

# Bauer-Furuta invariants and Galois symmetries

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The Bauer-Furuta invariants of smooth 4-manifolds are investigated from a functorial point of view. This leads to a definition of equivariant Bauer-Furuta invariants for compact Lie group actions. These are studied in Galois covering situations. We show that the ordinary invariants of all quotients are determined by the equivariant invariants of the covering manifold. In the case where the Bauer-Furuta invariants can be identified with the Seiberg-Witten invariants, this implies relations between the invariants in Galois covering situations, and these can be illustrated through elliptic surfaces. It is also explained that the equivariant Bauer-Furuta invariants potentially contain more information than the ordinary invariants.

## Introduction

We make a step towards a structural understanding of the Bauer-Furuta invariants [2, 3] of smooth 4-manifolds with complex spin structures, which refine the Seiberg-Witten invariants. The preface of [6] points out the need for such a venture in the related context of Donaldson invariants. Our investigations here lead to a definition of an equivariant version of the Bauer-Furuta invariants in the context of compact Lie group actions. This can be related to the family version of the Bauer-Furuta invariants defined in [25], as explained in Section 7 of *loc. cit.*. See also [24].

Any attempt to define an equivariant extension of the Seiberg-Witten invariants will have to face the problem that invariant perturbations are not generic, that equivariant transversality does not hold in general, see for example the discussion in [21]. These problems can be circumvented by the homotopical approach. We show how the equivariant Bauer-Furuta invariants provide insights into Galois covering situations for any finite Galois group, and illustrate this by examples for groups of prime order.

Sections 1 and 2 discuss the functoriality of the monopole map and the resulting definition of equivariant Bauer-Furuta invariants. See [14] for motivation of the use of categorical language in global differential geometry. Theorem 1.3 shows that the monopole map is functorial on a certain gauge category

which takes the complex spin structures on the 4-manifolds into account. Theorems 2.1 and 2.2 concern the existence of equivariant Bauer-Furuta invariants well-related to the ordinary ones. Our discussion is kept as detailed and elementary as possible in order to emphasise the simplicity of the homotopical approach.

From Section 3 on the focus is on Galois covering situations. If  $G$  is a finite group, and  $X \rightarrow X/G$  is a  $G$ -covering, the Bauer-Furuta invariants of  $X/G$  can be recovered from the equivariant invariants of  $X$ , see Theorem 3.3. This suggests the definition of a ghost map which allows the comparison of the  $G$ -invariants of  $X$  with the ordinary invariants of all its quotients  $X/H$  for  $H \leq G$ . In nice situations, see Theorem 3.7, the ghost map is an isomorphism away from the order of  $G$ .

While Section 4 contains some preparatory calculations, Sections 5 and 6 study the ghost map integrally in the case when the order of  $G$  is a prime number  $p$ . In the case when the complex spin structure on  $X/G$  comes from an almost complex structure, the equivariant invariants of  $X$  can be computed from the ordinary invariants of  $X$  and  $X/G$ , i.e. from Seiberg-Witten invariants, see Theorem 5.1, which also says that the latter satisfy a mod  $p$  congruence. This is illustrated through certain elliptic surfaces, where these relations are equivalent to congruences between binomial co-efficients. Finally, there may be more to the equivariant invariants than just the ordinary invariants of the quotients: by Theorem 6.1, the ghost map is not injective in general.

All 4-manifolds considered will be closed and oriented. Except where explicitly mentioned, they will be connected. The first Betti number  $b^1$  is assumed to vanish. The notation  $e$  and  $s$  will denote the Euler number and the signature, respectively. The circle group will be denoted by  $\mathbb{T}$ , and  $L\mathbb{T}$  will be its Lie algebra.

This work is based on the author's 2002 thesis [22]. I would like to take this opportunity to express my deep gratitude to Stefan Bauer. I would also like to apologise for the delayed publication; the manuscript had been stalled at two other journals for two years each. In the meantime, the theory developed here has been successfully applied, see for example the work [17, 18, 19, 20] of Liu and Nakamura.

## 1 Functoriality of the monopole map

In this section, a certain gauge category  $\mathcal{G}$  is defined such that the monopole map – which will also be reviewed – is functorial on it.

### 1.1 The categories

Let  $\mathcal{M}$  be the following category of manifolds. The objects are closed oriented 4-manifolds  $X$  with a Riemannian metric. The metric will usually be omitted from the notation. The orientation and the metric provide an  $\mathrm{SO}(4)$ -bundle  $\mathrm{SO}(X)$  of oriented orthonormal frames over  $X$ . A morphism  $f$  from  $X$  to  $Y$  is to be

a local diffeomorphisms for which the differential preserves the orientation and the metric in each tangent space. Note that such an  $f$  induces an isomorphism of  $\mathrm{SO}(X)$  with  $f^*\mathrm{SO}(Y)$  over  $X$ .

Let  $\mathcal{G}$  be the following gauge category. The objects are objects  $X$  of  $\mathcal{M}$  together with a complex spin structure  $\sigma_X$  on  $X$ , and an Hermitian connection on the determinant line bundle of  $\sigma_X$ . This line bundle will usually be denoted by  $L(\sigma_X)$ , the connection by  $A(\sigma_X)$  or just  $A$ . The reference to the connection will be omitted from the notation so that  $(X, \sigma_X)$  will be a typical object of  $\mathcal{G}$ . A complex spin structure may be thought of as a  $\mathrm{Spin}^c(4)$ -principal bundle  $\mathrm{Spin}^c(X, \sigma_X)$  on  $X$  and an isomorphism

$$\mathrm{Spin}^c(X, \sigma_X) \times_{\mathrm{Spin}^c(4)} \mathrm{SO}(4) \longrightarrow \mathrm{SO}(X).$$

Using this description, morphisms  $(X, \sigma_X) \rightarrow (Y, \sigma_Y)$  in the gauge category  $\mathcal{G}$  are pairs  $(f, u)$  where  $f$  is a morphism  $f : X \rightarrow Y$  in  $\mathcal{M}$  and  $u$  is an isomorphism of the bundle  $\mathrm{Spin}^c(X, \sigma_X)$  with the pullback  $f^*\mathrm{Spin}^c(Y, \sigma_Y)$  over  $X$  such that the induced isomorphism of  $\mathrm{SO}(4)$ -bundles is the one coming from  $f$ . Furthermore, the map  $u$  is to be compatible with the connections.

The forgetful functor from the gauge category  $\mathcal{G}$  to the category of manifolds  $\mathcal{M}$  is a fibration. The fibre  $\mathcal{G}(X)$  over an object  $X$  of  $\mathcal{M}$  is by definition the subcategory of  $\mathcal{G}$  consisting of those objects which map to  $X$  and those morphisms which map to  $\mathrm{id}_X$ .

Given an object  $(X, \sigma_X)$  of the gauge category  $\mathcal{G}$ , there is an exact sequence

$$1 \longrightarrow \mathrm{Aut}_{\mathcal{G}(X)}(X, \sigma_X) \longrightarrow \mathrm{Aut}_{\mathcal{G}}(X, \sigma_X) \longrightarrow \mathrm{Aut}_{\mathcal{M}}(X) \quad (1.1)$$

of groups. The automorphism group  $\mathrm{Aut}_{\mathcal{M}}(X)$  is the group of orientation preserving isometries of  $X$ . This is a compact Lie group. The map from the group  $\mathrm{Aut}_{\mathcal{G}}(X, \sigma_X)$  to  $\mathrm{Aut}_{\mathcal{M}}(X)$  need not be surjective, i.e. not every isometry  $f$  of  $X$  needs to appear in a morphism  $(f, u)$  in  $\mathcal{G}$ . By definition, this will be the case if and only if  $f^*\sigma_X$  is isomorphic to  $\sigma_X$ . On the other hand, if such a  $u$  exists, it is not determined by  $f$ : there are non-identity morphisms covering the identity of  $X$ . These are the elements in the group  $\mathrm{Aut}_{\mathcal{G}(X)}(X, \sigma_X)$ , which is isomorphic to  $\mathbb{T}$ .

## 1.2 The monopole map

The gauge category  $\mathcal{G}$  has been constructed so that the objects  $(X, \sigma_X)$  have all the structure needed to define the monopole map. The background connection  $A$  gives an identification of the vector space  $\Omega^1(L\mathbb{T})$  with the space of Hermitian connections on  $L(\sigma_X)$  via  $a \mapsto A + a$ . Every such element gives a Dirac operator  $D_{A+a}$  between the two spinor bundles  $W^\pm(\sigma_X)$ . The self-dual part  $F_{A+a}^+$  of the curvature lives in the space  $\Omega^+(L\mathbb{T})$  of self-dual 2-forms. The quadratic map from  $\Omega^0(W^+(\sigma_X))$  to that space will be denoted by  $\phi \mapsto \phi^2$ . Finally, let  $\bar{\Omega}^0(L\mathbb{T})$  be the quotient of the space of functions on  $X$  modulo the constant functions. (The class of a function will be denoted by square brackets.)

The Seiberg-Witten equations read

$$D_{A+a}(\phi) = 0 \quad \text{and} \quad F_{A+a}^+ = \phi^2$$

in this context. With the usual gauge fixing, the solutions correspond to zeros of the monopole map

$$\begin{aligned} \Omega^0(W^+(\sigma_X)) \oplus \Omega^1(L\mathbb{T}) &\longrightarrow \Omega^0(W^-(\sigma_X)) \oplus \Omega^+(L\mathbb{T}) \oplus \bar{\Omega}^0(L\mathbb{T}) \\ (\phi, a) &\longmapsto (D_{A+a}(\phi), F_{A+a}^+ - \phi^2, [d^*a]). \end{aligned}$$

The image of  $(\phi, a)$  decomposes as a sum

$$(0, F_A^+, 0) + (D_A(\phi), d^+a, [d^*a]) + (a\phi, -\phi^2, 0).$$

Thus, the monopole map is polynomial in  $(\phi, a)$ : the first summand is constant, the second summand is linear, it is the *linearisation* of the monopole map, and the remaining term is quadratic.

While an easy computation shows that the monopole map is  $\mathbb{T}$ -equivariant, this will also follow from its functoriality, which will be proven next.

