

The rational stable homology of mapping class groups of universal nil-manifolds

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We compute the rational stable homology of the automorphism groups of free nilpotent groups. These groups interpolate between the general linear groups over the ring of integers and the automorphism groups of free groups, and we employ bifunctor homology to reduce to the abelian case. As an application we compute the rational stable homology of the outer automorphism groups and the mapping class groups of the associated aspherical nil-manifolds in the TOP, PL, and DIFF categories.

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1 Introduction

For any integer $r \geq 1$, let F_r denote the free group on r generators. The automorphism groups $\text{Aut}(F_r)$ of these groups have been the subject of many studies,

beginning with Nielsen's work about a century ago, and leading, more recently, to substantial progress in the computation of their homology, in particular after stabilization: We can pass from r to $r + 1$ by sending the new generator to itself. In the (co)limit $r \rightarrow \infty$, we get the stable automorphism group $\text{Aut}(F_\infty)$. Note that this notation does *not* refer to an automorphism group of a group F_∞ . The homology of the group $\text{Aut}(F_\infty)$ is the colimit of the homologies of the groups $\text{Aut}(F_r)$. This stable homology has been computed by Galatius [Gal11]. In particular, it is rationally trivial, as had been conjectured by Hatcher and Vogtmann [HV98b].

The lower central series of the free group F_r is defined by the descending sequence of normal subgroups that is given by $\Gamma_1(F_r) = F_r$ and $\Gamma_{c+1}(F_r) = [F_r, \Gamma_c(F_r)]$. The quotient groups $N_c^r = F_r/\Gamma_{c+1}(F_r)$ are the free nilpotent groups of class $c \geq 1$.

For instance, the first class gives the free abelian groups $N_1^r \cong \mathbb{Z}^r$. Their automorphism groups are the general linear groups $\text{GL}_r(\mathbb{Z})$ over the ring \mathbb{Z} of integers, and their homology is related to the algebraic K-theory of the ring \mathbb{Z} . In contrast to the case of the groups $\text{Aut}(F_r)$, only very few stable homology groups are known integrally. This changes when one is willing to stabilize as above and work rationally. In that case a complete computation has been achieved by Borel [Bor74] using analytic methods. There is an isomorphism

$$H^*(\text{GL}_\infty(\mathbb{Z}); \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(r_5, r_9, r_{13}, \dots)$$

between the rational stable cohomology ring and a rational exterior algebra on generators in degrees $5, 9, 13, \dots, 4n + 1, \dots$. In contrast, the methods that we will use here are entirely algebraic and topological. They could, in principle, also be used to obtain information on torsion in homology. At present, such computations are out of reach, however. Compare Remark 5.4.

In this writing, we address the general (hence non-abelian) free c -nilpotent groups N_c^r of class $c \geq 2$ (and rank $r \geq 0$). This class contains the famous Heisenberg group of unit upper-triangular $(3, 3)$ -matrices with integral entries as N_2^2 . Its classifying space is the 3-dimensional Heisenberg manifold X_2^2 . In general, the groups N_c^r arise as fundamental groups of certain aspherical nil-manifolds X_c^r that can be presented as iterated torus bundles over tori.

We are here interested in the symmetries of these groups and manifolds, beginning with the automorphism groups $\text{Aut}(\mathbb{N}_c^r)$ of the groups \mathbb{N}_c^r , and their homology. Again, we can stabilize by passing from r to $r + 1$ by sending the new generator to itself. In the (co)limit $r \rightarrow \infty$, we get the stable automorphism group $\text{Aut}(\mathbb{N}_c^\infty)$. Again, this notation is *not* used to refer to the automorphism group of a group \mathbb{N}_c^∞ . The homology of the group $\text{Aut}(\mathbb{N}_c^\infty)$ is the colimit of the homologies of the groups $\text{Aut}(\mathbb{N}_c^r)$. The following result is a special case of Theorem 5.1 in the main text; The general case involves polynomial coefficients.

Theorem 1.1. *For all integers $c \geq 1$ the canonical homomorphism*

$$\text{Aut}(\mathbb{N}_c^r) \longrightarrow \text{GL}_r(\mathbb{Z})$$

given by abelianization is a rational homology isomorphism in the stable range.

As a geometric application, we consider the automorphism groups of the aspherical nil-manifolds X_c^r with fundamental groups isomorphic to \mathbb{N}_c^r . The following result summarizes Theorems 6.1 and 7.1.

Theorem 1.2. *For all integers $c \geq 1$ the canonical homomorphisms*

$$\text{Aut}(\mathbb{N}_c^r) \rightarrow \text{Out}(\mathbb{N}_c^r) \cong \pi_0 \mathbf{G}(X_c^r) \leftarrow \pi_0 \mathbf{TOP}(X_c^r) \leftarrow \pi_0 \mathbf{PL}(X_c^r) \leftarrow \pi_0 \mathbf{DIFF}(X_c^r)$$

of groups are rational homology isomorphisms in the stable range.

We need to comment on the “stable range” in the preceding two results: Stable homology computations such as those in the present paper produce most impact in the presence of homological stability: This guarantees that the stable homology determines infinitely many unstable values. Luckily, such results are already in place for the groups of interest here: Rational homological stability for the general linear groups over the ring \mathbb{Z} of integers is due to Borel [Bor74]. With integral coefficients, the result is due to Charney [Cha80], Maazen (unpublished), van der Kallen [vdK80], and Suslin [Sus82]. Homological stability results for the automorphism groups of free groups are more recent, see [Hat95] and [HV98a],

and have they been revisited by Bestvina [Bes] lately. For the automorphism groups of free nilpotent groups of an arbitrary class c , homological stability has been proven by the author in [Szy14].

