STABLE DIFFEOMORPHISM GROUPS OF 4-MANIFOLDS

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ABSTRACT. A localisation of the category of n-manifolds is introduced by formally inverting the connected sum construction with a chosen n-manifold Y. On the level of automorphism groups, this leads to the stable diffeomorphism groups of n-manifolds. In dimensions 0 and 2, this is connected to the stable homotopy groups of spheres and the stable mapping class groups of Riemann surfaces. In dimension 4 there are many essentially different candidates for the n-manifold Y to choose from. It is shown that the Bauer-Furuta invariants provide invariants in the case $Y = \overline{\mathbb{C}P}^2$, which is related to the birational classification of complex surfaces. This will be the case for other Y only after localisation of the target category. In this context, it is shown that the $K3$-stable Bauer-Furuta invariants determine the $S^2 \times S^2$-stable invariants.

INTRODUCTION

One of the main objects of geometric topology is to classify n-dimensional manifolds up to diffeomorphism – and their diffeomorphisms. However, the study of diffeomorphism groups has turned out to be difficult so far, and one is tempted to address simpler but related problems first. This has led, for example, to the introduction of the block diffeomorphism groups, which are relatively accessible via surgery theory, see [WW01] for a survey. In this work, the focus will be on a different way of simplifying things: localisation – in the sense of inverting operations which need not be invertible originally. In contrast to arithmetic localisation techniques, which long have their place in geometric topology, see [Sul70], we will consider inverting the operation of connected summation with a chosen n-dimensional manifold Y.

In the first four sections, after describing the general setup, I will discuss examples in low dimensions $n = 0$ and $n = 2$, which show that this has led

to interesting mathematics already, related to the stable homotopy groups of spheres and the stable mapping class groups of Riemann surfaces.

The final four sections concentrate on stabilisation of 4-manifolds, with particular emphasis on the question of how the Bauer-Furuta invariants relate to this. An essential feature in this dimension is that there are several reasonable choices of "directions" Y in which to stabilise. Section 5 discusses some of them. The cases $Y = \overline{\mathbb{C}P^2}$, $Y = S^2 \times S^2$, and $Y = K3$ will play a rôle later on.

It will turn out that the Bauer-Furuta invariants allow for the definition of *stable* characteristic classes for families of 4-manifolds. The reader may find it useful to keep following analogy with the corresponding approach for vector bundles in mind: The universal Chern classes live in $\mathrm{H}^*(\mathrm{B}U(n))$, where $\mathrm{B}U(n)$ is the classifying space for n-dimensional complex vector bundles, and the fact that Chern classes do not change when a trivial bundle is added means that there are stable universal Chern classes living in $\mathrm{H}^*(\mathrm{B}U(\infty))$, where $\mathrm{B}U(\infty)$ is the classifying space for stable complex vector bundles. However, caution should be taken to not to confuse the approach in this article with the stabilisation process by taking products of manifolds with $\mathbb{R}$ as in [Maz61], which is closer in spirit to the Chern class picture, but changes the dimension of the manifolds involved; the connected sum construction preserves dimension while possibly changing the rank of the intersection form.

The stable Bauer-Furuta invariants are worked out in Section 6 in the case $Y = \overline{\mathbb{C}P^2}$, which is simplified by the fact that the Bauer-Furuta invariant of $\overline{\mathbb{C}P^2}$ is the identity, leading to a stable universal characteristic class in the group

$$\pi_{\mathbb{T}}^0(\mathrm{BSpiff}^c(X \# \infty \overline{\mathbb{C}P}^2, y)^{\lambda(\sigma_X)}),$$

see Theorem 1. In general, one will have to localise the target category of the invariants as well. This done in Section 7, see Theorem 2 there, while the final section discusses the examples $Y = S^2 \times S^2$ and $Y = K3$. These two cases turn out to be related in the sense that the $K3$-stable invariants determine the $S^2 \times S^2$-stable invariants, see Theorem 3. This would be a trivial result if $S^2 \times S^2$ were a connected summand of $K3$; but it is not.

1. BASED MANIFOLDS

The aim of this section is to give a description of a category of manifolds where a common construction, connected summation with another manifold, is well-defined – not up to diffeomorphism, but on the nose. There will be reasons to restrict the topology of the manifolds involved. For example, one might only want to consider connected or simply-connected manifolds. And there will be reasons to consider manifolds with orientations, even in the generalised sense of spin structures or framings. However, it should be clear how to adapt the following to the specified contexts if necessary.