### 1.3 Functoriality

Let us introduce notation for the characters introduced in the previous subsection. For every object  $(X, \sigma_X)$  of  $\mathcal{G}$ , there are vector spaces

$$\begin{aligned} \mathcal{U}(X, \sigma_X) &= \Omega_X^0(W^+(\sigma_X)) \oplus \Omega_X^1(L\mathbb{T}) \\ \mathcal{V}(X, \sigma_X) &= \Omega_X^0(W^-(\sigma_X)) \oplus \Omega_X^+(L\mathbb{T}) \oplus \bar{\Omega}_X^0(L\mathbb{T}), \end{aligned}$$

and the monopole map

$$\mu(X, \sigma_X) : \mathcal{U}(X, \sigma_X) \longrightarrow \mathcal{V}(X, \sigma_X)$$

is a map between them. (We shall silently pass to suitable Sobolev  $L^2$ -completions from now on.) It will now be explained how all this behaves functorially on the gauge category  $\mathcal{G}$ . First of all, the source and the target will be addressed.

Given a morphism  $(f, u)$  from  $(X, \sigma_X)$  to  $(Y, \sigma_Y)$  in the gauge category  $\mathcal{G}$ , there are linear maps

$$(f, u)^* : \mathcal{U}(Y, \sigma_Y) \longrightarrow \mathcal{U}(X, \sigma_X) \tag{1.2}$$

$$(f, u)^* : \mathcal{V}(Y, \sigma_Y) \longrightarrow \mathcal{V}(X, \sigma_X). \tag{1.3}$$

The construction is as follows. Every section of the bundle  $W^\pm(\sigma_Y)$  pulls back to give a section of the pullback bundle  $f^*W^\pm(\sigma_Y)$ . Under the isomorphism  $u$ , this corresponds to a section of  $W^\pm(\sigma_X)$ . This gives maps  $\Omega_X^0(W_X^\pm) \leftarrow \Omega_Y^0(W_Y^\pm)$ . Functions can be pulled back along  $f$ , and constant functions pull back to constant functions. This describes a map  $\bar{\Omega}_X^0(L\mathbb{T}) \leftarrow \bar{\Omega}_Y^0(L\mathbb{T})$ . Similarly, the usual pullback of 1-forms yields a map  $\Omega_X^1(L\mathbb{T}) \leftarrow \Omega_Y^1(L\mathbb{T})$ . Finally, another

map  $\Omega_X^\pm(L\mathbb{T}) \leftarrow \Omega_Y^\pm(L\mathbb{T})$  is induced by pulling back 2-forms: all there is left to remark is that the pullback of a self-dual 2-form is self-dual as well.

This finishes the description of the maps  $(f, u)^*$ . They are almost isometries in the sense that the norms are preserved up to a factor: the degree of  $f$ . As functoriality of the maps  $(f, u)^*$  is easy to check, the following proposition summarises the discussion.

**Proposition 1.1.** *Given a morphism  $(f, u)$  from  $(X, \sigma_X)$  to  $(Y, \sigma_Y)$  in the gauge category  $\mathcal{G}$ , the isometries (1.2) and (1.3) make  $\mathcal{U}$  and  $\mathcal{V}$  into contravariant functors from  $\mathcal{G}$  to the category of Hilbert spaces and continuous linear maps.*

Note that, as a special case, for any object  $(X, \sigma_X)$  of the gauge category  $\mathcal{G}$ , there is an action of the group  $\text{Aut}_{\mathcal{G}}(X, \sigma_X)$  on the Hilbert spaces  $\mathcal{U}(X, \sigma_X)$  and  $\mathcal{V}(X, \sigma_X)$  by isometries. The subgroup  $\text{Aut}_{\mathcal{G}(X)}(X, \sigma_X) \cong \mathbb{T}$  acts by scalar multiplication on the sections of the complex bundles and trivially on the other ones.

Now that its source and target have been dealt with, the monopole map itself will be addressed. From the maps defined above one can build an obvious diagram, and the following proposition states that it commutes. The proof can be left to the reader.

**Proposition 1.2.** *Consider a morphism from  $(X, \sigma_X)$  to  $(Y, \sigma_Y)$  in the gauge category  $\mathcal{G}$ . If one uses the maps defined in the previous proposition as vertical arrows, the diagram*

$$\begin{array}{ccc} \mathcal{U}(X, \sigma_X) & \xrightarrow{\mu(X, \sigma_X)} & \mathcal{V}(X, \sigma_X) \\ \uparrow & & \uparrow \\ \mathcal{U}(Y, \sigma_Y) & \xrightarrow{\mu(Y, \sigma_Y)} & \mathcal{V}(Y, \sigma_Y). \end{array}$$

*commutes.*

As a special case again, for any object  $(X, \sigma_X)$  of the gauge category  $\mathcal{G}$ , the monopole map  $\mu(X, \sigma_X)$  is  $\text{Aut}_{\mathcal{G}}(X, \sigma_X)$ -equivariant. In particular, it always is  $\mathbb{T}$ -equivariant.

The previous proposition might suggest the question whether or not the collection of the monopole maps is a natural transformation between the functors  $\mathcal{U}$  and  $\mathcal{V}$ . But this would ignore the fact that the monopole maps are of a different nature than the linear maps of Proposition 1.1. A better way of describing the situation is as follows. Let  $\mathcal{F}$  be the following category of non-linear Fredholm maps. An object is a continuous map  $\mu$  between Hilbert spaces  $\mathcal{U}$  and  $\mathcal{V}$  which satisfies the conditions of the construction in [3]. The morphisms from  $\mu_1 : \mathcal{U}_1 \rightarrow \mathcal{V}_1$  to  $\mu_2 : \mathcal{U}_2 \rightarrow \mathcal{V}_2$  are pairs of continuous linear maps  $\mathcal{U}_1 \rightarrow \mathcal{U}_2$  and  $\mathcal{V}_1 \rightarrow \mathcal{V}_2$  between Hilbert spaces such that the evident diagram commutes. The two preceding propositions imply the following.

**Theorem 1.3.** *The monopole maps gives rise to a contravariant functor from the gauge category  $\mathcal{G}$  to the category  $\mathcal{F}$  of non-linear Fredholm maps.*

Composition with the construction in [3], which is functorial as well, gives a functor into a category of stable homotopy classes of maps. This will be explained in the next section.

## 2 Equivariant Bauer-Furuta invariants

In this section, after a brief review of the ordinary Bauer-Furuta invariants, an equivariant extension of them is defined and commented on.

The notation from equivariant stable homotopy theory used here will be fairly standard, see [16] for example. For a compact Lie group  $\mathbb{G}$  and a  $\mathbb{G}$ -representation  $V$ , the one-point-compactification will be denoted by  $S^V$ . Given two finite pointed  $\mathbb{G}$ -CW-complexes  $M$  and  $N$ , the group of  $\mathbb{G}$ -equivariant stable maps from  $M$  to  $N$  with respect to a  $\mathbb{G}$ -universe  $\mathcal{V}$  will be denoted by  $[M, N]_{\mathcal{V}}^{\mathbb{G}}$ . If the universe is understood, the reference to it will be omitted.

### 2.1 Ordinary Bauer-Furuta invariants

The definition of the invariants from the monopole map rests on the following construction. Let  $\mathcal{U}$  and  $\mathcal{V}$  be two Hilbert spaces on which  $\mathbb{G}$  acts via isometries. One may assume that  $\mathcal{V}$  is a  $\mathbb{G}$ -universe, although that is not strictly necessary. Let  $\mu$  be a  $\mathbb{G}$ -map from  $\mathcal{U}$  to  $\mathcal{V}$  which admits a decomposition  $\mu = \lambda + \kappa$  into a sum of a linear map  $\lambda$  which has finite-dimensional kernel and cokernel, and a continuous map  $\kappa$  which maps bounded sets into compact sets. Furthermore, it will be required that pre-images under  $\mu$  of bounded sets are bounded. Let  $\text{Cok}(\lambda)$  denote the orthogonal complement of the image of  $\lambda$  in  $\mathcal{V}$ . In this situation, an equivariant stable homotopy class in  $[S^{\text{Ker}(\lambda)}, S^{\text{Cok}(\lambda)}]_{\mathcal{V}}^{\mathbb{G}}$  can be defined, see [3].

The monopole maps fit into this framework, such that this construction can be used to define invariants. Let  $(X, \sigma_X)$  denote an object of  $\mathcal{G}$ . The source  $\mathcal{U} = \mathcal{U}(X, \sigma_X)$  and the target  $\mathcal{V} = \mathcal{V}(X, \sigma_X)$  of the monopole map are  $\mathbb{T}$ -universes, containing only representations isomorphic to  $\mathbb{R}$  or  $\mathbb{C}$ . It is shown in [3] that the monopole map  $\mu = \mu(X, \sigma_X)$  and its linearisation  $\lambda = \lambda(X, \sigma_X)$  satisfy the assumptions required by the construction mentioned in the preceding paragraph. This allows to define an element  $m(X, \sigma_X)$  in  $[S^{\text{Ker}(\lambda)}, S^{\text{Cok}(\lambda)}]_{\mathcal{V}}^{\mathbb{T}}$ , the *Bauer-Furuta invariant* of  $(X, \sigma_X)$ . It is independent of the metric and the reference connection.

### 2.2 A first application of functoriality

Let  $(X, \sigma_X)$  be an object of the gauge category  $\mathcal{G}$ . Let a compact Lie group  $G$  act on  $X$  preserving the orientation. One may assume that  $G$  also preserves the metric. Thus, if the action is faithful, the group  $G$  is a subgroup of  $\text{Aut}_{\mathcal{M}}(X)$ . The complex spin structure is called *G-invariant* if for all  $g$  in  $G$  the complex spin structure  $g^*\sigma_X$  is isomorphic to  $\sigma_X$ . (This is the case if and only if  $G$  is in

the image of the rightmost map in the exact sequence (1.1).) In this case, there is even an isomorphism  $g^*\sigma_X \cong \sigma_X$  which respects the reference connection.

