Alexander–Beck modules detect the unknot

Markus Szymik

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We introduce the Alexander–Beck module of a knot as a canonical refinement of the classical Alexander module, and we prove that this new invariant is an unknot-detector.

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Introduction

One of the most basic and fundamental invariants of knots K inside the 3-sphere is the knot group πK, the fundamental group of the knot complement. Any regular projection (i.e. any 'diagram') of the knot gives rise to a presentation of the group πK in terms of generators and relations, the Wirtinger presentation. As a consequence of Papakyriakopoulos' work [14], the knot group πK detects the unknot, but there are many pairs of distinct knots that have isomorphic knot groups. In addition to this, groups are also rather complicated objects to manipulate, because of their non-linear nature. There are, therefore, more than enough reasons to look for other knot invariants.

It turns out that the knot complement is actually a classifying space of the knot group, see again [14]. As a consequence, the homology of the group is isomorphic to the homology of the complement. By duality, this homology (and even the stable homotopy type) is easy to compute, and independent of the knot. Therefore, homology and other abelian invariants of groups and spaces do not give rise to interesting knot invariants unless we also find a way to refine the strategy to some extent.

For instance, the Alexander polynomial of a knot can be extracted from the homology of the canonical infinite cyclic cover of the knot complement. There are knots, such as the Conway knot [5], that have the same Alexander polynomial as the unknot. (At the time of writing, the corresponding problem for the Jones polynomial appears to be open.) This state of affairs may suggest that groups and the invariants derived from their abelianizations are not the most efficient algebraic means to provide invariants of knots.

Building on fundamental work [21] of Waldhausen, Joyce [9] and Matveev [12] have independently shown that there is an algebraic structure that gives rise to a complete invariant of knots K: the knot quandle QK. As with the knot group, it can be described in terms of paths in the knot complement, and it can also be presented by means of any of the knot's diagrams.

The knot quandle functorially determines the knot group, and the classical Alexander module of a knot K has a comeback as the (absolute) abelianization of the knot quandle QK.

In this paper, we use a relative version of the abelianization functor that goes back to Beck [1] in order to introduce a refinement of the classical Alexander module. The following will reappear as Definition 6.1, after an explanation of the terms involved.

Definition. Let K be a knot with knot quandle QK. The *Alexander–Beck module* of K is the value of the left adjoint of the forgetful functor from QK-modules to quandles over QK at the terminal object.

As the name suggests, the Alexander–Beck module of a knot is a linear algebraic object and therefore easier to manipulate than the knot group. We can also compute it from any diagram of the knot.

While knots are classified in theory, as recalled above, by means of their associated quandles, there is still considerable interest in finding weaker invariants. Of course, we do not want the invariants to be too weak, like homology or the Alexander polynomial. They should at least be unknot-detectors. As we have noted above, the knot group is such an invariant, and in [10] this property is established for Khovanov homology. We will see that there is no need to introduce homology. This is the main result of this paper, Theorem 6.4.

Theorem. *A knot is trivial if and only if its Alexander–Beck module is free.*

The paper is outlined as follows. In the following Section 1, we will review the categories of abelian group objects for algebraic theories. In Section 2, we apply this to the theories of racks and quandles. In Section 3, we specialize this further to the knot quandles and explain how the classical Alexander invariants can be interpreted from this point of view. Section 4 contains a review of Beck modules over objects for any algebraic theory. In Section 5, we employ this for the theories of racks and quandles. In Section 6, we specialize again to knot quandles. We introduce the Alexander–Beck modules in Definition 6.1, and we prove the main result of this paper, Theorem 6.4.

Homology and the derived functors of the abelianization will be addressed elsewhere [19].

1 Abelian group objects

Racks and quandles can be studied in the context of algebraic theories in the sense of Lawvere [11]. This was done, for instance, in [18]. Other more standard examples of such theories are given by the theory of groups, the theory of rings, the theory of sets with an action of a given group G, the theory of modules over a given ring A, the theory of Lie algebras, and not to forget the initial theory of sets. In this section we review the categories of abelian group objects in algebraic theories.