We will consider closed manifolds of a fixed dimension d. Let $\mathcal{M}_d$ be the category of pairs (X, x), where X is closed d-manifold and

$$x \colon \mathbb{D}^d \longrightarrow X$$

is an embedding of the closed d-disk into X. The space of embeddings $\mathbb{D}^d \to X$ is homotopy equivalent to the frame bundle of X. Thus, these embeddings x can be thought of as manifolds with a framed base point $x(0)$. However, it will be important to have an actual embedding as part of the structure. The morphisms in $\mathcal{M}_d$ from (X, x) to (X', x') are the diffeomorphisms from X to X' which send x to x' in the sense that the triangle

$$
\begin{array}{ccc}
 & \mathbb{D}^d & \\
{\scriptstyle x} \swarrow & & \searrow {\scriptstyle x'} \\
X & \xrightarrow[\cong]{} & X'
\end{array}
$$

commutes. By construction, the category $\mathcal{M}_d$ is a groupoid. This will hold for all categories considered here. The class of objects could be considered with the topology from the embedding spaces, but this will not be done here. However, the automorphism group of (X, x) in $\mathcal{M}_d$ can also be considered with its natural topology, and this will be done here. That automorphism group is the group $\mathrm{Diff}(X, x)$ of all diffeomorphisms of X which fix x.

Let us compare the groupoid $\mathcal{M}_d$ of based manifolds with the groupoid of all closed d-manifolds and diffeomorphisms. If two objects (X, x) and (X', x')

are isomorphic in $\mathcal{M}_d$, then X and X' are diffeomorphic. The converse holds if X (and therefore also X') is connected.

If X is connected, this automorphism group in $\mathcal{M}_d$ is only a frame bundle away from the group $\mathrm{Diff}(X)$ itself, in the sense that there is a fibration sequence

$$\mathrm{Diff}(X, x) \longrightarrow \mathrm{Diff}(X) \longrightarrow \mathrm{Emb}(\mathbb{D}^d, X),$$

and that $\mathrm{Emb}(\mathbb{D}^d, X)$ is homotopy equivalent to the frame bundle of X. For example, take $X = S^d$. Then the composition

$$SO(d+1) \longrightarrow \mathrm{Diff}(S^d) \longrightarrow \mathrm{Emb}(\mathbb{D}^d, S^d)$$

is an equivalence. In fact, for $d = 1, 2, 3$ both arrows are equivalences, and $\mathrm{Diff}(S^2, x)$ is contractible, see [Sma59] and [Hat83]. This is false for $d \geqslant 5$ [Mil84], and unknown (at present) for $d = 4$.

To sum up, the category $\mathcal{M}_d$ of based manifolds is sufficiently close to the category of unbased manifolds that their difference is under control. It is time to see what the base is good for.

2. CONNECTED SUMS

Let us fix a closed d-manifold Y and an embedding

$$(y', y) \colon \mathbb{D}^d + \mathbb{D}^d \longrightarrow Y.$$

If Y is connected, the choice of y and y' will not matter and will be omitted from the notation. In any case, this yields an endofunctor F_Y of $\mathcal{M}_d$ which is given on objects by connected summation:

$$F_Y(X, x) = (X \# Y, y),$$

where the connected sum $X \# Y$ is constructed from $X \backslash x(0)$ and $Y \backslash y'(0)$ by the usual identifications. (It is here, where the actual embedding is used.) On morphisms, the functor F_Y acts by extending a diffeomorphism of X which fixes x over $X \# Y$ by the identity on Y.

The question arises whether this functor is invertible or not. Of course, it will rarely be invertible in the strict sense: here and in the following, functors will be considered only up to natural isomorphism.

PROPOSITION 1. *The functor F_Y is invertible if and only Y is a homotopy sphere.*

PROOF. Let Y be an homotopy sphere. Then there is another homotopy sphere Z such that $Y \# Z \cong S^d$. Connected sum with Z gives the inverse.

Let F_Y be invertible. Then there is a manifold X such that $X \# Y \cong S^d$. It follows that X and Y are homotopy spheres. See [Mil59]. $\square$

In any case, if the functor F_Y is not invertible, it can be formally inverted, and this it what will be done next.

3. FORMALLY INVERTING ENDOFUNCTORS

Given a category $\mathcal{C}$ with an endofunctor F, there is a universal category $\mathcal{C}[F^{-1}]$ with an autofunctor, denoted by $\overline{F}$, and a functor $\mathcal{C} \to \mathcal{C}[F^{-1}]$ compatible with the functors F and $\overline{F}$, which is universal (initial) among such functors: if $(\mathcal{D}, G)$ is another category with an autofunctor, and if $\Phi \colon (\mathcal{C}, F) \to (\mathcal{D}, G)$ is a functor compatible with F and G, then there is a unique functor $\varphi \colon (\mathcal{C}[F^{-1}], \overline{F}) \to (\mathcal{D}, G)$ such that the diagram

$$
\begin{array}{ccc}
(\mathcal{C}, F) & \longrightarrow & (\mathcal{C}[F^{-1}], \overline{F}) \\
\Phi \downarrow & \swarrow \varphi & \\
(\mathcal{D}, G) & &
\end{array}
$$

commutes.

