Overflaten av enhetssfæren i d dimensjoner

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In d dimensions, we have a function $f(k)$, that is only dependent on the absolute value of $\mathbf{k}$. Then we have that

$$\int \mathrm{d}^d\mathbf{k}\ f(k) = S_d \int_0^\infty k^{d-1}\mathrm{d}k\ f(k), \tag{1}$$

where we have made the transformation to spherical coordinates, and where S_d is the surface area of the unit sphere in d dimensions. If we insert $f(k) = e^{-\frac{1}{2}k^2}$, then we have for the left hand side of (1):

$$\begin{aligned}\int \mathrm{d}^d\mathbf{k}\ e^{-\frac{1}{2}k^2} &= \prod_{i=1}^{d} 2\int_0^\infty \mathrm{d}k_i\ e^{-\frac{1}{2}k_i^2} \\ &= \sqrt{2\pi}^d. \end{aligned} \tag{2}$$

For the right hand side, we have

$$S_d \int_0^\infty k^{d-1}\mathrm{d}k\ e^{-\frac{1}{2}k^2} = S_d\ 2^{\frac{d}{2}-1}\int_0^\infty \mathrm{d}x\ e^{-x}x^{\frac{d}{2}-1}, \tag{3}$$

where we have made a change of variables:

$$\begin{aligned} k &\to \sqrt{2x} \\ \Rightarrow \mathrm{d}k &\to \frac{\mathrm{d}x}{\sqrt{2x}}. \end{aligned} \tag{4}$$

Now as we know, the integral on the right hand side of (3) is by definition equal to $\Gamma(\frac{d}{2})$. Hence, putting the two sides of (1) back together, we find

$$\begin{aligned} \sqrt{2\pi}^d &= S_d\ 2^{\frac{d}{2}-1}\Gamma(\frac{d}{2}) \\ \Rightarrow S_d &= \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}. \end{aligned} \tag{5}$$