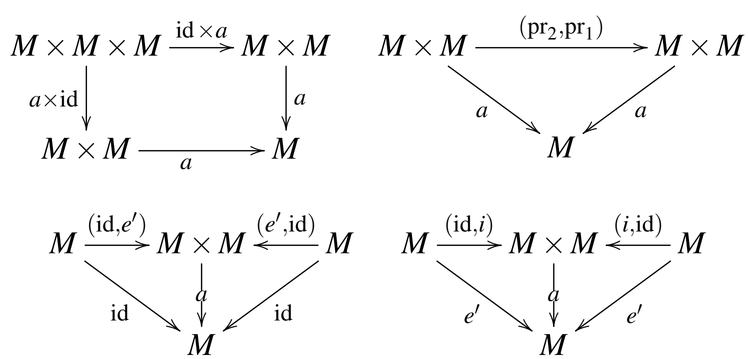
For any algebraic theory, the category $\mathbf{T}$ of its models (or algebras) is complete, cocomplete, and has a 'small' and 'projective' generator: a free model on one generator. The class of 'effective epimorphisms' agrees with the class of surjective homomorphisms. We will write $\mathbf{S}$, $\mathbf{G}$, $\mathbf{R}$, and $\mathbf{Q}$ for the category of sets, groups, racks (see Definition 2.1), and quandles (see Definition 2.5), respectively. Whenever we pick a category $\mathbf{T}$ of models for an algebraic theory, the reader is invited to choose any of these for guidance.

There are forgetful functors between these categories that all have left adjoints.

$$\mathbf{S} \longrightarrow \mathbf{R} \longrightarrow \mathbf{Q} \longrightarrow \mathbf{G}$$

In particular, the left adjoint $\mathbf{S} \to \mathbf{T}$ to the forgetful functor sends a set S to the free model $\mathrm{F}\mathbf{T}(S)$ on the set S of generators. We will write $\mathrm{F}\mathbf{T}_n$ if S is the set $\{1,\dots,n\}$ with n elements.

Definition 1.1. If $\mathbf{C}$ is a category with finite products, an *abelian group object* in $\mathbf{C}$ is an object M together with operations $e\colon \star \to M$ (the unit), $i\colon M \to M$ (the inverse), and $a\colon M \times M \to M$ (the addition) such that, writing e' for the composition of e with the unique map $M \to \star$, the four diagrams



commute.

See Beck's thesis [1] and Quillen's summary in [16].

Remark 1.2. A useful way of rephrasing Definition 1.1 goes as follows: An abelian group structure on M is a lift of the set-valued resheaf $C \mapsto \mathbf{C}(C,M)$ on $\mathbf{C}$ that is represented by M to an abelian presheaf, that is to a presheaf that takes values in the category of abelian groups.

Let $\mathbf{T}$ be the category of models (or algebras) for an algebraic theory. The category $\mathrm{Ab}(\mathbf{T})$ of abelian group objects in $\mathbf{T}$ is equivalent to the category of models for the tensor product of the given theory with the theory of abelian groups [7]. The category $\mathrm{Ab}(\mathbf{T})$ is also equivalent to the category of modules over a ring, the endomorphism ring of its generator. We will denote this ring by $\mathbb{Z}\mathbf{T}$.

The category $\mathrm{Ab}(\mathbf{T})$ of abelian group objects in $\mathbf{T}$ comes with a faithful forget functor $\mathrm{Ab}(\mathbf{T}) \to \mathbf{T}$, and that functor has a left adjoint

$$\Omega\colon \mathbf{T} \longrightarrow \mathrm{Ab}(\mathbf{T}).$$

This will be referred to as the *(absolute) abelianization* functor.

Example 1.3. If $\mathbf{T} = \mathbf{G}$ is the category of groups, then $\mathrm{Ab}(\mathbf{G})$ is the full subcategory of abelian groups, or $\mathbb{Z}$-modules, so that $\mathbb{Z}\mathbf{G} = \mathbb{Z}$. The abelianization of a group G in the abstract sense discussed above is just the abelianization G^{ab} of that group in the sense of group theory: the quotient of G by its commutator subgroup.

There is a standard recipe to compute the abelianization $\Omega(X)$, at least in principle.