The paper is outlined as follows. Section 2 contains the necessary background from homological algebra in functor categories and provides for a fundamental rational vanishing result, Proposition 2.5. Section 3 is about stable K-theory and recalls Scorichenko's theorem. In Section 4 we review the free nilpotent groups and their automorphisms. Section 5 contains a proof of Theorem 1.1. Sections 6 and 7 contain the applications to the outer automorphism groups and TOP, PL, and DIFF mapping class groups of the associated aspherical nil-manifolds.

2 Functor homology

In this section we briefly review some homological algebra in functor (and bifunctor) categories to the extent that we will need in the following sections. We also prove Proposition 2.5, a rational vanishing result that enters fundamentally into the proof of our main result.

2.1 Mac Lane homology

If \mathbf{C} is a small (or essentially small) category, then we write $\mathcal{F}(\mathbf{C})$ for the category of functors $\mathbf{C} \rightarrow \mathbf{Ab}$ from the category \mathbf{C} to the category \mathbf{Ab} of all abelian groups. This is an abelian category with enough projective objects to do homological algebra in.

We are particularly interested in the case when $\mathbf{C} = \mathbf{ab}$ is the essentially small category \mathbf{ab} of finitely generated free abelian groups. Then we have the inclusion functor $I: \mathbf{ab} \rightarrow \mathbf{Ab}$ at our disposal.

The following definition is due to Jibladze and Pirashvili [JP91, Def. 1.2].

Definition 2.1. The *Mac Lane homology* of the ring \mathbb{Z} of integers with coefficients in a functor $F \in \mathcal{F}(\mathbf{ab})$ is defined as

$$\mathrm{HML}_*(\mathbb{Z}; F) = \mathrm{Tor}_*^{\mathcal{F}(\mathbf{ab})}(\mathrm{I}^\vee, F),$$

where $\mathrm{I}^\vee : \mathbf{ab}^{\mathrm{op}} \rightarrow \mathbf{Ab}$ is the dual of the inclusion I .

Remark 2.2. A functor $F \in \mathcal{F}(\mathbf{ab})$ is additive if and only if it is given (up to isomorphism) by tensoring with an abelian group. In that case, Mac Lane homology is Hochschild homology of the \mathbb{Q} -construction $\mathbb{Q}(\mathbb{Z})$ with coefficients in that abelian group, thought of as a (\mathbb{Z}, \mathbb{Z}) -bimodule, see [JP91]. Also in that case, Mac Lane homology agrees with topological Hochschild homology (THH) of the Eilenberg–Mac Lane spectrum HZ with coefficients in the associated bimodule spectrum, see [PW92] and [FPSVW95].

We will now explain that the Mac Lane homology is a special case of the Hochschild (bifunctor) homology.

2.2 Hochschild (bifunctor) homology

If $D : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \rightarrow \mathbf{Ab}$ is a bifunctor, then the Hochschild homology of the category \mathbf{C} with coefficients in the bifunctor D is defined as

$$\mathrm{HH}_*(\mathbf{C}; D) = \mathrm{Tor}_*^{\mathcal{F}(\mathbf{C}^{\mathrm{op}} \times \mathbf{C})}(\mathrm{J}, D),$$

where $\mathrm{J} : (\mathbf{C}^{\mathrm{op}} \times \mathbf{C})^{\mathrm{op}} \rightarrow \mathbf{Ab}$ is the bifunctor with

$$\mathrm{J}(X, Y) = \mathbb{Z}\mathbf{C}(X, Y).$$

See [Lod98, App. C.10].

Example 2.3. If $\mathrm{pr}_2 : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \rightarrow \mathbf{C}$ is the projection, and $F : \mathbf{C} \rightarrow \mathbf{Ab}$ is a functor, then

$$\mathrm{HH}_*(\mathbf{C}; \mathrm{pr}_2^* F) \cong \mathrm{H}_*(\mathbf{C}; F), \tag{2.1}$$

so that this generalizes the homology of categories. One way to find an isomorphism (2.1) is to note that

$$\mathrm{pr}_2^* F = \mathbb{Z} \boxtimes F,$$

so that both sides of (2.1) are isomorphic to $\mathrm{Tor}_*^{\mathcal{F}(\mathbf{C})}(\mathbb{Z}, F)$ by [FP03, Prop. 2.10]. Here \mathbb{Z} denotes the constant functor with value \mathbb{Z} .