There are several (naturally equivalent) constructions of $\mathcal{C}[F^{-1}]$ available. In one of them, the objects are the pairs (C, n), where C is an object of $\mathcal{C}$ and n is an integer. The set of morphisms from (C, n) to (C', n') is

$$
(1) \qquad \operatorname*{colim}_{m} \operatorname{Mor}_{\mathcal{C}}(F^{m+n}C, F^{m+n'}C'),
$$

where the colimit is formed using the maps induced by F. For example, the identity of C represents a natural isomorphism

$$
(2) \qquad (C, 1) \cong (FC, 0)
$$

in $\mathcal{C}[F^{-1}]$. The functor from $\mathcal{C}$ to $\mathcal{C}[F^{-1}]$ sends C to $(C, 0)$, and the functor F on $\mathcal{C}$ extends to $\mathcal{C}[F^{-1}]$ by acting on the first component. An isomorphism like (2) shows that this extension of F is naturally isomorphic to the functor $\overline{F}$ on $\mathcal{C}[F^{-1}]$ which sends (C, n) to $(C, n + 1)$. As the latter is clearly invertible, so is the former. As for the universal property, φ must be defined on objects by $\varphi(C, n) = G^n \Phi C$. If $c \colon F^{m+n}C \to F^{m+n'}C'$ represents a morphism $[c] \colon (C, n) \to (C', n')$, then $\varphi[c]$ is to be defined such that $G^m \varphi[f] = \Phi(f)$. See [Mar83] for all this in a similar context.

A different model for $\mathcal{C}[F^{-1}]$ is a special case of Quillen's construction, see [Gra76], namely the category $\mathbb{N}^{-1}\mathcal{C}$, where the monoid $\mathbb{N}$ is interpreted as a (discrete) monoidal category, acting on $\mathcal{C}$ via F. As it turns out, this model is literally the same as the Grothendieck construction in the case of the diagram $\mathcal{C} \xrightarrow{F} \mathcal{C} \xrightarrow{F} \mathcal{C} \xrightarrow{F} \dots$ of categories. As Thomason proved, in [Tho79], there is an equivalence

$$\mathrm{B}\mathcal{C}[F^{-1}] \simeq \mathrm{hocolim}(\mathrm{B}\mathcal{C} \xrightarrow{\mathrm{B}F} \mathrm{B}\mathcal{C} \xrightarrow{\mathrm{B}F} \mathrm{B}\mathcal{C} \xrightarrow{\mathrm{B}F} \dots),$$

which also implies that this construction is well-behaved with respect to homology.

Two objects C and C' of $\mathcal{C}$ are called F-*stably isomorphic* if their images in $\mathcal{C}[F^{-1}]$ are isomorphic; this is the case if and only if there is a non-negative integer n such that $F^n C$ and $F^n C'$ are isomorphic in $\mathcal{C}$. Two objects C and C' of $\mathcal{C}$ are called F-*stably equivalent* if there are non-negative integers n and n' such that $F^n C$ and $F^{n'} C'$ are isomorphic in $\mathcal{C}$. Clearly, two F-stably isomorphic objects are F-stably equivalent, and the converse need not hold. The importance of the notion of equivalence lies in the following fact.

PROPOSITION 2. *The isomorphism type of the automorphism group of C in $\mathcal{C}[F^{-1}]$ depends only on its F-stable equivalence class.*

PROOF. It suffices to prove that C and FC have isomorphic automorphism groups in $\mathcal{C}[F^{-1}]$. But that follows immediately from the definition (1). $\square$

4. STABLE DIFFEOMORPHISM GROUPS

The abstract construction of the previous section can now be applied to the category $\mathcal{M}_d$ with the endofunctor F_Y. Let $\mathcal{M}_d[Y^{-1}]$ be the category which is obtained from $\mathcal{M}_d$ by formally inverting F_Y. Given any object (X, x, n) in $\mathcal{M}_d[Y^{-1}]$, its automorphism group is

(3) $$\mathrm{Diff}(X \# \infty Y) \stackrel{\mathrm{def}}{=} \operatorname*{colim}_n \mathrm{Diff}(X \# nY),$$

where again the embeddings have been suppressed from the notation. See (5) below for a geometric interpretation of the maps involved in the colimit on the level of classifying spaces. The groups (3) are the *stable diffeomorphism groups* of X with respect to Y to which the title refers. As has been pointed out in Proposition 2, up to isomorphism, the Y-stable diffeomorphism group of X depends only on the Y-stable equivalence class of X.

In this rest of this section, manifolds of dimension 0 and 2 will be studied from the point of view of their stable diffeomorphism groups. The underlying mathematics is well-known, and the only point of repeating it here is to illustrate the fact that the abstract setup from the previous section leads to interesting mathematics even in the simplest cases. The remaining sections will discuss stable diffeomorphism groups of 4-manifolds.