Proposition 1.4. *If the diagram*

$$X \longleftarrow \mathrm{F}\mathbf{T}(S) \leftleftarrows \mathrm{F}\mathbf{T}(R)$$

displays X as a coequalizer of free objects $\mathrm{F}\mathbf{T}(S)$ and $\mathrm{F}\mathbf{T}(R)$, then there is a diagram

$$\Omega(X) \longleftarrow \mathbb{Z}\mathbf{T}(S) \longleftarrow \mathbb{Z}\mathbf{T}(R)$$

that displays $\Omega(X)$ as a cokernel of the difference of the induced maps between free $\mathbb{Z}\mathbf{T}$-modules.

In order to find such a coequalizer diagram, choose a presentation of X by generators S and relations R, or simply take the canonical one, with $S = X$ and $R = \mathrm{F}\mathbf{T}X$.

Proof. If the model $X = \mathrm{F}\mathbf{T}(S)$ happens to be free on a set S, then we want an abelian model $\Omega(\mathrm{F}\mathbf{T}(S))$ together with natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Ab}(\mathbf{T})}(\Omega(\mathrm{F}\mathbf{T}(S)), M) \cong \mathbf{T}(\mathrm{F}\mathbf{T}(S), M) \cong \mathbf{S}(S, M)$$

for all abelian models M. There exists a free abelian model on any given set S, because such a model corresponds to a free $\mathbb{Z}\mathbf{T}$-module $\mathbb{Z}\mathbf{T}(S)$. It is then clear that such a free abelian model solves our problem in this special case, that is when X is free.

In general, we assume that the object X as a colimit of free objects, and use that the functor Ω, as a left adjoint, has to preserve these. Specifically, we assume that we have a diagram

$$X \longleftarrow \mathrm{F}\mathbf{T}(S) \leftleftarrows \mathrm{F}\mathbf{T}(R)$$

that displays X as a coequalizer of free objects $\mathrm{F}\mathbf{T}(S)$ and $\mathrm{F}\mathbf{T}(R)$. Then we get a coequalizer diagram

$$\Omega(X) \longleftarrow \Omega(\mathrm{F}\mathbf{T}(S)) \leftleftarrows \Omega(\mathrm{F}\mathbf{T}(R))$$

of abelian models, so that the abelianization $\Omega(X)$ is the cokernel of the difference of the parallel maps. This presents $\Omega(X)$ in terms of free abelian models, as desired. □

Remark 1.5. On the level of automorphism groups, abelianization induces homomorphisms

$$\mathrm{Aut}_{\mathbf{T}}(\mathrm{F}\mathbf{T}_n) \longrightarrow \mathrm{GL}_n(\mathbb{Z}\mathbf{T}) \tag{1.1}$$

from the automorphism groups of the free models into the general linear groups over the ring $\mathbb{Z}\mathbf{T}$. As we will see below in Remark 2.11, these can be thought of as generalizations of the Burau representations.

Remark 1.6. Let $\mathbf{T}$ be a category of models for an algebraic theory, let X be an object of $\mathbf{T}$, and let M in $\mathrm{Ab}(\mathbf{T})$ be an abelian model. The sets

$$\mathbf{T}(X,M) \cong \mathrm{Hom}_{\mathrm{Ab}(\mathbf{T})}(\Omega(X),M)$$

are actually abelian groups, see Remark 1.2. We will write $\mathrm{Der}(X;M)$ for either of them. The elements are the *derivations* in the sense of Beck. See again [1] and [16]. There is a universal derivation $X \to \Omega(X)$, adjoint to the identity.

2 Abelian racks and quandles

In this section we show how the general concepts from the previous section apply to the theory of racks and quandles. Although there is no claim to originality here, there is nevertheless reason to require a reasonably self-contained exposition: We can use it to fix the notation used throughout the text; it is instructive to see the general concepts of the previous section worked out in the case of interest to us; and the exact statements given here cannot be conveniently referenced from the literature.

Definition 2.1. A *rack* $(R,\rhd)$ is a set R together with a binary operation $\rhd$ such that all left multiplications

$$R \longrightarrow R,\, y \longmapsto x \rhd y$$

are automorphisms, i.e. they are bijective and satisfy

$$x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z)$$

for all x, y, and z.

See the papers by Brieskorn [2] and Fenn–Rourke [4].

Definition 2.2. An *abelian rack* is an abelian group object in the category $\mathbf{R}$ of racks, or equivalently, a rack object in abelian groups.