The most important case for us is this: If $\mathbf{C} = \mathbf{ab}$ and $F: \mathbf{ab} \rightarrow \mathbf{Ab}$ is an ordinary functor, let $\mathbf{I}^\vee \boxtimes F$ be the bifunctor with

$$(\mathbf{I}^\vee \boxtimes F)(X, Y) = \mathrm{Hom}(X, FY).$$

Proposition 2.4. *For all functors $F \in \mathcal{F}(\mathbf{ab})$ there is an isomorphism*

$$\mathrm{HH}_*(\mathbf{ab}; \mathbf{I}^\vee \boxtimes F) \cong \mathrm{HML}_*(\mathbb{Z}; F).$$

Proof. Again, both sides are isomorphic to $\mathrm{Tor}_*^{\mathcal{F}(\mathbf{ab})}(\mathbf{I}^\vee, F)$, see [Lod98, 13.1.7, 13.2.17], for instance. \square

2.3 A rational vanishing results

The following rational vanishing result will be the main computational input in our proof of Theorem 1.1. It refers to the Lie functors $\mathrm{Lie}^b: \mathbf{ab} \rightarrow \mathbf{Ab}$, i.e. the degree b homogeneous part of the free Lie algebra functor. See Section 4 for more background. We will abbreviate

$$\mathrm{Lie}^{[a,c]} = \bigoplus_{b=a}^c \mathrm{Lie}^b,$$

so that $\mathrm{Lie}^{[1,c]}$ is nothing but the free nilpotent Lie algebra of class c .

Proposition 2.5. *For all $c \geq 2$ and $q \geq 1$, the Hochschild homology groups*

$$\mathrm{HH}_*(\mathbf{ab}; (\mathbf{I}^\vee \boxtimes \mathrm{Lie}^{[2,c]})^{\otimes q})$$

vanish rationally.

Proof. The first case $q = 1$ is easy to check directly, because

$$\mathrm{HH}_*(\mathbf{ab}; \mathbb{I}^\vee \boxtimes \mathrm{Lie}^{[2,c]}) \cong \mathrm{HML}_*(\mathbb{Z}; \mathrm{Lie}^{[2,c]})$$

is isomorphic to the Mac Lane homology of the ring \mathbb{Z} of integers with coefficients in the functor $\mathrm{Lie}^{[2,c]}$ by Proposition 2.4. The Lie functors Lie^b are rationally retracts of the tensor powers \otimes^b : There are morphisms

$$\mathrm{Lie}^b \longrightarrow \otimes^b \longrightarrow \mathrm{Lie}^b$$

such that their composition is multiplication with b [Sch66, Prop. 3.3]. Since the functor \otimes^b has vanishing Mac Lane homology by Pirashvili's Lemma (compare [FP98, Lem. 2.2]), so have the functors Lie^b , and therefore also their direct sum

$$\mathrm{Lie}^{[2,c]} = \bigoplus_{b=2}^c \mathrm{Lie}^b.$$

If we are in the other cases, so that $q \geq 2$, then there are isomorphisms

$$\mathrm{HH}_*(\mathbf{ab}; (\mathbb{I}^\vee \boxtimes \mathrm{Lie}^b)^{\otimes q}) \cong \mathrm{HH}_*(\mathbf{ab}^{\times q}; (\mathbb{I}^\vee \boxtimes \mathrm{Lie}^b)^{\boxtimes q}) \cong \mathrm{HH}_*(\mathbf{ab}; (\mathbb{I}^\vee \boxtimes \mathrm{Lie}^b)^{\otimes q})$$

by the Künneth theorem. (We are working rationally.) Using what has been shown in the first part of the proof, this vanishes rationally. By additivity again, this proves our claim in the other cases as well. \square

Remark 2.6. In the case $c = 2$ and $q \geq 1$ of the preceding result, the Mac Lane cohomology of the ring of integers with coefficients in the functor $\mathrm{Lie}^2 = \Lambda^2$ has been computed integrally by Franjou and Pirashvili [FP98, Cor. 2.3]. It is all 2-torsion.

3 Stable K-theory

There is a simple trick (due to Waldhausen, compare [Wal79, Sec. 6]) that helps us compute the stable homology

$$\mathrm{H}_*(\mathrm{GL}_\infty(\mathbb{Z}); D_\infty)$$

of the general linear groups with twisted coefficients in a bifunctor D : We can use the Serre spectral sequence for the fibration

$$X \longrightarrow \mathrm{BGL}_\infty(\mathbb{Z}) \longrightarrow \mathrm{BGL}_\infty(\mathbb{Z})^+,$$

where the space X is the homotopy fiber of the Quillen plus construction. The groups on the E^2 page take the form $H_*(\mathrm{BGL}_\infty(\mathbb{Z})^+; H_*(X; D))$.

Remark 3.1. In $H_*(X; D)$ the action of the fundamental group on the coefficients is now trivial, see [Kas82, Thm. 3.1] and [Kas83, Pf. of Thm. 2.16]. We will not need this fact, since we will show that in our situation the coefficients itself will be trivial.

Since the Quillen plus construction is an acyclic map, it induces homology isomorphisms with respect to all coefficients (twisted and untwisted). Therefore, we can remove the plus to get a spectral sequence

$$E_{s,t}^2 \implies H_{s+t}(\mathrm{GL}_\infty(\mathbb{Z}); D_\infty), \quad (3.1)$$

with

$$E_{s,t}^2 = H_s(\mathrm{BGL}_\infty(\mathbb{Z}); K_t^{\mathrm{st}}(\mathbb{Z}; D)) \quad (3.2)$$

where, by definition,

$$K_*^{\mathrm{st}}(\mathbb{Z}; D) = H_*(X; D) \quad (3.3)$$

is the *stable K-theory* of the ring of integers with coefficients in the bifunctor D .

Remark 3.2. It has been pointed out by Bökstedt that this spectral sequence often degenerates. See also [BP94, Pf. of Thm. 2]. Again, this will be obvious in our situation, because the entire E^2 page will be trivial.