4.1. DIMENSION 0. The category $\mathcal{M}_0$ is the category of finite pointed sets and their pointed bijections. The classifying space is

$$\mathrm{B}\mathcal{M}_0 \simeq \coprod_{n \geqslant 0} \mathrm{B}\Sigma_n.$$

For stabilisation, one needs Y to have at least two elements. As shown in Proposition 1, the case $Y = S^0$ is uninteresting. One could use $S^0 \times S^0$, but that turns out to be a connected sums in this case: it is the connected double of the set Y with three elements. For any pointed finite set, the connected sum $X \# Y$ has one element more than X. There is just one Y-stable equivalence class of objects, and its automorphism group is the infinite symmetric group Σ_∞. This gives

$$\mathrm{B}\mathcal{M}_0[Y^{-1}] \simeq \mathbb{Z} \times \mathrm{B}\Sigma_\infty,$$

and in particular, the unstable homotopy groups are the stable homotopy groups of spheres. Note that a group completion of $\mathrm{B}\mathcal{M}_0$ agrees with the plus construction of $\mathrm{B}\mathcal{M}_0[Y^{-1}]$, see [Ada78].

4.2. DIMENSION 2. Let us now consider 2-manifolds which are connected and oriented only, retaining the notation $\mathcal{M}_2$ for the corresponding full subcategory. These manifolds are classified by their genus g. As we consider only diffeomorphisms which fix an embedded disk, the classifying spaces of the diffeomorphism groups are homotopically discrete, see [ES70] or [Gra73]. In fact, they are homotopy equivalent to the classifying spaces of the corresponding mapping class groups $\Gamma_{g,1}$. This gives

$$\mathrm{B}\mathcal{M}_2 \simeq \coprod_{g \geqslant 0} \mathrm{B}\Gamma_{g,1}.$$

Stabilisation with respect to the torus $Y = S^1 \times S^1$ gives

$$\mathrm{B}\mathcal{M}_2[Y^{-1}] \simeq \mathbb{Z} \times \mathrm{B}\Gamma_{\infty,1}.$$

Also in this case, the a completion of $\mathrm{B}\mathcal{M}_2$ agrees with the plus construction of $\mathrm{B}\mathcal{M}_2[Y^{-1}]$, see [MW04] and the references within.

In contrast to the previous, low-dimensional cases, there does not seem to be an obvious monoidal structure on $\mathcal{M}_d$ in the high-dimensional situation; so *a priori* it does not make much sense to ask about a group completion of $\mathrm{B}\mathcal{M}_d$ in general. The results above suggest that a reasonable substitute and object of study could be given by the plus construction of some $\mathrm{B}\mathcal{M}_d[Y^{-1}]$.

5. STABILISATION IN DIMENSION 4

In this section, we will consider simply-connected oriented 4-manifolds. There is not even a good conjecture what the set of isomorphism classes could be, and the entire space $\mathrm{B}\mathcal{M}_4$ seems to be far beyond reach at present. In order to simplify this, there are several different directions Y in which one could try to stabilise. The following discusses some choices which are reasonable from one or another perspective.

5.1. THE CASE $Y = \overline{\mathbb{C}P}^2$. Stabilisation with respect to $Y = \overline{\mathbb{C}P}^2$ is motivated by complex algebraic geometry. Two complex algebraic surfaces are birationally equivalent if and only if they are related by a sequence of blow-ups. From the point of view of differential topology, the blow-up of a surface X is diffeomorphic to $X \# \overline{\mathbb{C}P}^2$.

PROPOSITION 3. *If two complex algebraic surfaces are birationally equivalent, then they are $\overline{\mathbb{C}P}^2$-stably equivalent, but (in general) not conversely.*

PROOF. The first part of the statement is clear from the discussion above.

While diffeomorphic surfaces will *a fortiori* be $\overline{\mathbb{C}P}^2$-stably equivalent, they need not be algebraically isomorphic. In fact, minimal models for non-ruled surfaces are algebraically unique, see III (4.6) in [BPV84] for example. Thus, it suffices to find two minimal surfaces which are diffeomorphic but not algebraically isomorphic, and there are plenty of those. □

In any case, this discussion leads to the question of finding smooth minimal models: representatives of the $\overline{\mathbb{C}P}^2$-stable equivalence classes. This is richer than the theory of complex algebraic surfaces:

PROPOSITION 4. *Not every $\overline{\mathbb{C}P}^2$-stable equivalence class is representable by a complex surface.*

PROOF. For complex surfaces, the sum of the Euler characteristic and the signature is divisible by 4 by Noether's formula. Therefore, the sphere S^4 (and the connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$ and...) is not $\overline{\mathbb{C}P}^2$-stably equivalent to a complex surface. □

5.2. THE CASE $Y = S^2 \times S^2$. This is the classical case, and it corresponds to stabilising the intersection form with respect to hyperbolic planes. It is known, by a result of Wall's, see [Wal64b], that two simply-connected 4-manifolds are $(S^2 \times S^2)$-stably diffeomorphic if and only their intersection forms are isomorphic. It is now easy to make a list of the $(S^2 \times S^2)$-stable equivalence classes.