In other words, an abelian rack is a rack M that is also an abelian group (with zero 0), and both structures are compatible in the sense that the map $M \times M \to M$ that sends (x,y) to $x \rhd y$ is a group homomorphism (with respect to the usual abelian group structure on the product). In equations, this means $0 \rhd 0 = 0$ and

$$(m+n) \rhd (p+q) = (m \rhd p) + (n \rhd q)$$

for all m,n,p,q in M. In particular, we have an automorphism $\alpha \colon M \to M$ of the abelian group M defined by

$$\alpha(x) = 0 \rhd x,$$

and an endomorphism $\varepsilon \colon M \to M$ defined by

$$\varepsilon(x) = x \rhd 0.$$

The equation $x \rhd y = x \rhd 0 + 0 \rhd y$ can then be rewritten

$$x \rhd y = \varepsilon(x) + \alpha(y). \tag{2.1}$$

We see that these two morphisms determine the rack structure and conversely. The calculation

$$\begin{aligned}\varepsilon(\alpha(y)) &= \varepsilon(0 \rhd y)\\ &= (0 \rhd y) \rhd 0\\ &= (0 \rhd y) \rhd (0 \rhd 0)\\ &= 0 \rhd (y \rhd 0)\\ &= \alpha(\varepsilon(y))\end{aligned}$$

shows that α and ε commute.

Proposition 2.3. *The category of abelian racks is equivalent to the category of modules over the ring*

$$\mathbb{Z}\mathbf{R} = \mathbb{Z}[\mathrm{A}^{\pm}, \mathrm{E}]/(\mathrm{E}^2 - \mathrm{E}(1 - \mathrm{A})).$$

Proof. Fenn and Rourke [4, Sec. 1, Ex. 6] have remarked that $\mathbb{Z}\mathbf{R}$-modules define racks, using (2.1). Conversely, the calculation

$$\begin{aligned}
\varepsilon(x) &= x \rhd 0 \\
&= x \rhd (0 \rhd 0) \\
&= (x \rhd 0) \rhd (x \rhd 0) \\
&= \varepsilon(x) \rhd \varepsilon(x) \\
&= (\varepsilon(x) \rhd 0) + (0 \rhd \varepsilon(x)) \\
&= \varepsilon^2(x) + \alpha\varepsilon(x)
\end{aligned}$$

shows that any abelian rack admits a module structure over that ring. □

Remark 2.4. If X is a rack, its (absolute) abelianization $\Omega(X)$ corresponds to the quotient of the free $\mathbb{Z}\mathbf{R}$-module with basis X by the relations

$$x \rhd y = \mathrm{E}x + \mathrm{A}y$$

for x and y in X. For instance, if $\star$ is the terminal rack, then it has precisely one element, and we get that $\Omega(\star)$ is the quotient of the ring $\mathbb{Z}\mathbf{R}$ by the ideal generated by the element $1 = \mathrm{E} + \mathrm{A}$. This is the ring $\mathbb{Z}[\mathrm{A}^{\pm}]$, with $\mathrm{E} = 1 - \mathrm{A}$.

Every rack R comes with a canonical automorphism F_R that is defined by the simple equation $\mathrm{F}_R(x) = x \rhd x$, see [18] for an exhaustive study.

Definition 2.5. A *quandle* is a rack such that its canonical automorphism is the identity.

The theory of quandles was born in the papers of Joyce [9] and Matveev [12], after a considerable embryonal phase for which we refer to the original papers and [4] again.

Proposition 2.6. *The category of abelian quandles is equivalent to the category of modules over the ring*

$$\mathbb{Z}\mathbf{Q} = \mathbb{Z}[\mathrm{A}^{\pm}].$$

Proof. Again, it is well known that $\mathbb{Z}\mathbf{Q}$-modules define quandles. See [9, Sec. 1, p. 38], [12, § 2, Ex. 1], or [4, Sec. 1, Ex. 5], for instance. Conversely, the quandle condition $x \rhd x = x$ implies

$$x = x \rhd x = x \rhd 0 + 0 \rhd x = \varepsilon(x) + \alpha(x),$$

or

$$\varepsilon = \mathrm{id} - \alpha.$$

This leads to the relation $\varepsilon^2 = \varepsilon(1-\alpha)$ for abelian quandles. □

Definition 2.7. A rack is *involutary* if the axiom $x \rhd (x \rhd y)$ is satisfied.