The definition (3.3) of stable K-theory is topological in nature. We will need the following result that gives an algebraic description of it.

Theorem 3.3 (Scorichenko). *For a ring R , let \mathbf{mod}_R be the category of finitely generated projective left R -modules, and let D be a bifunctor on it. If D has finite degree with respect to both variables, then there is an isomorphism between Waldhausen's stable K -theory and the Hochschild homology of \mathbf{mod}_R :*

$$K_*^{\text{st}}(R; D) \cong \text{HH}_*(\mathbf{mod}_R; D).$$

See the exposition [FP03] by Franjou and Pirashvili. The result is stated as Theorem 1.1 there, and the proof given is complete for rings R with the property that submodules of finitely generated projective left R -modules are still finitely generated and projective. This property is satisfied for $R = \mathbb{Z}$, which is the only case that we will be using here. See also [Dja12, 5.2] for an enlightening discussion.

4 Free nilpotent groups and their automorphisms

In this section, we present some basic results about the free nilpotent groups and their automorphisms. Most of this must be well-known, and we can refer to the exposition in [Szy14], for instance. We will, however, prove the two Propositions 4.1 and 4.3 which will need later be used.

Let G be a (discrete) group. For integers $n \geq 1$, the subgroups $\Gamma_n(G)$ are defined inductively by $\Gamma_1(G) = G$ and $\Gamma_{n+1}(G) = [G, \Gamma_n(G)]$. We also set $\Gamma_\infty(G)$ to be the intersection of all the $\Gamma_n(G)$. This gives a series

$$G = \Gamma_1(G) \geq \Gamma_2(G) \geq \dots \geq \Gamma_\infty(G)$$

of normal subgroups, the *descending/lower central series* of G . The associated graded group is abelian, and the commutator bracket induces the structure of a graded Lie algebra $\text{Lie}(G)$ on it.

Let F_r denote a free group on a set of r generators. In this case, an old theorem of Magnus says that the subgroup $\Gamma_\infty(F_r)$ is trivial, so that the free groups F_r are

residually nilpotent. The canonical homomorphism from a free Lie algebra on a set of r generators to the associated graded Lie algebra of F_r is an isomorphism. In particular

$$\Gamma_n(F_r)/\Gamma_{n+1}(F_r) \cong \text{Lie}^n(\mathbb{Z}^r) \quad (4.1)$$

is a free abelian group, the degree n homogeneous part of the free Lie algebra on a set of r generators. For instance, we have $\text{Lie}^1 = \text{Id}$ and $\text{Lie}^2 = \Lambda^2$.

4.1 Free nilpotent groups

The universal examples of nilpotent groups of class $c \geq 1$ are the quotients

$$N_c^r = F_r/\Gamma_{c+1}(F_r).$$

As the two extreme cases, we obtain $N_1^r \cong \mathbb{Z}^r$ and $N_\infty^r \cong F_r$. It follows from (4.1) that there are extensions

$$0 \longrightarrow \text{Lie}^c(\mathbb{Z}^r) \longrightarrow N_c^r \longrightarrow N_{c-1}^r \longrightarrow 1 \quad (4.2)$$

of groups.

We will later need the following structural result on the rational homology of the groups N_c^r .

Proposition 4.1. *For every class $c \geq 1$ and every degree d , the rational homology group $H_d(N_c^r; \mathbb{Q})$ is polynomial (of degrees at most cd) in r .*

Proof. Since we are only interested in the rational homology of a nilpotent group, we may just as well consider the rationalization $N_c^r \otimes \mathbb{Q}$. We have a natural isomorphism

$$H_d(N_c^r; \mathbb{Q}) \cong H_d(N_c^r \otimes \mathbb{Q}; \mathbb{Q})$$

by construction. The category of uniquely divisible nilpotent groups is equivalent to the category of nilpotent Lie algebras over \mathbb{Q} , and we get a natural isomorphism

$$H_d(N_c^r; \mathbb{Q}) \cong H_d(\text{Lie}(N_c^r \otimes \mathbb{Q}); \mathbb{Q}),$$

as demonstrated by Nomizu [Nom54], see also Pickel [Pic78]. Now this Lie algebra $\text{Lie}(\mathbb{N}_c^r \otimes \mathbb{Q})$ of the group of interest is nothing else than the free nilpotent Lie algebra $\text{Lie}^{[1,c]}(\mathbb{Q}^r)$ on r generators, and this is clearly functorial in \mathbb{Q}^r (and in particular in \mathbb{Z}^r) of degree c . The d -th homology of a Lie algebra \mathfrak{g} is the homology of a resolution

$$\dots \longleftarrow \Lambda^d \mathfrak{g} \longleftarrow \dots$$

As a subquotient of the functor $\Lambda^d \text{Lie}(\mathbb{N}_c^r \otimes \mathbb{Q})$, which is of degree cd , the homology is then also functorial of degree at most cd . \square

4.2 Automorphisms

We are interested in the groups $\text{Aut}(\mathbb{N}_c^r)$ of automorphisms. In the case $c = 1$, these are the general linear groups $\text{GL}_r(\mathbb{Z})$, and in the limiting case $c = \infty$, these are the automorphism groups $\text{Aut}(F_r)$ of free groups.