PROPOSITION 5. *The different* $(S^2 \times S^2)$*-stable equivalence classes are*

$$S^4 \qquad mK3 \qquad m\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \qquad \mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2,$$

where $m \geqslant 1$ *and* mX *is again short for the* m*-fold connected sum of* X *with itself.*

PROOF. We have to represent all the stable isomorphism classes of intersection forms of 4-manifolds with respect to orthogonal summation with hyperbolic planes. In case the form is even, it is stably determined be the (even) number $2m$ of E_8 summands. These are stably represented by $mK3$ (or S^4 if $m = 0$). In case the form is odd, the orthogonal sum with the hyperbolic plane is indefinite, so these are stably represented by $m\mathbb{C}P^2 \# n\overline{\mathbb{C}P}^2$ with some $m, n \geqslant 1$. But the existence of a diffeomorphism

$$(4) \qquad (S^2 \times S^2) \# (\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2) \cong (\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2) \# (\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2)$$

implies that, stably, one of m and n may be chosen to be 1. $\qquad \square$

As for the $(S^2 \times S^2)$-stable diffeomorphism groups, only the groups of components, the stable mapping class groups, have been studied so far, again initiated by Wall [Wal64a]. See also [Qui86].

The $(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2)$-stable case is related, but different. Note that the existence of a diffeomorphism (4) implies that $(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2)$-stabilisation is coarser than stabilisation with respect to $(S^2 \times S^2)$; it neglects the distinction between (real) spin and non-spin 4-manifolds. See [Gia] for more on this case.

5.3. THE CASE $Y = K3$. This is related to the previous example, since the intersection form of $S^2 \times S^2$ is a direct summand in that of $K3$, but different, since $S^2 \times S^2$ is not a connected summand of $K3$. The final section discusses this at the level of the (stable) Bauer-Furuta invariants, see Theorem 2. At present, it is unknown whether any two homotopy equivalent, simply-connected 4-manifolds are $K3$-stably diffeomorphic, see Problem 4.7 in [Bau04].

6. BAUER-FURUTA INVARIANTS

The aim of this section is to show that the Bauer-Furuta invariants, see [BF04], can be used to define *stable* characteristic classes for families of 4-manifolds. Let me first recall how they have been extended, in [Szy], to define *unstable* characteristic classes for families of 4-manifolds.

The Bauer-Furuta invariant of a 4-manifold X depends on the choice of a complex spin structure σ_X on X. Complex spin families are classified by an extension $\mathrm{BSpiff}^c(X, \sigma_X)$ of $\mathrm{BDiff}(X)$ by the gauge group, which in this case is equivalent to $\mathbb{T}$ again. Let $\lambda = \lambda(\sigma_X)$ be the virtual vector bundle over $\mathrm{BSpiff}^c(X, \sigma_X)$ which is the difference of the index bundle of the Dirac operator (associated to σ_X) and the bundle of self-dual harmonic 2-forms, and denote its Thom spectrum by $\mathrm{BSpiff}^c(X, \sigma_X)^\lambda$. Then there is a universal characteristic class for complex spin families with typical fibre (X, σ_X) living in the 0-th stable cohomotopy group $\pi_{\mathbb{T}}^0(\mathrm{BSpiff}^c(X, \sigma_X)^\lambda)$ of this Thom spectrum. The family X over the singleton is classified by a map $S^\lambda \to \mathrm{BSpiff}^c(X, \sigma_X)^\lambda$, along which the universal class pulls back to the Bauer-Furuta invariant of (X, σ_X) living in $\pi_{\mathbb{T}}^0(S^\lambda)$.

By Bauer's connected sum theorem, see [Bau04], the Bauer-Furuta invariant of a connected sum $(X\#Y, \sigma_{X\#Y})$ is the smash product

$$S^{\lambda(X\#Y)} \cong S^{\lambda(X)} \wedge S^{\lambda(Y)} \longrightarrow S^0 \wedge S^0 = S^0$$

of the invariants of the summands. We will need an extension of that result to families in order to define stable characteristic classes below. This will involve based manifolds, so to keep notation reasonable, the complex spin structure will be omitted from it if confusion seems unlikely.

Let (X, x) be a based complex spin 4-manifold, and (Y, y', y) be a complex spin 4-manifold with respect to which we want to stabilise. If $\mathcal{X}$ is a based family of complex spin 4-manifolds with typical fibre (X, x) over B, the base yields a thickened section $B \times \mathbb{D}^4 \to \mathcal{X}$ of the projection $\mathcal{X} \to B$. Such families are classified by maps from B to $\mathrm{BSpiff}^c(X, x)$. Similarly, the product family Y_B over B with fibre Y comes with two disjoint thickened sections. The fibrewise connected sum of $\mathcal{X}$ and Y_B is another based family over B, say $\mathcal{X}\#_B Y_B$, this time with typical fibre $X\#Y$. If $\mathcal{X}$ is the universal

family over $B = \mathrm{BSpiff}^c(X, x)$, the family $\mathfrak{X} \#_B Y_B$ is classified by a map

$$(5) \qquad\qquad \mathrm{BSpiff}^c(X, x) \longrightarrow \mathrm{BSpiff}^c(X \# Y, y).$$

This gives a geometric interpretation, on the level of classifying spaces, of the maps in the colimit (3) defining $\mathrm{BSpiff}^c(X \# \infty Y, y)$.