If M is an abelian involutary rack, then we have $\alpha^2 = \mathrm{id}$. This implies the following two results.

Proposition 2.8. *The category of abelian involutary racks is equivalent to the category of modules over the ring*

$$\mathbb{Z}\mathbf{I} = \mathbb{Z}[\mathrm{A}^{\pm}, \mathrm{E}]/(\mathrm{E}^2 - \mathrm{E}(1-\mathrm{A}), \mathrm{A}^2 - 1).$$

Proposition 2.9. *The category of abelian kei is equivalent to the category of modules over the ring*

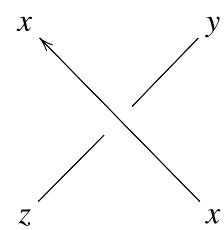
$$\mathbb{Z}\mathbf{K} = \mathbb{Z}[\mathrm{A}]/(\mathrm{A}^2 - 1).$$

Remark 2.10. Note that in the two idempotent cases (when the canonical automorphism is trivial) we get group rings $\mathbb{Z}\mathbf{Q} \cong \mathbb{Z}[\mathrm{C}_\infty]$ and $\mathbb{Z}\mathbf{K} \cong \mathbb{Z}[\mathrm{C}_2]$, where C_n denotes a cyclic group of order n.

Remark 2.11. The Burau representations (1.1) for the theories **R**, **Q**, **I**, and **K** take the following form.

$$\begin{aligned}
\mathrm{Aut}_{\mathbf{R}}(\mathrm{F}\mathbf{R}_n) &\longrightarrow \mathrm{GL}_n(\mathbb{Z}[\mathrm{A}^{\pm}, \mathrm{E}]/(\mathrm{E}^2 - \mathrm{E}(1-\mathrm{A})))\\
\mathrm{Aut}_{\mathbf{Q}}(\mathrm{F}\mathbf{Q}_n) &\longrightarrow \mathrm{GL}_n(\mathbb{Z}[\mathrm{A}^{\pm 1}])\\
\mathrm{Aut}_{\mathbf{I}}(\mathrm{F}\mathbf{I}_n) &\longrightarrow \mathrm{GL}_n(\mathbb{Z}[\mathrm{A}^{\pm}, \mathrm{E}]/(\mathrm{E}^2 - \mathrm{E}(1-\mathrm{A}), \mathrm{A}^2 - 1))\\
\mathrm{Aut}_{\mathbf{K}}(\mathrm{F}\mathbf{K}_n) &\longrightarrow \mathrm{GL}_n(\mathbb{Z}[\mathrm{A}^{\pm}]/(\mathrm{A}^2 - 1))
\end{aligned}$$

Figure 3.1: A crossing in a knot diagram



The name comes from the fact that the braid group on n strands embeds into the group $\mathrm{Aut}_{\mathbf{Q}}(\mathrm{F}\mathbf{Q}_n)$ in such a way that the restriction of the representation above is the classical Burau representation [3].

3 The classical Alexander modules of knots

In this section we apply the theory of abelian quandles to the quandles arising in knot theory. We will also see how the classical Alexander invariants can be interpreted in these terms.

Let K be a knot with knot quandle $\mathrm{Q}K$. This has been introduced by Joyce [9] and Matveev [12], who have also shown, building on deep results of Waldhausen's [21], that the knot quandle is a complete invariant of the knot.

We can find a presentation of $\mathrm{Q}K$ from any diagram of the knot: The generators are the arcs, and there is a relation of the form $x \rhd y = z$ whenever x, y, and z meet in a crossing, with x as the overpass, and y turns into z under it, as in Figure 3.1.

In other words, we can present the quandle $\mathrm{Q}K$ as a coequalizer

$$\mathrm{Q}K \longleftarrow \mathrm{F}\mathbf{Q}\{\text{arcs}\} \leftleftarrows \mathrm{F}\mathbf{Q}\{\text{crossings}\}, \tag{3.1}$$

where one of the two parallel arrows sends a crossing involving the arcs x, y, and z as in Figure 3.1 to the element $x \triangleright y$ in the free quandle on the set of arcs, and the other one sends it to the arc z.