Proposition 4.2. *There are extensions*

$$0 \longrightarrow \text{Hom}(\mathbb{Z}^r, \text{Lie}^c(\mathbb{Z}^r)) \longrightarrow \text{Aut}(\mathbb{N}_c^r) \longrightarrow \text{Aut}(\mathbb{N}_{c-1}^r) \longrightarrow 1 \quad (4.3)$$

of groups.

See again [Szy14, Prop. 3.1], for instance. Note that any element of the quotient group $\text{Aut}(\mathbb{N}_{c-1}^r)$ acts on the kernel group via restriction along the canonical projection $\text{Aut}(\mathbb{N}_{c-1}^r) \rightarrow \text{Aut}(\mathbb{N}_1^r) = \text{GL}_r(\mathbb{Z})$. For that reason, it is sometimes more efficient to study the extensions

$$1 \longrightarrow \text{IA}_c^r \longrightarrow \text{Aut}(\mathbb{N}_c^r) \longrightarrow \text{GL}_r(\mathbb{Z}) \longrightarrow 1 \quad (4.4)$$

of groups. The kernels IA_c^r are known to be nilpotent of class $c - 1$. See for instance Andreadakis [And65, Cor. 1.3.]. By what has already been said, it is clear that they differ by free abelian groups in the sense that there are extensions

$$0 \longrightarrow \text{Hom}(\mathbb{Z}^r, \text{Lie}^c(\mathbb{Z}^r)) \longrightarrow \text{IA}_c^r \longrightarrow \text{IA}_{c-1}^r \longrightarrow 1. \quad (4.5)$$

We can now put this together to get the following result that will later help us to estimate homology.

Proposition 4.3. *The $\mathrm{GL}_\infty(\mathbb{Z})$ -module $H_q(\mathrm{IA}_c^\infty)$ is a subquotient of the $\mathrm{GL}_\infty(\mathbb{Z})$ -module $\Lambda^q(\mathbf{I}^\vee \boxtimes \mathrm{Lie}^{[2,c]})_\infty$.*

Proof. We consider the Lyndon–Hochschild–Serre spectral sequences associated with the extensions (4.5). We have

$$E_{s,t}^2 \implies H_{s+t}(\mathrm{IA}_c^r)$$

with

$$E_{s,t}^2 = H_s(\mathrm{IA}_{c-1}^r; H_t \mathrm{Hom}(\mathbb{Z}^r, \mathrm{Lie}^c(\mathbb{Z}^r))).$$

As remarked above, it is evident that the action of the kernels IA_{c-1}^r of the canonical projections $\mathrm{Aut}(\mathbb{N}_c^r) \rightarrow \mathrm{GL}_r(\mathbb{Z})$ on the groups $\mathrm{Hom}(\mathbb{Z}^r, \mathrm{Lie}^c(\mathbb{Z}^r))$ are trivial. Therefore, we can write

$$E_{s,t}^2 \cong H_s(\mathrm{IA}_{c-1}^r) \otimes \Lambda^t \mathrm{Hom}(\mathbb{Z}^r, \mathrm{Lie}^c(\mathbb{Z}^r)),$$

using that the homology of the free abelian groups is given by the exterior powers. Passing to the colimit, we see that $H_q(\mathrm{IA}_c^\infty)$ is a subquotient of the degree q part of

$$H_s(\mathrm{IA}_{c-1}^\infty) \otimes \Lambda^t(\mathbf{I}^\vee \boxtimes \mathrm{Lie}^c)_\infty.$$

By induction on c , this is isomorphic to the degree q part of

$$\Lambda^*(\mathbf{I}^\vee \boxtimes \mathrm{Lie}^2)_\infty \otimes \cdots \otimes \Lambda^*(\mathbf{I}^\vee \boxtimes \mathrm{Lie}^c)_\infty,$$

which is $\Lambda^q(\mathbf{I}^\vee \boxtimes \mathrm{Lie}^{[2,c]})_\infty$, as claimed. \square

Remark 4.4. If F is a (polynomial) functor from the category of finitely generated free abelian groups to the category of \mathbb{Q} -vector spaces, say, then Proposition 4.3 generalizes as follows: The $\mathrm{GL}_\infty(\mathbb{Z})$ -module $H_q(\mathrm{IA}_c^\infty) \otimes F$ is a subquotient of the $\mathrm{GL}_\infty(\mathbb{Z})$ -module $\Lambda^q(\mathbf{I}^\vee \boxtimes \mathrm{Lie}^{[2,c]})_\infty \otimes F$. The proof is the same as above, except that F has to be inserted in the appropriate places as coefficients.

5 A proof of Theorem 1.1

We will now prove the following homological version of Theorem 1.1 from the introduction with polynomial coefficients.

Theorem 5.1. *Let F be a polynomial functor from the category \mathbf{ab} of finitely generated free abelian groups to the category of \mathbb{Q} -vector spaces. For every integer $c \geq 1$ the canonical homomorphism*

$$H_*(\mathrm{Aut}(\mathbb{N}_c^\infty); F) \longrightarrow H_*(\mathrm{GL}_\infty(\mathbb{Z}); F) \quad (5.1)$$

is an isomorphism.

Remark 5.2. By Betley's work [Bet92] on the homology of the general linear groups with twisted coefficients, which can be seen as a predecessor of Scorichenko's result [FFSS99], both sides of (5.1) vanish if the functor F is reduced, that is if we have $F(0) = 0$.