PROPOSITION 6. *With the notation from the paragraph above, the family invariant of $\mathfrak{X} \#_B Y_B$ is the fibrewise smash product of the family invariants of $\mathfrak{X}$ and Y_B.*

PROOF. As long as long the glueing happens over product bundles of cylinders $B \times (S^3 \times I)$, Bauer's proof of his connected sum theorem given in §2 and §3 of [Bau04] can be adapted to families using the same homotopies extended constant in the B-direction. This is exactly what the thickened sections have been chosen for in the case at hand. $\qquad\square$

The map (5) induces a map

$$\mathrm{BSpiff}^c(X, x)^{\lambda(X \# Y)} \longrightarrow \mathrm{BSpiff}^c(X \# Y, y)^{\lambda(X \# Y)}$$

between Thom spectra. The family invariant of the product family Y_B is just the pullback of the Bauer-Furuta invariant of Y along the map induced by the map from B to the singleton. In particular, the virtual vector bundle $\lambda(Y)$ is trivial. This leads to an identification

$$\mathrm{BSpiff}^c(X, x)^{\lambda(X \# Y)} \simeq \mathrm{BSpiff}^c(X \# Y, y)^{\lambda(X)} \wedge S^{\lambda(Y)}.$$

Using this and Proposition 6 above, it follows by naturality of the family invariants that the induced map in cohomotopy sends the invariant of the universal family over $\mathrm{BSpiff}^c(X \# Y, y)$ to the smash product of the invariant of the universal family over $\mathrm{BSpiff}^c(X, x)$ with the invariant of the product family Y_B.

The easiest case would be that the invariant of the product family Y_B is the identity over B. This happens for $Y = \overline{\mathbb{C}P}^2$ with its standard complex spin structure.

PROPOSITION 7. *The Bauer-Furuta invariant of $\overline{\mathbb{C}P}^2$ is the class of the identity $S^0 \to S^0$.*

This is well-known, see [Bau04]. Thus, the map induced by

$$\mathrm{BSpiff}^c(X, x) \longrightarrow \mathrm{BSpiff}^c(X \# \overline{\mathbb{C}P}^2, y)$$

in the cohomotopy of the Thom spectra sends the invariant of the universal family, which lives over $\mathrm{BSpiff}^c(X \# \overline{\mathbb{C}P}^2, y)$, to the invariant of the universal family over $\mathrm{BSpiff}^c(X, x)$. These classes therefore define an element of the limit of the $\pi_{\mathbb{T}}^0(\mathrm{BSpiff}^c(X \# n\overline{\mathbb{C}P}^2, y)^{\lambda(X)})$, so that there is a universal classes over the colimit of these Thom spectra. Summing up, this proves the following result.

THEOREM 1. *The Bauer-Furuta invariants define a stable universal characteristic class which lives in the group* $\pi_{\mathbb{T}}^0(\mathrm{BSpiff}^c(X \# \infty\overline{\mathbb{C}P}^2, y)^{\lambda(X)})$.

As a corollary of this, which may also be deduced from [Bau04] directly, the Bauer-Furuta invariants of X and $X \# \overline{\mathbb{C}P}^2$ agree:

COROLLARY. *The Bauer-Furuta invariants only depend on* $\overline{\mathbb{C}P}^2$*-stable equivalence classes.*

The strength of the theorem as compared to its corollary lies in the fact that it gives information on the entire classifying space, not just on the set of its components.

7. LOCALISATION

In the previous section, we have seen that the Bauer-Furuta classes are invariants of the $\overline{\mathbb{C}P}^2$-stable diffeomorphism category of complex spin 4-manifolds. The reason for this was that the invariant of $\overline{\mathbb{C}P}^2$ itself is the identity map. However, for the general Y-stable case, the invariant of Y need not be the identity. In fact, it need not even be invertible; but, it can be formally inverted as in Section 3. This will be done in this section, in order to define invariants of the Y-stable diffeomorphism type of X even in the case when the invariant of Y is not invertible. For clarity, the focus will be on plain manifolds here; general families should be treated using the notation from the previous section.

If $f\colon A \to B$ is a stable map, smashing with f defines the morphism groups

$$[M, N]_f \overset{\text{def}}{=} \operatorname{colim}_k [M \wedge A^{\wedge k}, N \wedge B^{\wedge k}]$$

in the localisation of the stable homotopy category with respect to the endofunctor $? \wedge f$. Similar notation will be used in the equivariant case. Localisation away from Euler classes of representations has a long tradition [tD71].