If  $\sigma_X$  is a  $G$ -invariant complex spin structure on  $X$ , the exact sequence (1.1) gives rise to an extension

$$1 \longrightarrow \mathbb{T} \longrightarrow \mathbb{G} \longrightarrow G \longrightarrow 1 \quad (2.1)$$

of  $G$  by  $\mathbb{T}$ . Since  $\mathbb{G}$  is a subgroup of  $\text{Aut}_{\mathfrak{g}}(X, \sigma_X)$ , functoriality implies the existence of a  $\mathbb{G}$ -action on the source and the target of the monopole map such that it will not only be  $\mathbb{T}$ -equivariant but in fact  $\mathbb{G}$ -equivariant. The construction in [3] then allows to define an equivariant invariant.

**Theorem 2.1.** *If a compact Lie group  $G$  acts on  $X$ , and if the complex spin structure  $\sigma_X$  is  $G$ -invariant, the monopole map defines an equivariant invariant  $m_G(X, \sigma_X)$  in the group  $[S^{\text{Ker}(\lambda)}, S^{\text{Cok}(\lambda)}]^{\mathbb{G}}$ .*

The class  $m_G(X, \sigma_X)$  does not depend on the  $G$ -invariant metric, as in the non-equivariant case. Neither matters the reference connection. As in the non-equivariant case, the universe in question should be  $\mathcal{V}$ . While this need not be a  $\mathbb{G}$ -universe in the first place, it will be after suitably enlarging the source and the target of the monopole map with  $\mathbb{G}$ -representations on which the map is defined to be the identity. A situation in which this correction is not necessary is that of free actions of finite groups, which is the subject of the following sections.

### 2.3 Forgetful maps

As soon as equivariant invariants have been defined, one may discuss the amount of information they contain – as compared to the ordinary invariants. It is obvious from the definition that one gets the ordinary invariant  $m(X, \sigma_X)$  back from the equivariant invariant  $m_G(X, \sigma_X)$  by forgetting the equivariance coming from the non-trivial elements of  $G$ .

**Theorem 2.2.** *The forgetful map*

$$[S^{\text{Ker}(\lambda)}, S^{\text{Cok}(\lambda)}]^{\mathbb{G}} \longrightarrow [S^{\text{Ker}(\lambda)}, S^{\text{Cok}(\lambda)}]^{\mathbb{T}}$$

*sends the equivariant invariant  $m_G(X, \sigma_X)$  to the ordinary Bauer-Furuta invariant  $m(X, \sigma_X)$ .*

While this is a trivial observation, it is important from a structural point of view, since it shows that the equivariant version of the invariant is really an extension of the ordinary invariant.

### 2.4 Two elementary examples

Given any 4-manifold  $X$ , there are two elementary ways of producing a 4-manifold with  $G$ -action. On the one hand,  $X$  itself carries the trivial action.

On the other hand, if  $G$  is finite, the group  $G$  acts on the (non-connected) 4-manifold  $G \times X$  by permuting the components. The equivariant invariants for these two cases will be discussed now.

In these two examples, the complex spin structure  $\sigma_X$  on  $X$  will not only be  $G$ -invariant, but even  $G$ -equivariant. This means that the short exact sequence (2.1) splits and that a splitting has been chosen:  $G$  is a subgroup of the group  $\text{Aut}_{\mathbb{G}}(X, \sigma_X)$ .

**Example 2.3.** If  $G$  acts trivially on  $X$ , every complex spin structure  $\sigma_X$  is  $G$ -invariant. One can also endow  $\sigma_X$  with the trivial  $G$ -action. As a consequence, the group  $\mathbb{G}$  can be identified with the product  $\mathbb{T} \times G$ . Since  $G$  acts trivially on the source and the target of the monopole map, the forgetful map can be split by the map which is the identity on representatives. In particular, the equivariant invariant  $m_G(X, \sigma_X)$  is just the ordinary invariant  $m(X, \sigma_X)$ , regarded as a  $\mathbb{G}$ -map.

**Example 2.4.** Let now  $G$  be finite. If one chooses a complex spin structure  $\sigma_X$  on  $X$ , this gives a complex spin structure  $G \times \sigma_X$  on  $G \times X$ . There is an obvious  $G$ -action on  $G \times \sigma_X$ , so that the symmetry group  $\mathbb{G}$  can be identified with the product  $\mathbb{T} \times G$  as above. The invariant of a sum is the smash product of the invariants of the summands, see [1]. It follows for the (non-connected) 4-manifold

$$(G \times X, G \times \sigma_X) = \coprod_{g \in G} (X, \sigma_X)$$

that the ordinary invariant  $m(G \times X, G \times \sigma_X)$  is the smash product

$$\bigwedge_{g \in G} m(X, \sigma_X) \in \left[ \bigwedge_{g \in G} S^{\text{Ker}(\lambda)}, \bigwedge_{g \in G} S^{\text{Cok}(\lambda)} \right]^{\mathbb{T}}. \quad (2.2)$$

The functor  $\bigwedge_{g \in G}$  really takes values in  $(\mathbb{T} \times G)$ -spaces and  $(\mathbb{T} \times G)$ -maps. See [5] for the construction and its properties. For example, as  $m(X, \sigma_X)$  is a map between compactified representations, the map (2.2) is a map between the induced representations:

$$\bigwedge_{g \in G} S^{\text{Ker}(\lambda)} \cong S^{\text{ind}_1^G(\text{Ker}(\lambda))}$$

and similarly for  $\text{Cok}(\lambda)$ . Since  $G$  acts on  $G \times X$  by permuting the factors, the equivariant invariant  $m_G(G \times X, G \times \sigma_X)$  is also given by (2.2), but considered as a  $(\mathbb{T} \times G)$ -map.

The two examples have a common flavour. They show how to relate certain constructions in equivariant stable homotopy theory to constructions of 4-manifolds. The main theorem in [1] has the same flavour, but is much deeper: it relates the smash product (or the composition) to connected sums. Here, it has been shown how, for a stable  $\mathbb{T}$ -map  $f$  which is realised by a 4-manifold, also the stable  $(\mathbb{T} \times G)$ -maps  $f$  and – if  $G$  is finite – also  $\bigwedge_{g \in G} f$  can be realised.

## 2.5 Reduction mod $\mathbb{T}$

As for the structure of the groups

$$[S^{\text{Ker}(\lambda)}, S^{\text{Cok}(\lambda)}]^{\mathbb{G}}$$

in which the equivariant invariants live, in nice situations one can pass to an isomorphic group of  $G$ -equivariant maps. For that, it will be necessary to assume that certain positivity conditions are satisfied. On the one hand, there is to be an actual  $\mathbb{G}$ -representation  $V$  such that

$$[V] = \text{ind}_{\mathbb{G}}(D_A) \tag{2.3}$$

in  $\text{RO}(\mathbb{G})$ . (In particular, the index must not have negative dimension. But this is not sufficient.) The projective space  $\mathbb{C}P(V)$  is a  $G$ -space, and the notation  $\mathbb{C}P(V)_+$  will be used for  $\mathbb{C}P(V)$  with a disjoint  $G$ -fixed base-point added. Let us write  $W$  for the  $G$ -representation  $H^+(X)$ . On the other hand, it will have to be assumed that

$$\dim_{\mathbb{R}}(W^G) \geq 2 \tag{2.4}$$

holds. In particular, one may choose a complement  $W - 1$  of a trivial subrepresentation in  $W$ . The following can be proven as in [3], taking care of the additional  $G$ -action.

**Proposition 2.5.** *Under the two conditions (2.3) and (2.4), there is an isomorphism*

$$[S^{\text{Ker}(\lambda)}, S^{\text{Cok}(\lambda)}]^{\mathbb{G}} \cong [\mathbb{C}P(V)_+, S^{W-1}]^G$$

*of groups.*

The requirement (2.3) will be met in relevant situations. On the one hand, geometry can come for help: For example, if  $G$  acts on a complex surface  $X$  via holomorphic maps, kernel and cokernel of the Dirac operator can be interpreted and (ideally) computed using coherent cohomology. If the cokernel vanishes, the kernel serves as  $V$ . On the other hand, if  $G$  is finite, and  $X$  is a  $G$ -Galois covering of  $Y$ , and  $\sigma_Y$  has non-negative index, then (2.3) is satisfied for the pullback of  $\sigma_Y$  to  $X$ . This will be the situation to which we turn next.

## 3 Galois symmetries

In the previous section, an invariant has been defined for 4-manifolds  $X$  with an action of a compact Lie group  $G$  and a  $G$ -invariant complex spin structure  $\sigma_X$ . From this section on, the group  $G$  will be finite and act freely on  $X$ , adding the aspect that the quotient  $X/G$  is a 4-manifold as well. This suggests to compare the invariants of  $X$  with those of  $X/G$ . The problem to face is that there need not be a compatible complex spin structure on the quotient; and if there is, it need not be unique. The following proposition clarifies the situation.

**Proposition 3.1.** *Let  $\sigma_X$  be a  $G$ -invariant complex spin structure on  $X$ , and let  $\mathbb{G}$  be the corresponding extension of  $G$  by  $\mathbb{T}$ . For any subgroup  $H$  of  $G$  there are canonical bijections between the following sets.*

- (1) *The set of  $H$ -actions  $j$  on  $\sigma_X$  compatible with the action of  $H$  on  $X$ .*
- (2) *The set of subgroups  $H(j)$  of  $\mathbb{G}$  which map isomorphically to  $H$  under the projection from  $\mathbb{G}$  to  $G$ .*
- (3) *The set of isomorphism classes of complex spin structures  $\sigma_{X/H}(j)$  on  $X/H$  such that the pullback along the quotient map  $q : X \rightarrow X/H$  is isomorphic to  $\sigma_X$ .*

*Proof.* The pullbacks of the complex spin structures have an obvious action. Conversely, given an action, one may pass to the quotient. This gives the bijection between the sets in (1) and (3). The bijection between the sets in (1) and (2) is given by the fact that actions on  $\sigma_X$  correspond to group homomorphism into the automorphism group, which is  $\mathbb{G}$  in this case.  $\square$

The notation  $J_H(\sigma_X)$  will be used for any of the sets from the proposition.

### 3.1 A second application of functoriality

Let us fix a subgroup  $H$  and an element  $j$  in  $J_H(\sigma_X)$ . Let  $\sigma_{X/H}$  be the induced complex spin structure on  $X/H$ . The first aim of this section is the identification of the monopole map – and therefore the invariants – for the pair  $(X/H, \sigma_{X/H})$  with fixed point data of  $(X, \sigma_X)$ . Note that there is still some symmetry downstairs on  $X/H$  which can be taken into account: if  $WH$  is the Weyl group of  $H$  in  $G$ , then  $WH$  acts on  $X/H$  and leaves  $\sigma_{X/H}$  invariant. The relevant extension of  $WH$  by  $\mathbb{T}$  is given by the Weyl group  $\mathbb{W}H(j)$  of  $H(j)$  in  $\mathbb{G}$ .