In the case of the constant functor F with $F(\mathbb{Z}^r) = \mathbb{Q}$, we see that the natural homomorphism $\mathrm{Aut}(\mathbb{N}_c^\infty) \rightarrow \mathrm{GL}_\infty(\mathbb{Z})$ is a rational homology equivalence for every integer $c \geq 1$, as claimed in the introduction.

By 2-out-of-3, we get the following corollary.

Corollary 5.3. *For every integer $c \geq 2$ the canonical homomorphism*

$$H_*(\mathrm{Aut}(\mathbb{N}_c^\infty); \mathbb{Q}) \longrightarrow H_*(\mathrm{Aut}(\mathbb{N}_{c-1}^\infty); \mathbb{Q}) \quad (5.2)$$

is an isomorphism.

Proof of Theorem 5.1. We will assume that the coefficients are constant, so that they do not clutter the notation. Remark 4.4 applies.

The homomorphism (5.1) in question is the edge homomorphism in the Lyndon–Hochschild–Serre spectral sequence

$$E_{p,q}^2 \implies H_{p+q}(\mathrm{Aut}(\mathbb{N}_c^\infty))$$

for the colimit of the extensions (4.4), with E^2 page

$$E_{p,q}^2 = H_p(\mathrm{GL}_\infty(\mathbb{Z}); H_q(\mathrm{IA}_c^\infty)). \quad (5.3)$$

In fact, the 0-line is just the homology $H_*(\mathrm{GL}_\infty(\mathbb{Z}))$ of the stable general linear group. We will show that the other groups (i.e. the lines with $q \geq 1$) on the E^2 page vanish rationally:

$$H_*(\mathrm{GL}_\infty(\mathbb{Z}); H_q(\mathrm{IA}_c^\infty)) = 0 \quad (5.4)$$

for all $c \geq 1$ and $q \geq 1$; This will imply the result.

We will prove (5.4) using very crude estimates (always originating from spectral sequences). The first one is Proposition 4.3, which implies that it suffices to be shown that

$$H_*(\mathrm{GL}_\infty(\mathbb{Z}); \Lambda^q(\mathbb{I}^\vee \boxtimes \mathrm{Lie}^{[2,c]})_\infty) = 0 \quad (5.5)$$

for all $c \geq 1$ and $q \geq 1$. As explained in Section 3, such twisted homology can be computed from the stable K-theory $K_*^{\mathrm{st}}(\mathbb{Z}; \Lambda^q(\mathbb{I}^\vee \boxtimes \mathrm{Lie}^{[2,c]}))$ using the spectral sequence (3.1) and (3.2):

$$E_{s,t}^2 \implies H_{s+t}(\mathrm{GL}_\infty(\mathbb{Z}); \Lambda^q(\mathbb{I}^\vee \boxtimes \mathrm{Lie}^{[2,c]})_\infty)$$

with

$$E_{s,t}^2 = H_s(\mathrm{GL}_\infty(\mathbb{Z}); K_t^{\mathrm{st}}(\mathbb{Z}; \Lambda^q(\mathbb{I}^\vee \boxtimes \mathrm{Lie}^{[2,c]}))).$$

The bifunctor $\Lambda^q(\mathbb{I}^\vee \boxtimes \mathrm{Lie}^{[2,c]})$ at hand is of finite degree in each variable, so that we can use Scorichenko's Theorem 3.3 to the effect that stable K-theory is bifunctor homology,

$$K_*^{\mathrm{st}}(\mathbb{Z}; \Lambda^q(\mathbb{I}^\vee \boxtimes \mathrm{Lie}^{[2,c]})) \cong \mathrm{HH}_*(\mathbf{ab}; \Lambda^q(\mathbb{I}^\vee \boxtimes \mathrm{Lie}^{[2,c]})).$$

Thus, it suffices to see that the right hand side is zero rationally. This is Proposition 2.5. \square

Remark 5.4. The preceding proof was written in a way that introduced rational coefficients only in the very last lines. Thus, it seems to suggest a strategy to

compute the *integral* homology of the group $\text{Aut}(\mathbf{N}_2^\infty)$, say. The reader is invited to try her or his luck, keeping in mind that, integrally, the homology of the stable general linear group $\text{GL}_\infty(\mathbb{Z})$ is unknown. Thus, integrally, even the 0-line of the spectral sequence (5.3) is unknown. Further up, only the 1-line is known, integrally, by the work of Franjou and Pirashvili, see Remark 2.6.

6 Outer automorphism groups

The outer automorphism group $\text{Out}(G)$ of a group G is the quotient in an extension

$$1 \longrightarrow \text{Inn}(G) \longrightarrow \text{Aut}(G) \longrightarrow \text{Out}(G) \longrightarrow 1,$$

with inner automorphism group $\text{Inn}(G) \cong G/Z(G)$ and the projection gives a factorization $\text{Aut}(G) \rightarrow \text{Out}(G) \rightarrow \text{Aut}(H_1G)$ of the canonical homomorphism to the automorphism group of the homology/abelianization: Inner automorphisms act trivially on homology.