This construction will now be applied in the case $f = m(Y)\colon S^{\lambda(Y)} \to S^0$, the Bauer-Furuta invariant of the 4-manifold Y with respect to which we want to stabilise. For every complex spin 4-manifold X as before, the Bauer-Furuta invariant of the connected sum $X \# Y$ is obtained from that of X by smashing it with $m(Y)$, by Bauer's theorem again. This implies that the sequence of invariants $m(X \# kY)$, for varying k, defines an element

$$(6) \qquad\qquad m(X \# \infty Y) \in [S^{\lambda(X)}, S^0]_{m(Y)}^{\mathbb{T}}$$

in the localisation of the $\mathbb{T}$-equivariant stable homotopy category with respect to $m(Y)$. It turns out, however, that this is practically useless, except for manifolds Y with $b^+(Y) = 0$, such as $Y = \overline{\mathbb{C}P}^2$, since the $\mathbb{T}$-equivariant Bauer-Furuta invariants are known to be nilpotent otherwise. See [FKM07], for example. And clearly, if f is nilpotent, then the localisation with respect to f leads to trivial groups.

However, if Y is real spin, it is known that there is a lift of the Bauer-Furuta invariant $m(Y)$ from the $\mathbb{T}$-equivariant to the $\mathbb{P}$-equivariant stable homotopy category, see [BF04]. Here and in the following, the group $\mathbb{P} = \operatorname{Pin}(2)$ is the normaliser of $\mathbb{T}$ inside $\operatorname{Sp}(1)$; it sits in an extension

$$1 \longrightarrow \mathbb{T} \longrightarrow \mathbb{P} \longrightarrow \mathbb{Z}/2 \longrightarrow 1.$$

One also has to specify a $\mathbb{P}$-universe, which in this case is of the form $\mathbb{R}^\infty \oplus D^\infty \oplus H^\infty$, where $\mathbb{R}$ is the trivial $\mathbb{P}$-line, D is the line with the action induced by the antipodal action of $\mathbb{P}/\mathbb{T} = \mathbb{Z}/2$, and H is the 4-dimensional tautological quaternion action. This universe is understood from now on. A useful reference for the homotopy theory in this context is [Sch03].

In the situation leading to (6), if X and Y are real spin, so is $X \# kY$ for all k. The same reasoning as above yields the following result.

THEOREM 2. *Let X and Y be real spin 4-manifolds. The sequence of invariants $m(X \# kY)$, for varying k, defines an element*

$$m(X \# \infty Y) \in [S^{\lambda(X)}, S^0]^{\mathbb{P}}_{m(Y)}$$

in the localisation of the $\mathbb{P}$-equivariant stable homotopy category with respect to $m(Y)$.

The invariant from the previous theorem will be referred to as the Y-*stable Bauer-Furuta invariant* of X. Before the general theory will be illustrated for two Y which are non-trivial in the sense that their $\mathbb{P}$-equivariant Bauer-Furuta invariants are non-nilpotent, $Y = S^2 \times S^2$ and $Y = K3$, the rest of this section will contain two general remarks.

First, the $\mathbb{P}$-equivariant Bauer-Furuta invariant of a real spin 4-manifold, stable or not, will by construction always restrict to the identity map of S^0 on $\mathbb{P}$-fixed points. In particular, it is never nilpotent. As the identity is invertible, there is an induced dashed arrow in the diagram

$$
\begin{array}{ccc}
[M, N]^{\mathbb{P}} & \longrightarrow & [M, N]^{\mathbb{P}}_{m(Y)} \\
& \searrow{\scriptstyle \Phi^{\mathbb{P}}} & \big\downarrow \\
& & [\Phi^{\mathbb{P}}(M), \Phi^{\mathbb{P}}(N)],
\end{array}
$$

and the observation above may be rephrased to say that the image of a Y-stable Bauer-Furuta invariant will always map down to the identity of S^0. This clearly gives restrictions on the possible values of the invariants.

Second, passage to $\mathbb{P}$-fixed points is also a localisation in the situation at hand.

PROPOSITION 8. *There is a natural isomorphism*

$$[\Phi^{\mathbb{P}} M, \Phi^{\mathbb{P}} N] \cong [M, N]^{\mathbb{P}}_{e(D \oplus H)}$$

for all $\mathbb{P}$-spectra M and N indexed on our $\mathbb{P}$-universe.