Note that the quotient map  $q : X \rightarrow X/H$  defines a morphism in the category  $\mathcal{G}$  from  $(X, \sigma_X)$  to  $(X/H, \sigma_{X/H})$ . It has been shown in the previous section that this leads to a commutative diagram

$$\begin{array}{ccc} \mathcal{U}(X, \sigma_X) & \xrightarrow{\mu(X, \sigma_X)} & \mathcal{V}(X, \sigma_X) \\ q^* \uparrow & & \uparrow q^* \\ \mathcal{U}(X/H, \sigma_{X/H}(j)) & \xrightarrow{\mu(X/H, \sigma_{X/H}(j))} & \mathcal{V}(X/H, \sigma_{X/H}(j)). \end{array}$$

As a consequence of functoriality, the images of the vertical arrows live in the  $H(j)$ -fixed subspaces. Thus there is a commutative diagram as above with the top arrow replaced by its restriction to the  $H(j)$ -fixed points. In a Galois covering situation as at hand, if  $E_{X/H}$  is a vector bundle on  $X/H$  and  $E_X$  is the pullback, the pullbacks of the sections of  $E_{X/H}$  are exactly the  $H$ -invariant sections of  $E_X$ . Thus, with this adaption of the target, the vertical arrows become isomorphisms:

**Proposition 3.2.** *The vertical arrows in the diagram*

$$\begin{array}{ccc}
\mathcal{U}(X, \sigma_X)^{H(j)} & \xrightarrow{\mu(X, \sigma_X)^{H(j)}} & \mathcal{V}(X, \sigma_X)^{H(j)} \\
q^* \uparrow & & \uparrow q^* \\
\mathcal{U}(X/H, \sigma_{X/H}(j)) & \xrightarrow{\mu(X/H, \sigma_{X/H}(j))} & \mathcal{V}(X/H, \sigma_{X/H}(j))
\end{array}$$

are isomorphisms.

This proposition identifies the monopole map  $\mu(X/H, \sigma_{X/H}(j))$  of the quotient with the restriction of  $\mu(X, \sigma_X)$  to the  $H(j)$ -fixed points. One may use the isomorphisms given by  $q^*$  as in the proposition above to get an isomorphism

$$[\mathcal{S}^{\text{Ker}(\lambda_{X/H}), \mathcal{S}^{\text{Cok}(\lambda_{X/H})}]^{\mathbb{W}H(j)} \xrightarrow{\cong} [\mathcal{S}^{\text{Ker}(\lambda_X)^{H(j)}, \mathcal{S}^{\text{Cok}(\lambda_X)^{H(j)}}]^{\mathbb{W}H(j)} \quad (3.1)$$

of groups. The previous proposition implies the following structural result on the Bauer-Furuta invariants.

**Theorem 3.3.** *The  $H(j)$ -fixed point map*

$$[\mathcal{S}^{\text{Ker}(\lambda_X), \mathcal{S}^{\text{Cok}(\lambda_X)}}]^{\mathbb{G}} \longrightarrow [\mathcal{S}^{\text{Ker}(\lambda_X)^{H(j)}, \mathcal{S}^{\text{Cok}(\lambda_X)^{H(j)}}]^{\mathbb{W}H(j)} \quad (3.2)$$

sends the equivariant invariant  $m_G(X, \sigma_X)$  of  $X$  to a class which can be identified with the invariant  $m_{\mathbb{W}H}(X/H, \sigma_{X/H}(j))$  of the quotient  $X/H$  by means of the isomorphism (3.1).

This theorem has an interesting consequence. Given a 4-manifold  $Y$  with finite fundamental group  $G$ , let  $X$  be a universal covering of  $Y$ . This will be a Galois covering of  $Y$  with group  $G$ . For each complex spin structure  $\sigma_Y$  on  $Y$ , the pullback  $\sigma_X$  of this to  $X$  will be  $G$ -equivariant. The theorem above implies that  $m(Y, \sigma_Y)$  can be obtained from the equivariant invariant  $m_G(X, \sigma_X)$  as the restriction to the  $G$ -fixed points.

**Corollary 3.4.** *The information of the ordinary Bauer-Furuta invariants of 4-manifolds with finite fundamental group is contained in the equivariant Bauer-Furuta invariants of their universal coverings.*

### 3.2 The ghost map – first version

In Corollary 3.4, the point of view of  $Y$  has been emphasised, answering the question how invariants of  $(Y, \sigma_Y)$  can be interpreted in terms of invariants of  $(X, \sigma_X)$ . Now one may also take the point of view of  $X$  and try to understand the information which is contained in its equivariant invariants.

Up to this point, maps out of the group  $[\mathcal{S}^{\text{Ker}(\lambda_X), \mathcal{S}^{\text{Cok}(\lambda_X)}}]^{\mathbb{G}}$  in two directions have been considered. On the one hand, in the foregoing section, it has been noted that the forgetful map

$$[\mathcal{S}^{\text{Ker}(\lambda_X), \mathcal{S}^{\text{Cok}(\lambda_X)}}]^{\mathbb{G}} \longrightarrow [\mathcal{S}^{\text{Ker}(\lambda_X), \mathcal{S}^{\text{Cok}(\lambda_X)}}]^{\mathbb{T}}$$

maps  $m_G(X, \sigma_X)$  to  $m(X, \sigma_X)$ . In fact, the image of the forgetful map lies in the  $G$ -invariants, so that we might even consider this as map

$$[S^{\text{Ker}(\lambda_X)}, S^{\text{Cok}(\lambda_X)}]^\mathbb{G} \longrightarrow \mathbb{H}^0(G; [S^{\text{Ker}(\lambda_X)}, S^{\text{Cok}(\lambda_X)}]^\mathbb{T}). \quad (3.3)$$

On the other hand, in this section, the relevance of the fixed point maps has been shown. These maps took the form

$$[S^{\text{Ker}(\lambda_X)}, S^{\text{Cok}(\lambda_X)}]^\mathbb{G} \longrightarrow [S^{\text{Ker}(\lambda_X)^{H(j)}}, S^{\text{Cok}(\lambda_X)^{H(j)}}]^\mathbb{W}H(j)$$

for a subgroup  $H$  of  $G$  and an element  $j$  in  $J_H(\sigma_X)$ . They can be composed with a forgetful map like (3.3) to get a map

$$[S^{\text{Ker}(\lambda_X)}, S^{\text{Cok}(\lambda_X)}]^\mathbb{G} \longrightarrow \mathbb{H}^0(WH; [S^{\text{Ker}(\lambda_X)^{H(j)}}, S^{\text{Cok}(\lambda_X)^{H(j)}}]^\mathbb{T}).$$

The image of the element  $m_G(X, \sigma_X)$  under this composition has been identified with the ordinary invariant  $m(X/H, \sigma_{X/H}(j))$  of the quotient  $X/H$  with respect to the complex spin structure corresponding to  $j$ . Notice that the map (3.3) is just the case where  $H$  is the trivial subgroup.

Let us now put all this information together in one map. One can sum over the  $j$  in  $J_H(\sigma_X)$  to obtain a map

$$[S^{\text{Ker}(\lambda_X)}, S^{\text{Cok}(\lambda_X)}]^\mathbb{G} \longrightarrow \bigoplus_j \mathbb{H}^0(WH; [S^{\text{Ker}(\lambda_X)^{H(j)}}, S^{\text{Cok}(\lambda_X)^{H(j)}}]^\mathbb{T})$$

for each subgroup  $H$  of  $G$ . If  $H_1$  and  $H_2$  are conjugate subgroups of  $G$ , their Weyl groups are isomorphic and there is a diffeomorphism between  $X/H_1$  and  $X/H_2$  which respects the actions of the Weyl groups. Therefore, it is not necessary to consider all subgroups  $H$  of  $G$ , but only a set of representatives of the conjugacy classes ( $H$ ). The product

$$[S^{\text{Ker}(\lambda_X)}, S^{\text{Cok}(\lambda_X)}]^\mathbb{G} \longrightarrow \bigoplus_{(H), j} \mathbb{H}^0(WH; [S^{\text{Ker}(\lambda_X)^{H(j)}}, S^{\text{Cok}(\lambda_X)^{H(j)}}]^\mathbb{T}) \quad (3.4)$$

of the above maps over such a set will be referred to as the *ghost map*. As is apparent from the interpretation given for the terms involved, it codifies the relationship between the equivariant and the ordinary invariants. Therefore, one would like to understand it as well as possible. This is the aim of the rest of this text.

### 3.3 The ghost map – second version

The following proposition gives a translation from the ghost map (3.4) to something which has smaller groups of equivariance: if  $G$  is finite, they will be finite, too.

**Proposition 3.5.** *Assume that the positivity conditions (2.3) and (2.4) are fulfilled, such that  $[S^{\text{Ker}(\lambda_X)}, S^{\text{Cok}(\lambda_X)}]^{\mathbb{G}} \cong [\mathbb{C}P(V)_+, S^{W-1}]^G$  as in Proposition 2.5. Then the ghost map (3.4) is isomorphic to a map*

$$[\mathbb{C}P(V)_+, S^{W-1}]^G \longrightarrow \bigoplus_{(H)} \mathbb{H}^0(WH; [\mathbb{C}P(V)_+^H, S^{W^H-1}]). \quad (3.5)$$

The map (3.5) is built similar to the ghost map: in each factor, first restrict to the fixed points, then forget the equivariance.

The proof of Proposition 3.5 starts with a remark on the indexing. At first sight, it seems that the indexing over  $j$  has disappeared in (3.5), but the components of the fixed point set  $\mathbb{C}P(V)_+^H$  are indexed by  $J_H(\sigma_X)$ :

**Proposition 3.6.** *Let  $H$  be a subgroup of  $G$ . Then*

$$\mathbb{C}P(V)^H = \coprod_{j \in J_H(\sigma_X)} \mathbb{C}P(V^{H(j)}).$$

If  $J_H(\sigma_X)$  is empty, this means that  $\mathbb{C}P(V)^H$  is empty as well.