6.1 Free nilpotent groups

In the case when $G = \mathbf{N}_c^r$ is a free nilpotent group, the center is the kernel of the canonical projection $\mathbf{N}_c^r \rightarrow \mathbf{N}_{c-1}^r$, so that we have

$$Z(\mathbf{N}_c^r) \cong \text{Lie}^c(\mathbb{Z}^r)$$

and

$$\text{Inn}(\mathbf{N}_c^r) \cong \mathbf{N}_{c-1}^r.$$

Consequently, we get an extension

$$1 \longrightarrow \mathbf{N}_{c-1}^r \longrightarrow \text{Aut}(\mathbf{N}_c^r) \longrightarrow \text{Out}(\mathbf{N}_c^r) \longrightarrow 1. \quad (6.1)$$

In parallel with (4.4), we define subgroups $\text{IO}_c^r \leq \text{Out}(\mathbf{N}_c^r)$ as the kernels in the extensions

$$1 \longrightarrow \text{IO}_c^r \longrightarrow \text{Out}(\mathbf{N}_c^r) \longrightarrow \text{GL}_r(\mathbb{Z}) \longrightarrow 1. \quad (6.2)$$

This gives extensions

$$1 \longrightarrow \mathbf{N}_{c-1}^r \longrightarrow \mathbf{IA}_c^r \longrightarrow \mathbf{IO}_c^r \longrightarrow 1 \quad (6.3)$$

of nilpotent groups of class $c - 1$.

This finishes our description of the outer automorphism groups $\text{Out}(\mathbf{N}_c^r)$ up to extensions. We can proceed to compute their homology.

6.2 Homology

Let F be a polynomial functor (of finite degree) from the category **ab** of finitely generated free abelian groups to the category of \mathbb{Q} -vector spaces. This defines twisted coefficients for the both groups $\text{Aut}(\mathbf{N}_c^\infty)$ and $\text{Out}(\mathbf{N}_c^\infty)$ via their homomorphisms to the general linear groups.

Theorem 6.1. *For every class $c \geq 1$ the canonical projection induces an isomorphism*

$$\mathbf{H}_*(\text{Aut}(\mathbf{N}_c^r); F(\mathbb{Z}^r)) \longrightarrow \mathbf{H}_*(\text{Out}(\mathbf{N}_c^r); F(\mathbb{Z}^r)) \quad (6.4)$$

in a stable range for all finite degree rational coefficient functors F .

Remark 6.2. The rational homology vanishes in the stable range if the functor F is reduced, by Theorem 5.1 and Remark 5.2 following it.

Remark 6.3. The existence of a stable range for the automorphism groups has been established in [Szy14], even integrally. The theorem implies the existence of a stable range for the outer automorphism groups in the same range, at least rationally.

Combining the specializations of both Theorems 5.1 and 6.1 to the case of constant coefficients we get:

Corollary 6.4. *The canonical projection induces an isomorphism*

$$\mathbf{H}_*(\text{Out}(\mathbf{N}_c^\infty); \mathbb{Q}) \longrightarrow \mathbf{H}_*(\text{GL}_\infty(\mathbb{Z}); \mathbb{Q})$$

in rational homology.

Proof of Theorem 6.1. We prove this by induction on the homological degree $*$.

We start the induction in degree $* = 0$. This case concerns the co-invariants. Since the groups $\text{Aut}(\mathbb{N}_c^r)$ surject onto the groups $\text{Out}(\mathbb{N}_c^r)$, the co-invariants are the same.

Let us now assume that the degree $*$ is positive, and that the result has already been proven for all smaller degrees than that. We consider the Lyndon–Hochschild–Serre spectral sequence

$$E_{s,t}^2(r) = H_s(\text{Out}(\mathbb{N}_c^r); H_t(\mathbb{N}_{c-1}^r; F(\mathbb{Z}^r))) \implies H_{s+t}(\text{Aut}(\mathbb{N}_c^r); F(\mathbb{Z}^r))$$

for the extension (6.1) in the stable range, that is for r so large that the homology groups under consideration represent the stable values. (There are only finitely many of them relevant at each given time.) We would like to show that the rows with $t \neq 0$ vanish, so that the edge homomorphism—which is the homomorphism (6.4) in question—is an isomorphism.

Since the action on the coefficients $F(\mathbb{Z}^r)$ factors through the quotient $\text{Out}(\mathbb{N}_c^r)$, the coefficients for the homology $H_t(\mathbb{N}_{c-1}^r; F(\mathbb{Z}^r))$ are actually untwisted.

$$H_t(\mathbb{N}_{c-1}^r; F(\mathbb{Z}^r)) \cong H_t(\mathbb{N}_{c-1}^r) \otimes F(\mathbb{Z}^r)$$

From Proposition 4.1 we deduce that this is a polynomial functor (of finite degree). And if $t \neq 0$, then this functor is reduced: Since the group \mathbb{N}_{c-1}^0 is trivial, so is the homology $H_t(\mathbb{N}_{c-1}^0)$ for $t \neq 0$. It follows by induction (and Remark 6.2) that the groups $E_{s,t}^2(r)$ vanish for $t \neq 0$, and we are left with an isomorphism

$$H_*(\text{Aut}(\mathbb{N}_c^r); F(\mathbb{Z}^r)) \cong E_{*,0}^2(r) = H_*(\text{Out}(\mathbb{N}_c^r); F(\mathbb{Z}^r)),$$

as desired. □

Remark 6.5. The basic set-up for this proof is the same as in [Szy14, Sec. 1]. There it was used to prove homological stability, whereas here it is used to compute stable homology.