PROOF. In general, there is an isomorphism

$$[\Phi^{\mathbb{P}} M, \Phi^{\mathbb{P}} N] \cong [M, N \wedge \mathrm{colim}_U S^U]^{\mathbb{P}},$$

where the colimit is over the subrepresentations U of the universe which satisfy $U^{\mathbb{P}} = 0$. See [May96], XVI.6. In the case at hand, we have $U^{\mathbb{P}} = 0$ if and only if all the irreducible summands of U are isomorphic to D or H. This gives an isomorphism

$$[M, N \wedge \mathrm{colim}_U S^U]^{\mathbb{P}} \cong [M, N]^{\mathbb{P}}_{\mathrm{e}(D \oplus H)}$$

by taking the colimit out of the brackets. $\square$

8. EXAMPLES: $Y = S^2 \times S^2$ AND $Y = K3$

In this section, the $\mathbb{P}$-equivariant Y-stable Bauer-Furuta invariants will be discussed in the two cases $Y = S^2 \times S^2$ and $Y = K3$. Computations in our $\mathbb{P}$-equivariant stable homotopy category are much easier than in the general case due to the simplicity of the universe at hand. The stabilisers occuring here are only 1, $\mathbb{T}$, and $\mathbb{P}$, with Weyl groups $W1 = \mathbb{P}$, $W\mathbb{T} = \mathbb{Z}/2$, and $W\mathbb{P} = 1$, respectively. Only the latter two are finite. It follows that the Burnside ring $[S^0, S^0]^{\mathbb{P}}$ of $\mathbb{P}$ has rank 2. The following result is an immediate application of the splitting theorem of tom Dieck and Segal.

PROPOSITION 9. *For $n \geqslant 1$, the group $[S^0, S^{nD}]^{\mathbb{P}}$ is free abelian, generated by the n-th power of the Euler class $\mathrm{e}(D) \colon S^0 \to S^D$. An isomorphism is given by the mapping degree of the $\mathbb{P}$-fixed points.*

As $\Phi^{\mathbb{P}}(\eta \mathrm{e}(H))$ is the identity, the previous proposition implies the following result.

COROLLARY. *The relation*

$$(7) \qquad\qquad\qquad\qquad \eta \mathrm{e}(H) = e(D)^3$$

holds.

A similar statement to Proposition 9 can be established for H instead of D. As it will not be needed in the following, let us turn towards the examples now.

PROPOSITION 10. *The $\mathbb{P}$-equivariant Bauer-Furuta invariant of $S^2 \times S^2$ is the Euler class $\mathrm{e}(D)$ of D.*

PROOF. The index computation shows that the invariant lives in $[S^0, S^D]^{\mathbb{P}}$. By Proposition 9, it suffices to know map induced on the $\mathbb{P}$-fixed points. As already remarked in the previous section, this is always the identity. $\square$

PROPOSITION 11. *The $\mathbb{P}$-equivariant Bauer-Furuta invariant of a $K3$ surface is the $\mathbb{P}$-Hopf map $\eta \colon S^H \to S^{3D}$.*

PROOF. This time, the index computation shows that the invariant lives in $[S^H, S^{3D}]^{\mathbb{P}}$. The sphere S^H sits in a cofibre sequence

$$S(H)_+ \longrightarrow S^0 \longrightarrow S^H \longrightarrow \Sigma S(H)_+.$$

It follows that there is an induced exact sequence

$$[S(H)_+, S^{3D}]^{\mathbb{P}} \longleftarrow [S^0, S^{3D}]^{\mathbb{P}} \longleftarrow [S^H, S^{3D}]^{\mathbb{P}} \longleftarrow [\Sigma S(H)_+, S^{3D}]^{\mathbb{P}}.$$

Here, the unit sphere $S(H)$ in H is a free, 2-dimensional $\mathbb{P}$-CW-complex with orbit space the real projective plane $\mathbb{R}P^2$. Therefore, there are isomorphisms

$$[\Sigma^t S(H)_+, S^{3D}] \cong [\mathbb{R}P^2_+, S^{3-t}],$$

and these groups vanish for $t = 0, 1$. As a consequence, the middle map in the exact sequence is an isomorphism $[S^H, S^{3D}]^{\mathbb{P}} \cong [S^0, S^{3D}]^{\mathbb{P}}$, the latter group being isomorphic to the integers thanks to Proposition 9. Under this isomorphism, both $m(K3)$ and η are sent to $\mathrm{e}(D)^3$. $\square$

The relation (7) from the preceding corollary shows that η is invertible if $\mathrm{e}(D)$ is invertible, so that there is an arrow

$$[M, N]^{\mathbb{P}}_{\eta} \longrightarrow [M, N]^{\mathbb{P}}_{\mathrm{e}(D)}$$

for all $\mathbb{P}$-spectra M and N, which is the identity on representatives. Therefore, it sends the $K3$-stable invariant of a real spin 4-manifold to its $S^2 \times S^2$-stable invariant:

THEOREM 3. *If X is a real spin 4-manifold, the $\mathbb{P}$-equivariant $K3$-stable Bauer-Furuta invariant $m(X \# \infty K3)$ of X determines the $S^2 \times S^2$-stable invariant $m(X \# \infty (S^2 \times S^2))$.*

This may come as a surprise, in view of the fact that, while $S^2 \times S^2$ is algebraically a summand of $K3$ (in terms of orthogonal sums of intersection forms), it is not so geometrically (in terms of connected sums). See 5.3. above.

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