Note that the summands  $\mathbb{C}P(V^{H(j)})$  are invariant under the  $WH$ -action. Therefore, the proposition gives a decomposition of  $\mathbb{C}P(V)^H$  as a  $WH$ -space, and enables us to prove Proposition 3.5.

*Proof.* As a consequence of Proposition 3.6, and since  $H(j)$  acts on  $W$  via  $H$ , we have  $W^H = W^{H(j)}$ , and consequently

$$[\mathbb{C}P(V)_+^H, S^{W^H-1}]^{WH} = \bigoplus_j [\mathbb{C}P(V^{H(j)})_+, S^{W^{H(j)}-1}]^{WH}.$$

Hence, the composition of this identification with the sum over  $j$  of the isomorphisms

$$[\mathbb{C}P(V^{H(j)})_+, S^{W^{H(j)}-1}]^{WH} \xrightarrow{\cong} [S^{V^{H(j)}}, S^{W^{H(j)}}]^{WH(j)}$$

from Proposition 2.5 can be used to define the dashed isomorphism on the right hand side of the diagram

$$\begin{array}{ccc} [S^V, S^W]^{\mathbb{G}} & \xrightarrow{\bigoplus_j (?)^{H(j)}} & \bigoplus_j [S^{V^{H(j)}}, S^{W^{H(j)}}]^{WH(j)} \\ \cong \uparrow & & \uparrow \cong \\ [\mathbb{C}P(V)_+, S^{W-1}]^G & \xrightarrow{(?)^H} & [\mathbb{C}P(V)_+^H, S^{W^H-1}]^{WH} \end{array}$$

so that this diagram commutes. Also, using the dashed arrow on the left hand side and another instance of the isomorphism from Proposition 2.5 for the right hand side, the diagram

$$\begin{array}{ccc}
\bigoplus_j [S^{V^{H(j)}}, S^{W^{H(j)}}]^{WH(j)} & \longrightarrow & \bigoplus_j H^0(WH; [S^{V^{H(j)}}, S^{W^{H(j)}}]_{\mathbb{T}}) \\
\cong \uparrow & & \uparrow \cong \\
[\mathbb{C}P(V)_+^H, S^{W^H-1}]^{WH} & \longrightarrow & H^0(WH; [\mathbb{C}P(V)_+^H, S^{W^H-1}])
\end{array}$$

commutes. By placing the two preceding diagrams next to each other and summing over the  $(H)$  one obtains the result.  $\square$

The rest of this text will be concerned with the ghost map in the form (3.5).

### 3.4 Localisation

Now that the main case of interest has been reduced to equivariant stable homotopy theory with a finite group of equivariance, one can use the fact that this theory is easier to understand as soon as the order of the group is inverted.

**Theorem 3.7.** *Under the assumptions of Proposition 3.5, the ghost maps (3.4) and (3.5), which compare the equivariant invariant with the family of the ordinary invariants, become isomorphisms after inverting the order of the group: kernel and cokernel are finite abelian groups whose order is some power of the order of the group.*

*Proof.* This is a consequence of the following general result: Let  $G$  be a finite group,  $M$  and  $N$  two pointed  $G$ -CW-complexes,  $M$  finite. Consider the *localisation map*

$$[M, N]^G \longrightarrow \bigoplus_{(H)} H^0(WH; [M^H, N^H]) \quad (3.6)$$

which is constructed as the ghost map by first passing to fixed points and then taking invariants. This is an isomorphism after inverting the order of  $G$ . (See for example Lemma 3.6 on page 567 of [15]. The reader might also like to have a look at [13].) The ghost map is the localisation map (3.6) for  $M = \mathbb{C}P(V)_+$  and  $N = S^{W-1}$ . The final comment follows since the groups in question are finitely generated.  $\square$

In other words, away from the group order, the  $G$ -equivariant Bauer-Furuta invariant of  $(X, \sigma_X)$  contains the same information as all of the ordinary invariants of all of its quotients  $(X/H, \sigma_{X/H})$  for subgroups  $H$  of  $G$  and complex spin structures  $\sigma_{X/H}$  on  $X/H$  which pull back to  $\sigma_X$ . In particular, the situation is understood rationally. The following sections will address the integral and torsion information.

## 4 Indices and groups

Recall from Theorem 2.1 that if  $X$  is a 4-manifold with an action of a compact Lie group  $G$ , and if  $\sigma_X$  is a  $G$ -invariant complex spin structure on  $X$ , there is an

equivariant Bauer-Furuta invariant which lives in the group  $[S^{\text{Ker}(\lambda)}, S^{\text{Cok}(\lambda)}]^{\mathbb{G}}$ . Up to isomorphism, this group depends only on the class  $[\text{Ker}(\lambda)] - [\text{Cok}(\lambda)]$  in  $\text{RO}(\mathbb{G})$ . This section contains a few remarks on these groups, starting with the case of the ordinary Bauer-Furuta invariants, where  $G = 1$  and  $\mathbb{G} = \mathbb{T}$ , and finishing with the case where  $G$  is a finite group of Galois symmetries.

#### 4.1 The ordinary situation

The linearisation of the monopole map decomposes as a sum of the complex Dirac operator, which sends a spinor  $\phi$  to the spinor  $D_A(\phi)$ , and the real map

$$\Omega^1(L\mathbb{T}) \rightarrow \Omega^+(L\mathbb{T}) \oplus \bar{\Omega}^0(L\mathbb{T}), \quad (4.1)$$

which maps  $a$  to  $(d^+a, [d^*a])$ . The complex index of the Dirac operator is

$$a(\sigma_X) = \frac{c_1^2(\sigma_X) - s}{8}.$$

Since the first Betti number is assumed to vanish, the real index of the operator (4.1) is  $-b^+$ . If we interpret the index of the linearisation as a  $\mathbb{T}$ -representation, the circle acting trivially on real vector spaces and by scalar multiplication on complex vector spaces, then the  $\mathbb{T}$ -index of the linearisation is the class  $a(\sigma_X)[\mathbb{C}] - b[\mathbb{R}]$  in  $\text{RO}(\mathbb{T})$ , with  $a(\sigma_X)$  as above and  $b = b^+$ . The real virtual dimension of the  $\mathbb{T}$ -index is given by  $2a(\sigma_X) - b$ , which equals

$$\frac{c_1^2(\sigma_X) - (2e + 3s)}{4} + 1. \quad (4.2)$$

The number

$$d(\sigma_X) = \frac{c_1^2(\sigma_X) - (2e + 3s)}{4}$$

will be called the *degree* of the complex spin structure. This number vanishes if and only if the complex spin structure comes from an almost complex structure. From a different point of view, the number  $d(\sigma_X)$  is the virtual dimension of the moduli space of solutions to the Seiberg-Witten equations. The virtual dimension (4.2) of the index is  $d(\sigma_X) + 1$ , since the monopole map describes the  $\mathbb{T}$ -equivariant situation, which corresponds to a  $\mathbb{T}$ -space over the moduli space.

The following discussion assumes that  $b \geq 2$ . If  $a \leq 0$ , one may reorder to show that the group  $[S^{\text{Ker}(\lambda)}, S^{\text{Cok}(\lambda)}]^{\mathbb{T}}$  in question is isomorphic to the group  $[S^0, S^{b\mathbb{R} - a\mathbb{C}}]^{\mathbb{T}}$ , where  $b\mathbb{R} - a\mathbb{C}$  is a  $\mathbb{T}$ -representation with  $b$ -dimensional trivial summand. It follows that this group vanishes. From now on, one can focus on the case where  $a$  is positive. Then one reorders the index of  $\lambda$  according to real and complex parts and sees that the group in question is isomorphic to the group  $[S^{a\mathbb{C}}, S^{b\mathbb{R}}]^{\mathbb{T}}$ . If  $a \geq 0$  and  $b \geq 2$ , there is an isomorphism

$$[S^{a\mathbb{C}}, S^{b\mathbb{R}}]^{\mathbb{T}} \cong [\mathbb{C}P_+^{a-1}, S^{b-1}] \quad (4.3)$$

of groups. (This is Proposition 2.5 in the case when  $G$  is trivial.) Therefore, one needs to know the structure of the groups  $[\mathbb{C}P_+^{a-1}, S^{b-1}]$ . Let us regard  $a$  as fixed and sort these groups by degree  $d$ , using the relation  $b - 1 = 2(a - 1) - d$ , so that the groups are  $[\mathbb{C}P_+^{a-1}, S^{2(a-1)-d}]$ . These groups are finitely generated and vanish for negative  $d$ . They vanish rationally, except for even integers  $d$  that satisfy  $0 \leq d \leq 2(a - 1)$ . In those cases, they have rank 1. Apart from divisibility properties of the Hurewicz image, about which nothing will be said here, the most interesting information is the structure of the torsion. If  $\ell$  is a prime, there is no  $\ell$ -power torsion in  $[\mathbb{C}P_+^{a-1}, S^{2(a-1)-d}]$  for  $d < 2\ell - 3$ . If  $d = 2\ell - 3$ , the  $\ell$ -power torsion in  $[\mathbb{C}P_+^{a-1}, S^{2(a-1)-d}]$  is a group of order  $\ell$  if  $a$  is a multiple of  $\ell$  and is trivial else.

## 4.2 The equivariant situation

As a starting point for later computations, it will be useful to have a more concrete description of the groups appearing in the ghost map (3.5). It will be assumed that  $b = b^+(X/G) \geq 2$ . This implies  $b^+(X/H) \geq 2$  for all subgroups  $H$  of  $G$ .