There is a forgetful functor from the category of groups to the category of quandles: Given a group G, the underlying set comes with the quandle structure given by $x \triangleright y = xyx^{-1}$. This functor admits a left adjoint, and that left adjoint sends $\mathrm{Q}K$ to the knot group πK. The presentation (3.1) above is mapped to Wirtinger's (unpublished) presentation of the knot group, where the relation $x \triangleright y = z$ now reads $xyx^{-1} = z$.

The abelianization $\pi^{\mathrm{ab}}K$ of the knot group is always infinite cyclic, independent of the knot K. There are different knots with isomorphic knot groups, but at least the 'kernel' of the knot group invariant is trivial: By [14], the knot group is abelian (hence infinite cyclic) if and only if the knot is trivial.

The weakness of the knot group as an invariant has its virtues: The kernel of the abelianization $\pi K \to \pi^{\mathrm{ab}}K$ defines an infinite cyclic covering of the knot complement, and the first homology of the covering space, as a module over the group ring $\mathbb{Z}[\mathrm{A}^{\pm 1}]$ of the covering group, is the classical Alexander module of the knot. The Alexander polynomial is the characteristic polynomial of the action of the generator A on the torsion part. See Milnor's concise summary [13].

The abelianization $\Omega(\mathrm{Q}K)$ of the knot quandle $\mathrm{Q}K$ is an abelian quandle, or equivalently, according to Proposition 2.6, it can also be thought of as a module over the Laurent polynomial ring $\mathbb{Z}\mathbf{Q} = \mathbb{Z}[\mathrm{A}^{\pm 1}]$. The general formalism of Section 1 allows us to compute the module $\Omega(\mathrm{Q}K)$ from a presentation given by a diagram of the knot as follows.

Proposition 3.1. *Given a diagram of a knot K, the $\mathbb{Z}[\mathrm{A}^{\pm 1}]$-module $\Omega(\mathrm{Q}K)$ is a cokernel of the homomorphism*

$$\mathbb{Z}[\mathrm{A}^{\pm 1}]\{\text{arcs}\} \longleftarrow \mathbb{Z}[\mathrm{A}^{\pm 1}]\{\text{crossings}\}$$

between free $\mathbb{Z}[\mathrm{A}^{\pm 1}]$*-modules that sends a crossing to the* $\mathbb{Z}[\mathrm{A}^{\pm 1}]$*-linear combination*

$$(1-\mathrm{A})x + \mathrm{A}y - z$$

of the arcs involved in that crossing as in Figure 3.1.

Proof. Recall that the knot quandle $\mathrm{Q}K$ has a presentation as a coequalizer (3.1). Abelianization Ω is a left-adjoint, and it therefore preserved colimits such as coequalizers. This immediately leads to the stated result. □

The following result has been established, in a different mathematical language, by Joyce [9, Sec. 17] and Matveev [12, §11].

Proposition 3.2. *The abelianization of* $\mathrm{Q}K$*, the* $\mathbb{Z}[\mathrm{A}^{\pm 1}]$*-module* $\Omega(\mathrm{Q}K)$*, is isomorphic to the classical Alexander module of the knot.*

Example 3.3. The unknot U has a diagram with one arc and no crossing. Therefore its quandle $\mathrm{Q}U$ is the free quandle on one generator. The free quandle on one generator is the terminal quandle with one element. Its abelianization is a free $\mathbb{Z}[\mathrm{A}^{\pm 1}]$-module on one generator. It can also be described as the cokernel of the homomorphism $\mathbb{Z}[\mathrm{A}^{\pm 1}] \longleftarrow 0$.

Example 3.4. For the trefoil knot T, the usual diagram with three arcs and three crossings leads to the presentation matrix

$$\begin{bmatrix} 1-\mathrm{A} & -1 & \mathrm{A} \\ \mathrm{A} & 1-\mathrm{A} & -1 \\ -1 & \mathrm{A} & 1-\mathrm{A} \end{bmatrix}$$

for the abelianization $\Omega(\mathrm{Q}T)$, the classical Alexander module of T, as a cokernel of a $\mathbb{Z}[\mathrm{A}^{\pm 1}]$-linear endomorphism of $\mathbb{Z}[\mathrm{A}^{\pm 1}]^{\oplus 3}$. We recognize that the Alexander module is isomorphic to $\mathbb{Z}[\mathrm{A}^{\pm 1}]/(\mathrm{A}^2-\mathrm{A}+1) \oplus \mathbb{Z}[\mathrm{A}^{\pm 1}]$.

