slowly varying with x, $h=h(x)$. If there is no reflected or dissipated energy (*e.g.* removed by friction), show that the amplitude, a, and the group velocity, c_g must satisfy

$$a^2(x)c_g(x) = \text{constant}.$$

A wave coming from the deep sea with amplitude 1m and wavelength λ= 300 m is moving as above into shallow water, 5 m deep. Determine the new wave amplitude and its wavelength. The *wave steepness* is defined by $2a/\lambda$. Determine these ratios for the wave above. (Answ: New amplitude 1.24 m, steepness, deep water = 0.0067, shallow water 0.026).

Exercise 9.2: Compute the maximum electric power that can be generated by a device that converts 10% of the energy from a plane wave of amplitude 0.5m and wavelength 100m if the device is 1 km, parallel to the wave crests. (Answ: $7.6\cdot 10^5$W).

Review questions A9:

1. What are the potential and kinetic energies per unit area for a regular wave?
2. How is the group velocity defined, and how does it relate to the phase velocity in deep and shallow water?
3. Derive the expression for the energy absorbed per unit time and unit length of an
4. absorber sitting orthogonal to the propagation direction of the wave.

10 REFERENCES

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