In general, if a group  $G$  acts on the 4-manifold  $X$ , one may consider the equivariant Euler characteristic

$$e_G(X) = [\mathbb{H}^0(X)] - [\mathbb{H}^1(X)] + [\mathbb{H}^2(X)] - [\mathbb{H}^3(X)] + [\mathbb{H}^4(X)]$$

and the equivariant signature

$$s_G(X) = [\mathbb{H}^+(X)] - [\mathbb{H}^-(X)]$$

of  $X$ , which are elements in the representation ring  $\text{RO}(G)$ . In the special case when the group  $G$  is finite and acts freely on  $X$ , these are simply given by  $e_G(X) = e(X/G) \cdot [\mathbb{R}G]$  and  $s_G(X) = s(X/G) \cdot [\mathbb{R}G]$ , as follows for example from the Atiyah-Bott-Lefschetz fixed point formula. Rather than only in  $e_G(X)$  and  $s_G(X)$ , one might also be interested in the  $G$ -subrepresentations  $\mathbb{H}^+(X)$  and  $\mathbb{H}^-(X)$  of  $\mathbb{H}^2(X)$ . The assumption  $b^1 = 0$  implies that we have the equation  $e_G(X) = 2[\mathbb{R}] + [\mathbb{H}^+(X)] + [\mathbb{H}^-(X)]$  holds in  $\text{RO}(G)$ . It follows that

$$[\mathbb{H}^\pm(X)] = \frac{e_G(X) \pm s_G(X)}{2} - [\mathbb{R}].$$

The  $G$ -representation  $W = \mathbb{H}^+(X)$  is isomorphic to  $(b + 1)\mathbb{R}G - 1$ . Therefore, one has  $S^{W-1} \cong S^{(b+1)\mathbb{R}G-2}$ . Starting from this, one can easily sort out the  $H$ -fixed points for the various subgroups  $H$  of  $G$ .

Let now  $\sigma_X$  be a  $G$ -invariant complex spin structure on  $X$  such that an actual  $\mathbb{G}$ -representation  $V$  represents the  $\mathbb{G}$ -index of the Dirac operator. It is not as easy to describe the  $G$ -space  $\mathbb{C}P(V)$ , since the  $G$ -action need not be induced by a  $G$ -action on  $V$ . But, more generally, if the restriction of  $\mathbb{C}P(V)$  from  $G$  to a subgroup  $H$  is not induced by an  $H$ -action on  $V$ , the fixed point set  $\mathbb{C}P(V)^H$  is empty, see Proposition 3.6. Therefore, these  $H$  do not contribute to the target

of the ghost map. One may therefore assume that the restriction of  $\mathbb{C}P(V)$  from  $G$  to  $H$  is induced by an  $H$ -action on  $V$ . The choices of the  $H$ -actions are indexed by the elements  $j$  in  $J_H(\sigma_X)$ . The complex  $H(j)$ -representation  $V$  represents the  $H(j)$ -equivariant index of the Dirac operator  $D_X$ . The usual indices  $\text{ind}(D_X)$  and  $\text{ind}(D_Y)$  are integers. Since  $D_X$  is  $H(j)$ -equivariant, the equivariant index  $\text{ind}_{H(j)}(D_X)$ , which by definition is  $[\text{Ker}(D_X)] - [\text{Cok}(D_X)]$ , lives in the complex representation ring  $\text{RU}(H(j))$ . One may deduce that the equality  $\text{ind}_{H(j)}(D_X) = \text{ind}(D_Y) \cdot [\mathbb{C}H(j)]$  holds in  $\text{RU}(H(j))$  as above. Thus,  $V$  is a multiple of the regular  $H(j)$ -representation  $\mathbb{C}H(j)$ . The multiplicity is the index  $a(\sigma_{X/H}(j))$  of the complex spin structure on the quotient  $X/H$  which pulls back to  $\sigma_X$ . While this description seems to depend on  $j$ , the  $H$ -space  $\mathbb{C}P(V)$  does not. Also, the indices  $a(\sigma_{X/H}(j))$  are independent of  $j$  so that one may write  $a(\sigma_{X/H})$  unambiguously. The  $H$ -fixed point set  $\mathbb{C}P(V)^H$  consists of a disjoint union – indexed by the set  $\text{Hom}(H, \mathbb{T})$  – of complex projective spaces of complex dimension  $a(\sigma_{X/H}) - 1$ , see Proposition 3.6 again.

For use in the following two sections, I will make explicit what the previous remarks mean if the order of  $G$  is a prime  $p$ .

### 4.3 Groups of prime order

In the case when the order of the group  $G$  is a prime  $p$ , any  $G$ -invariant complex spin structure  $\sigma_X$  on  $X$  can be made  $G$ -equivariant. There are  $p$  complex spin structures on the quotient  $X/G$  which pull back to  $\sigma_X$ . Let  $a = a(\sigma_{X/G})$  and  $d = d(\sigma_{X/G})$  be the index and the degree, respectively, of the corresponding complex spin structures on the quotient  $X/G$ . Recall that  $2a - d = b + 1$  for  $b = b^+(X/G) \geq 2$ .

**Proposition 4.1.** *If the order of the group  $G$  is a prime number  $p$ , up to isomorphism, the source of the ghost map is*

$$[\mathbb{C}P(a\mathbb{C}G)_+, S^{(2a-d)\mathbb{R}G-2\mathbb{R}}]^G.$$

The target is

$$\mathbb{H}^0(G; [\mathbb{C}P_+^{ap-1}, S^{(2a-d)p-2}] \oplus [\mathbb{C}P_+^{a-1}, S^{2a-d-2}]^{\oplus p},$$

up to isomorphism.

In the statement, the  $G$ -action on  $[\mathbb{C}P_+^{ap-1}, S^{(2a-d)p-2}]$  comes from the identification of that group with  $[\mathbb{C}P(a\mathbb{C}G)_+, S^{(2a-d)\mathbb{R}G-2\mathbb{R}}]$ .

This might be a good point to insert a comment on the structure of  $G$ -modules such as  $[M, N]$  for finite  $G$ -CW-complexes  $M$  and  $N$ . The  $G$ -action on a group like that is trivial if the group  $G$  acts on  $M$  and  $N$  via homotopically trivial maps. The latter will always be the case for complex projective spaces  $M = \mathbb{C}P(V)_+$  with a linear action. For spheres  $N = S^W$  it is the case if and only if  $W$  is orientable, i.e. if  $\det(W)$  is trivial. In particular, for odd order groups the action is always trivial.

The ghost map in the two cases  $d = 0$  and  $d = 1$  will be discussed in more detail in the following two sections. As both phenomena – a non-trivial cokernel and a non-trivial kernel – already occur in these two examples, it does not seem to be illuminating to proceed and discuss the cases where  $d \geq 2$ .

## 5 Degree zero

In this section, some applications of calculations in equivariant stable homotopy theory to the Bauer-Furuta invariants will be described in the case when all complex spin structures involved have degree zero. Since in this case the ordinary Bauer-Furuta invariants can be identified with the Seiberg-Witten invariants, the results of this section apply to those as well. As examples, some elliptic surfaces will be discussed. This will lead to congruences between certain binomial co-efficients.

### 5.1 General results

Let  $G$  be a finite group of prime order  $p$  which may be even or odd. Let  $X$  be a 4-manifold with a  $G$ -action which preserves a complex spin structure  $\sigma_X$  on  $X$ . In the previous section, a ghost map has been assembled which sends  $m_G(X, \sigma_X)$  to  $m(X, \sigma_X)$  and the family of the invariants  $m(X/G, \sigma_{X/G}(j))$  for  $j$  in  $J_G(\sigma_X)$ . Since one has  $d(\sigma_X) = p \cdot d(\sigma_{X/G}(j))$ , the degree zero case is the case where the degrees of all the complex spin structures involved are zero.

Assume as before that  $b^+(X/G) \geq 2$  holds. This implies that  $a(\sigma_{X/G}) \geq 2$ . In particular, the hypotheses of Proposition 3.5 are fulfilled, so that the ghost map takes the form (3.5). Proposition 4.1 then implies that the target of the ghost map is isomorphic to a free abelian group of rank  $p+1$ : As regards the first summand, note that  $[\mathbb{C}P_+^{ap-1}, S^{2ap-2}] \cong \mathbb{Z}$ , and the action of  $G$  on it is trivial. This is clear for odd  $p$ . If  $p = 2$  it follows from the fact that  $W \cong a\mathbb{C}G - \mathbb{C}$  has the structure of a complex representation, so that the action on the sphere  $S^W$  preserves a chosen orientation. For the other  $p$  summands, note that there is an isomorphism  $[\mathbb{C}P_+^{a-1}, S^{2a-d-2}] \cong \mathbb{Z}$ .

There is an equivariant stable Hopf theorem, see [23], which tells us that in this situation the ghost map is injective with a cokernel of order  $p$ , and that the elements in the image are characterised by a certain congruence. In our situation this reads as follows.

**Theorem 5.1.** *Assume that  $b^+(X/G) \geq 2$ . In the degree zero case, the equivariant invariant  $m_G(X, \sigma_X)$  is determined by the ordinary invariants  $m(X, \sigma_X)$  and  $m(X, \sigma_{X/G}(j))$ , where  $j$  ranges over  $J_G(\sigma_X)$ . The relation*

$$m(X, \sigma_X) \equiv \sum_{j \in J_G(\sigma_X)} m(X/G, \sigma_{X/G}(j)) \pmod{p}$$

*is satisfied by the latter.*

As has been shown in [3], in the degree zero case, the Bauer-Furuta invariants can be identified with the integer valued Seiberg-Witten invariants. Therefore, the latter must satisfy the same relations. See [21] for the case of involutions. Theorem 5.1 will now be illustrated with elliptic surfaces, a class of examples where the Seiberg-Witten invariants are known.

## 5.2 Elliptic surfaces

Relatively minimal regular elliptic surfaces with at most two multiple fibres will be considered. (For background on elliptic surfaces see the articles [7] and [27] or the books [9], [8] and [12].) Let  $m_1$  and  $m_2$  denote the multiplicities of the two fibres  $F^1$  and  $F^2$ , respectively. Setting  $p = \gcd(m_1, m_2)$ , the fundamental group of the surface is cyclic of order  $p$ . In particular  $b^1 = 0$ . The other invariants can be computed from the geometric genus  $p_g$  as follows. The Euler characteristic is  $e = 12(p_g + 1)$ . It follows that  $b^2 = 12p_g + 10$ . In fact  $b^+ = 2p_g + 1$  and  $b^- = 10p_g + 9$ . The signature is given by  $s = -8(p_g + 1)$ . The assumption  $b^+ \geq 2$  translates into  $p_g \geq 1$ . In particular, Dolgachev surfaces are not allowed since they have  $p_g = 0$ .