7 Mapping class groups

The free nilpotent groups N_c^r of class c arise as fundamental groups of aspherical nil-manifolds: Upon passage to classifying spaces, the extensions (4.2) gives rise to iterated torus bundles over a tori. This manifold will be denoted by X_c^r here. We can now spell out some implications of our preceding results for the mapping class groups of the X_c^r in the various categories. Note that $\dim(X_c^r)$ increases with r , so that—in the stable range—surgery and concordance/pseudo-isotopy theory apply to compute the mapping class groups in the TOP, PL, and DIFF categories. To get there, we need to begin with the homotopy category:

For any discrete group G , the group $\text{Out}(G)$ is isomorphic to the group of components of the topological monoid $G(X)$ of self-equivalences of the classifying space X of G . For the groups $G = N_c^r$ with classifying spaces X_c^r , we therefore have an isomorphism

$$\pi_0 G(X_c^r) \cong \text{Out}(N_c^r) \tag{7.1}$$

of groups, and Corollary 6.4 directly gives the stable rational homology of the symmetries of the X_c^r manifolds in the homotopy category. We now explain how this extends to the TOP, PL, and DIFF categories:

Theorem 7.1. *For the nil-manifolds X_c^r , the canonical homomorphisms*

$$\text{DIFF}(X_c^r) \longrightarrow \text{PL}(X_c^r) \longrightarrow \text{TOP}(X_c^r) \longrightarrow G(X_c^r)$$

induce rational homology isomorphisms in the stable range for all integers $c \geq 1$.

The case $c = 1$ follows from the (independent) work of Hsiang–Sharpe [HS76] and Hatcher [Hat78].

Remark 7.2. The homology in the stable range can be computed using (7.1), Corollary 6.4, and Theorem 5.1.

Proof of Theorem 7.1. We start with the homeomorphisms. It is clear that the canonical homomorphism $\pi_0 G(X_c^r) \rightarrow \pi_0 \text{TOP}(X_c^r)$ is surjective: This follows

from surgery theory, for instance [Wal70, Thm. 15B.1], but it can also be seen explicitly, because the projection $\text{Aut}(\mathbb{N}_c^r) \rightarrow \text{Out}(\mathbb{N}_c^r)$ is evidently surjective, and $\text{Aut}(\mathbb{N}_c^r)$ tautologically acts on \mathbb{N}_c^r , on its completion $\mathbb{N}_c^r \otimes \mathbb{R}$, and therefore also on the quotient $X_c^r \cong \mathbb{N}_c^r \otimes \mathbb{R} / \mathbb{N}_c^r$. (This lift is actually algebraic, hence smooth.) We need to estimate the kernel of $\pi_0 \text{G}(X_c^r) \rightarrow \pi_0 \text{TOP}(X_c^r)$, which is isomorphic to the fundamental group $\pi_1(\text{G}(X_c^r) / \text{TOP}(X_c^r))$. It follows from surgery theory again that $\text{G}(X_c^r) / \widetilde{\text{TOP}}(X_c^r)$ is contractible, where the notation $\widetilde{\text{TOP}}$ as usual refers to the block homeomorphism group, so that

$$\pi_1(\text{G}(X_c^r) / \text{TOP}(X_c^r)) \cong \pi_1(\widetilde{\text{TOP}}(X_c^r) / \text{TOP}(X_c^r)),$$

and the latter can be determined from concordance/pseudo-isotopy theory: Work of Hatcher [Hat78] and Igusa [Igu84] shows that that group is dominated by π_0 of the concordance space of X_c^r . Waldhausen has shown in [Wal78] that the Whitehead space of the poly- \mathbb{Z} group \mathbb{N}_c^r is contractible (in the TOP and PL categories). It follows that the only possible contribution to π_0 is a quotient of the 2-torsion group $\mathbb{F}_2[\mathbb{N}_2^r] / \mathbb{F}_c$. We see that this is vanishing rationally (in fact, away from 2), so that

$$\pi_0 \text{TOP}(X_c^r) \longrightarrow \pi_0 \text{G}(X_c^r)$$

is a rational homology isomorphism.

Since the group $\pi_1(\widetilde{\text{CAT}}(X_c^r) / \text{CAT}(X_c^r))$ is independent of the category CAT of symmetries considered, we see that it vanishes rationally for the remaining two cases $\text{CAT} = \text{PL}$ and $\text{CAT} = \text{DIFF}$ as well. For this reason, the kernels of the homomorphisms

$$\pi_0 \text{DIFF}(X_c^r) \longrightarrow \pi_0 \text{PL}(X_c^r) \longrightarrow \pi_0 \text{TOP}(X_c^r) \tag{7.2}$$

between the mapping class groups in the various categories of structures are the same as the kernels between the corresponding blocked groups $\pi_0 \widetilde{\text{CAT}}(X_c^r)$, and these kernels are rationally trivial. This follows again from surgery theory: On the one hand, we have an equivalence between the quotient $\widetilde{\text{TOP}}(X_c^r) / \text{PL}(X_c^r)$ and the space of maps $X_c^r \rightarrow \text{TOP/PL}$, and the target TOP/PL is an Eilenberg–Mac Lane space of type $(\mathbb{Z}/2, 3)$. On the other hand, we similarly have an

equivalence between $\widetilde{\text{PL}}(X_c^r)/\text{DIFF}(X_c^r)$ and the space of maps $X_c^r \rightarrow \text{PL}/\text{O}$, and the target PL/O also has finite homotopy groups, essentially the groups of exotic spheres of Kervaire and Milnor. In summary, the homomorphisms (7.2) are rational homology equivalences in the stable range. \square

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