Let me abuse (additively written) divisors to denote the corresponding line bundles, the canonical divisor  $K$  corresponding to  $\Lambda^2 T^*$ . For complex surfaces the isomorphism classes of complex spin structures are canonically parametrised by the isomorphism classes of line bundles, the trivial line bundle corresponding to a complex spin structure with  $W^+ = \mathbb{C} \oplus \Lambda^2 T$  and  $W^- = T$ , having determinant line bundle  $-K$ . It follows that  $p_g + 1$  is the index of the Dirac operator for the canonical complex spin structure.

The Seiberg-Witten invariants of the elliptic surfaces have been computed, see for example [4], [10] and [11]. In order to describe the result, some notation needs to be introduced. For an integer  $i \geq 0$  let  $[i] = \{0, \dots, i\}$ . Write  $[i_1, i_2, i_3]$  for  $[i_1] \times [i_2] \times [i_3]$ . For  $(a, b, c)$  in  $[p_g - 1, m_1 - 1, m_2 - 1]$ , there is the effective divisor

$$D(a, b, c) = aF + bF^1 + cF^2$$

on the elliptic surface. At the two extremes,  $D(p_g - 1, m_1 - 1, m_2 - 1)$  is the canonical divisor  $K$ , and  $D(0, 0, 0)$  is the trivial divisor. Later,  $D(a)$  will be written instead of  $D(a, 0, 0)$ . The line bundles leading to non-trivial Seiberg-Witten invariants are of the form  $K - 2D(a, b, c)$  for triples  $(a, b, c)$  in the cube  $[p_g - 1, m_1 - 1, m_2 - 1]$ . The value of the invariant for the corresponding complex spin structure is  $(-1)^a \binom{p_g - 1}{a}$ , independent of  $b$  and  $c$ .

Now let us consider a Galois covering situation. The notation  $E(n)$  is used for an elliptic surface with  $p_g = n - 1$  and no multiple fibres. Multiplicities will appear as indices:  $E(n)_{m_1, m_2}$ . Note that  $n \geq 2$  by our assumption on  $p_g$ . The shift from  $p_g$  to  $n$  is justified by the fact that  $n = p_g + 1$  behaves well under coverings. In fact, if  $p$  divides  $m_1$  and  $m_2$ , there is a Galois covering

$$E(pn)_{m_1/p, m_2/p} \longrightarrow E(n)_{m_1, m_2}$$

with Galois group of order  $p$ . If  $p = \gcd(m_1, m_2)$ , this is a universal covering. For example, the universal covering of an Enriques surface  $E(1)_{2,2}$  is a K3-surface  $E(2)$ ; however, the condition  $n \geq 2$  is not satisfied by the Enriques surfaces. As the simplest examples, I would like to discuss the Galois coverings  $E(pn) \rightarrow E(n)_{p,p}$  for  $n \geq 2$  and any prime number  $p$ .

The Seiberg-Witten invariants of the covering surface  $X = E(pn)$  take the value  $(-1)^d \binom{pn-2}{d}$  on the classes  $K_X - 2D_X(d)$  for  $d$  in  $[pn-2]$ . On the covered surface  $Y = E(n)_{p,p}$  they are given by the number  $(-1)^a \binom{n-2}{a}$  on the classes  $K_Y - 2D_Y(a, b, c)$  for a triple  $(a, b, c)$  in the cube  $[n-2, p-1, p-1]$ . Since the canonical divisor  $K_Y$  pulls back to  $K_X$ , the class  $D_Y(a, b, c)$  pulls back to  $D_X(pa + b + c)$ . The relations from Theorem 5.1 are now equivalent to a congruence between binomial co-efficients. In order to make these explicit, let again  $p$  be a prime number, and  $n$  be an integer,  $n \geq 2$ . Then, for any integer  $d$  such that  $0 \leq d \leq pn-2$ , the relations are equivalent to the congruence

$$(-1)^d \binom{pn-2}{d} \equiv \sum_{(a,b,c)} (-1)^a \binom{n-2}{a} \pmod{p}, \quad (5.1)$$

where the sum ranges over the triples  $(a, b, c)$  in  $[n-2, p-1, p-1]$  which satisfy the relation  $pa + b + c = d$ . Note that the terms on the right hand side of (5.1) do not depend on  $b$  and  $c$ ; these enter only in the summation set. Also, the sets summed over do not always have  $p$  elements. That reflects the fact that for some of the pre-images of  $K_X - 2D_X(d)$  the Seiberg-Witten invariant vanishes. For example, this is the case for  $d = 0$  and  $d = pn-2$ , in other words for the canonical and the anti-canonical complex spin structures. The reader is encouraged to find her or his own elementary proof of (5.1) so as to verify this instance of Theorem 5.1 by hand.

## 6 Degree one

As in the previous section, let  $G$  be a finite group whose order is a prime  $p$ . Again, calculations in equivariant stable homotopy theory will be applied to study Bauer-Furuta invariants of Galois coverings  $X \rightarrow X/G$ . This time, however, the complex spin structures on the quotient  $X/G$  will have degree one.

### 6.1 General results

If the complex spin structures on the quotient  $X/G$  have degree one, those on  $X$  consequently have degree  $p$ . We will show that the class of the equivariant invariant  $m_G(X, \sigma_X)$  is in general not determined by the classes of the ordinary invariants

By Proposition 4.1, the target of the ghost map is isomorphic to

$$H^0(G; [\mathbb{C}P(a\mathbb{C}G)_+, S^{(2a-1)\mathbb{R}G-2\mathbb{R}}]) \oplus [\mathbb{C}P_+^{a-1}, S^{2a-3}]^{\oplus p}.$$

If  $\ell$  is a prime number, the  $\ell$ -power torsion of  $[\mathbb{C}P_+^{a-1}, S^{2a-3}]$  is trivial except maybe for  $\ell = 2$ . The  $\ell$ -power torsion of

$$[\mathbb{C}P(a\mathbb{C}G)_+, S^{(2a-1)\mathbb{R}G-2\mathbb{R}}] = [\mathbb{C}P_+^{ap-1}, S^{(2a-1)p-2}]$$

is trivial except maybe if  $2\ell - 3 \leq p$ . If  $p \geq 5$ , this means that only  $\ell$ -power torsion for  $\ell < p$  appears in the target of the ghost map. This is the main case. Let me briefly comment on the other two primes  $p = 2$  and  $p = 3$  before I return to it in more detail.

In the case  $p = 2$ , the 2-torsion in  $[\mathbb{C}P_+^{a-1}, S^{2a-3}]$  becomes relevant, and the  $G$ -action on  $[\mathbb{C}P(a\mathbb{C}G)_+, S^{(2a-1)\mathbb{R}G-2\mathbb{R}}]$  will have to be discussed. The latter group sits in an extension

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow [\mathbb{C}P(a\mathbb{C}G)_+, S^{(2a-1)\mathbb{R}G-2\mathbb{R}}] \longrightarrow \mathbb{Z} \longrightarrow 0$$

which comes from the Hurewicz map. This time, however, the action of the group  $G$  on the sphere  $S^{(2a-1)\mathbb{R}G-2\mathbb{R}}$  is not orientation preserving, so the action on  $\mathbb{Z}$  is non-trivial. As a consequence, the target of the ghost map is isomorphic to  $\mathbb{Z}/2$  for odd  $a$  and to  $(\mathbb{Z}/2)^{\oplus 3}$  for even  $a$ . If  $p = 3$ , the target of the ghost map is isomorphic to one copy of  $\mathbb{Z}/3$  and maybe some 2-torsion.

Now back to the main case. As mentioned above, if  $p \geq 5$ , the target of the ghost map is torsion of order away from the group order  $p$ . In particular, by Theorem 3.7, the ghost map is surjective in this case. The kernel is equal to the  $p$ -power torsion in the source  $[\mathbb{C}P(a\mathbb{C}G)_+, S^{(2a-1)\mathbb{R}G-2\mathbb{R}}]^G$  of the ghost map. Calculations with an Adams spectral sequence show, see [22] or [26], that the ghost map is not injective in this case. This means for the Bauer-Furuta invariants that there are several elements which the ghost map sends to the collection of the ordinary invariants.

**Theorem 6.1.** *The homotopy classes of all the ordinary invariants  $m(X, \sigma_X)$  and  $m(X/G, \sigma_{X/G}(j))$  for all  $j$  in  $J_G(\sigma_X)$  do not determine the homotopy class of the equivariant invariant  $m_G(X, \sigma_X)$  in general.*

This raises the question of whether there are 4-manifolds with  $G$ -action realising the different possibilities opened up by homotopy theory or not: Do there exist  $(X, \sigma_X)$  and  $(X', \sigma_{X'})$  with free  $G$ -actions which have different equivariant invariants but which have the same ordinary invariants? In particular, if there is a pair  $(X, \sigma_X)$  with free  $G$ -action for which the ordinary invariants  $m(X, \sigma_X)$  and  $m(X/G, \sigma_{X/G}(j))$  are zero for all  $j$ , does it follow that the equivariant invariant  $m_G(X, \sigma_X)$  is zero as well? A natural place to look for examples of complex spin structures of degree one is among connected sums of those of degree zero; but, as will be shown in the following final subsection, in that case the invariants will all be determined by non-equivariant data. As at the time of writing there do not seem to be any other relevant examples known which do not fit into this pattern, new geometric constructions seem to be called for to settle this question; Theorem 6.1 marks the scope of homotopy theory here.

## 6.2 Connected sums

Recall from [1] the following. Let  $X_1$  and  $X_2$  be two 4-manifolds with complex spin structures  $\sigma_{X_1}$  and  $\sigma_{X_2}$ . Then there is a canonical complex spin structure  $\sigma_{X_1\#X_2}$  on the connected sum  $X_1\#X_2$  which restricts to the given one on each summand. The connected sum theorem says that  $m(X_1\#X_2, \sigma_{X_1\#X_2})$  can be identified with the smash product  $m(X_1, \sigma_{X_1}) \wedge m(X_2, \sigma_{X_2})$ . More generally, let  $X^\pm$  be two 4-manifolds, oriented and compact, but not necessarily connected. The boundaries  $\partial X^\pm$  are to be identified with a collection  $[-L, L] \times S^3 \times \Lambda$  of necks of length  $2L$ . Here  $\Lambda$  is a finite index set. Given a permutation  $\tau$  of  $\Lambda$ , one may build an oriented closed 4-manifold  $X^- \cup_\tau X^+$  by gluing as indicated by  $\tau$ . This carries a complex spin structure if the  $X^\pm$  do so in a way compatible with the identification over the necks. Then, if  $\tau_1$  and  $\tau_2$  are even permutations and the first Betti numbers of  $X^- \cup_{\tau_1} X^+$  and  $X^- \cup_{\tau_2} X^+$  are zero, there is an (explicitly described) identification

$$m(X^- \cup_{\tau_1} X^+) \cong m(X^- \cup_{\tau_2} X^+), \quad (6.1)$$

omitting the evident complex spin structures from the notation. See [1] again.

In order to describe an equivariant extension of this result, let us assume that  $G$  acts freely on  $X^\pm$ . (For the matter of this paragraph,  $G$  can be any finite group.) We will also assume that  $G$  acts on the complex spin structures in a compatible way, so that  $\mathbb{G}$  is identified with  $\mathbb{T} \times G$  for all components. This time the boundaries  $\partial X^\pm$  are required to be identified with a collection  $G \times [-L, L] \times S^3 \times \Lambda$  of necks. As above, given a permutation  $\tau$  of  $\Lambda$ , one may build an oriented closed 4-manifold  $X^- \cup_\tau X^+$  by gluing. This carries a free  $G$ -action. Note that an even permutation of  $\Lambda$  induces an even permutation of  $G \times \Lambda$ . Therefore, if the condition on the first Betti numbers is satisfied, there is an identification (6.1). It is easily checked that the identification maps and homotopies used in [1] are  $G$ -equivariant. This implies that also the equivariant invariants  $m_G(X^- \cup_{\tau_1} X^+)$  and  $m_G(X^- \cup_{\tau_2} X^+)$  can be identified. As in the non-equivariant setting, this leads to results on connected sums, as will now be exemplified.

Let  $X \rightarrow Y$  be a Galois  $G$ -covering. If a complex spin structure  $\sigma_Y$  on  $Y$  with degree zero is given, the pullback  $\sigma_X$  on  $X$  has degree zero as well. Again, the notation  $\sigma_Y(j)$  will be used for the different complex spin structures on  $Y$  which pull back to  $\sigma_X$  on  $X$ . Let us choose an additional 4-manifold  $Z$  and a complex spin structure  $\sigma_Z$  of degree zero. To be on the safe side, let us also assume that  $b^+ \geq 2$ . (For example,  $Z$  may be taken to be a K3-surface and  $\sigma_Z$  the canonical spin structure.) For the rest of this section let us work with these chosen complex spin structures and their connected sums. If confusion is unlikely, they can be suppressed from the notation. In the same vein, to improve legibility, let us write  $Y(j)$  for  $Y$  with the complex spin structure  $\sigma_Y(j)$  and similarly  $Y(j)\#Z$  to indicate the relevant complex spin structures on  $Y\#Z$ . If  $S^3$  denotes the separating 3-sphere in  $Y\#Z$ , there is an equivariant connected sum  $X\#_G(G \times Z)$  along  $G \times S^3$ , and one may consider the  $G$ -coverings

$$X\#_G(G \times Z) \longrightarrow Y(j)\#Z. \quad (6.2)$$

Using the connected sum theorem of Bauer, which identifies  $m(Y(j)\#Z)$  with the smash product  $m(Y(j)) \wedge m(Z)$ , we see that the ordinary invariants of each of the  $Y(j)\#Z$  can be described in terms which shall be assumed to be known, namely the ordinary invariants of the  $Y(j)$  and  $Z$ . That theorem also identifies the ordinary invariant  $m(X\#_G(G \times Z))$  with the smash product  $m(X) \wedge m(Z)^{\wedge p}$ . This is zero as soon as  $p \geq 5$ . (If  $a \geq 2$  then every  $m$  in  $[S^{a\mathbb{C}}, S^{2a-1}]^{\mathbb{T}}$  satisfies  $m^{\wedge 5} = 0$ .) To sum up, the ordinary invariants of all the 4-manifolds involved in the coverings (6.2) can be computed from those of  $X$ , the  $Y(j)$  and  $Z$ .

Let us turn towards the equivariant invariants. It has been shown in Theorem 5.1 that the equivariant invariant  $m_G(X)$  of  $X$  is determined by the ordinary invariants of  $X$  and the  $Y(j)$ . Hence the only thing which has not been determined yet is the equivariant invariant  $m_G(X\#_G(G \times Z))$  of  $X\#_G(G \times Z)$ . Of course, there is again the forgetful map which sends that to the ordinary invariants  $m(X\#_G(G \times Z))$  and all the  $m(Y(j)\#Z)$ . By Theorem 6.1, this map is not injective. So at this point one cannot deduce the equivariant invariant  $m_G(X\#_G(G \times Z))$  from the ordinary invariants. But it can be deduced from the equivariant invariants  $m_G(X)$  and  $m_G(G \times Z)$ : using the general remarks on equivariant connected sums above, there is an identification

$$m_G(X\#_G(G \times Z)) \cong m_G(X) \wedge m_G(G \times Z).$$

As described in Example 2.4, the equivariant invariant  $m_G(G \times Z)$  of  $G \times Z$  is given by

$$m_G(G \times Z) \cong \bigwedge_{g \in G} m(Z).$$

The following result summarises the discussion.

**Proposition 6.2.** *In the Galois covering situation (6.2), the equivariant Bauer-Furuta invariant is given as*

$$m_G(X\#_G(G \times Z)) \cong m_G(X) \wedge \bigwedge_{g \in G} m(Z),$$

and  $m_G(X)$  is determined by the ordinary invariants  $m(X)$  and  $m(Y(j))$ .

To sum up, (6.2) is a situation, where an equivariant extension of the connected sum theorem allows one to work around the difficulties posed by the non-injectivity of the ghost map, so that the equivariant invariant can nevertheless be determined from ordinary invariants. It would be interesting to see other examples where this holds (or not).

## References

- [1] S. Bauer, A stable cohomotopy refinement of Seiberg-Witten invariants: II, *Invent. Math.* **155** (2004) 21–40.
- [2] S. Bauer, Refined Seiberg-Witten invariants. Different faces of geometry, 1746, *Int. Math. Ser. (N.Y.)*, 3, Kluwer/Plenum, New York, 2004.

- [3] S. Bauer M. Furuta, A stable cohomotopy refinement of Seiberg-Witten invariants: I, *Invent. Math.* **155** (2004) 1–19.
- [4] R. Brussee, The canonical class and the  $C^\infty$  properties of Kähler surfaces, *New York J. Math.* **2** (1996) 103–146.
- [5] T. tom Dieck, *Transformation groups*, de Gruyter, Berlin, 1987.
- [6] S.K. Donaldson P.B. Kronheimer, *The geometry of four-manifolds*, Clarendon Press, Oxford, 1990.
- [7] I. Dolgachev, Algebraic surfaces with  $p_g = q = 0$ , in: *Algebraic Surfaces*, 97–214, Centro Internazionale Matematico Estivo, Napoli, 1981.
- [8] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*, Springer, Berlin, 1998.
- [9] R. Friedman J.W. Morgan, *Smooth four-manifolds and complex surfaces*, Springer, Berlin, 1994.
- [10] R. Friedman J.W. Morgan, Algebraic surfaces and Seiberg-Witten invariants, *J. Algebraic Geom.* **6** (1997) 445–479.
- [11] R. Friedman J.W. Morgan, Obstruction bundles, semi-regularity and Seiberg-Witten invariants, *Comm. Anal. Geom.* **7** (1999) 451–495.
- [12] R.E. Gompf A.I. Stipsicz, *4-manifolds and Kirby calculus*, American Mathematical Society, Providence, 1999.
- [13] J.P.C. Greenlees, Homotopy equivariance, strict equivariance and induction theory, *Proc. Edinburgh Math. Soc.* (2) **35** (1992) 473–492.
- [14] N. Hitchin, Global differential geometry, in: *Mathematics unlimited – 2001 and beyond*, 577–591, Springer, Berlin, 2001.
- [15] M.J. Hopkins, N.J. Kuhn D.C. Ravenel, Generalized group characters and complex oriented cohomology theories, *J. Amer. Math. Soc.* **13** (2000) 553–594.
- [16] L.G. Lewis, J.P. May, M. Steinberger J.E. McClure, *Equivariant stable homotopy theory*, *Lecture Notes in Mathematics* 1213, Springer, Berlin, 1986.
- [17] X. Liu, N. Nakamura, Nonsmoothable group actions on elliptic surfaces, *Topology Appl.* **155** (2008) 946–964.
- [18] N. Nakamura, Mod  $p$  vanishing theorem of Seiberg-Witten invariants for 4-manifolds with  $\mathbb{Z}/p$ -actions, *Asian J. Math.* **10** (2006) 731–748.
- [19] N. Nakamura, Bauer-Furuta invariants under  $\mathbb{Z}/2$ -actions, *Math. Z.* **262** (2009) 219–233.

- [20] N. Nakamura, Smoothability of  $\mathbb{Z} \times \mathbb{Z}$ -actions on 4-manifolds. Proc. Amer. Math. Soc. 138 (2010) 2973–2978
- [21] Y. Ruan S. Wang, Seiberg-Witten invariants and double covers of 4-manifolds, Comm. Anal. Geom. 8 (2000) 477–515.
- [22] M. Szymik, Galois symmetries of 4-manifolds and equivariant stable homotopy theory, Dissertation, Bielefeld, 2002.
- [23] M. Szymik, A stable approach to the equivariant Hopf theorem, Topology and its Applications 154 (2007) 2323–2332.
- [24] M. Szymik, Stable diffeomorphism groups of 4-manifolds, Math. Res. Lett. 15 (2008) 1003–1016.
- [25] M. Szymik, Characteristic cohomotopy classes for families of 4-manifolds, Forum Math. 22 (2010) 509–523.
- [26] M. Szymik, A non-trivial ghost kernel for the equivariant stable cohomotopy of projective spaces, Preprint.  
[www.ruhr-uni-bochum.de/topologie/szymik/szymik.ghosts.pdf](http://www.ruhr-uni-bochum.de/topologie/szymik/szymik.ghosts.pdf)
- [27] M. Ue, On the topology of elliptic surfaces – a survey, Amer. Math. Soc. Transl. **160** (1994) 95–123.

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