For a general knot K, the Alexander module will be isomorphic to

$$\Omega(\mathrm{Q}K) \cong \mathbb{Z}[\mathrm{A}^{\pm 1}]/(\Delta_K(\mathrm{A})) \oplus \mathbb{Z}[\mathrm{A}^{\pm 1}],$$

where Δ_K is the Alexander polynomial of K.

4 Beck modules

Many categories $\mathbf{T}$ of models for algebraic theories do not have interesting abelian group objects. For instance, this is the case for the theory of commutative rings. And even for those theories that have interesting abelian group objects, one can do better than just looking at the absolute abelianization functor: One can use a relative version of it. This leads to abelian invariants that are tailored to a given object X of $\mathbf{T}$. For instance, given a group G, this naturally leads us to consider G-modules, that is modules over the integral group ring $\mathbb{Z}G$. In general, the situation is more complicated though.

Let us choose an object X in $\mathbf{T}$, and let $\mathbf{T}_X$ denote the slice category of objects of $\mathbf{T}$ over X. Products in the category $\mathbf{T}_X$ are pullbacks in the category $\mathbf{T}$. Let again $\mathrm{Ab}(\mathbf{T}_X)$ denote the category of abelian group objects in the category $\mathbf{T}_X$. According to Quillen [16, p. 69], this category is abelian.

Definition 4.1. The objects in $\mathrm{Ab}(\mathbf{T}_X)$ are the *X-modules* in the sense of Beck [1].

Example 4.2. If $\mathbf{T} = \mathbf{G}$ is the category of groups, and G is any group then the category $\mathrm{Ab}(\mathbf{G}_G)$ is equivalent to the category of $\mathbb{Z}G$-modules. An equivalence is given by associating with every abelian group over G the kernel of its structure homomorphism to G.

Remark 4.3. Contrary to what one might wish for, the abelian category $\mathrm{Ab}(\mathbf{T}_X)$ is not always equivalent to the category of modules over a ring. For instance, when $\mathbf{T} = \mathbf{S}$ is the algebraic theory of sets, then $\mathbf{T}_X$ is the category of sets over X, or equivalently–by passage to fibers–the category of X-graded sets. Then $\mathrm{Ab}(\mathbf{T}_X)$ is the category of X-graded abelian groups, and I only know how to realize this (up to equivalence) as a category of modules over a ring if X is finite. This problem disappears if one is willing to work with 'ringoids.'

Definition 4.4. The left adjoint

$$\Omega_X \colon \mathbf{T}_X \longrightarrow \mathrm{Ab}(\mathbf{T}_X).$$

to the forgetful functor $\mathrm{Ab}(\mathbf{T}_X) \to \mathbf{T}_X$ is the *(relative) abelianization* functor.

Remark 4.5. In the relative situation, when Y in $\mathbf{T}_X$ is an object over X, and N in $\mathrm{Ab}(\mathbf{T}_X)$ is an X-module in the sense of Beck, we will write $\mathrm{Der}_X(Y;N)$ for the module

$$\mathbf{T}_X(Y,N) \cong \mathrm{Hom}_{\mathrm{Ab}(\mathbf{T}_X)}(\Omega_X(Y),N)$$

of *X-derivations* from Y into N. We will mostly be interested in the case when $Y = X$ is the terminal object over X, for reasons that will become apparent in Remark 4.7 below.

Let us pause to see how the X-module $\Omega_X(Y)$ might be computed, at least in principle.

Proposition 4.6. *If the diagram*

$$Y \longleftarrow \mathrm{F}\mathbf{T}(S) \leftleftarrows \mathrm{F}\mathbf{T}(R)$$

displays Y as a coequalizer of free objects $\mathrm{F}\mathbf{T}(S)$ and $\mathrm{F}\mathbf{T}(R)$, then there is a diagram

$$\Omega_X(Y) \longleftarrow \Omega_X(\mathrm{F}\mathbf{T}(S)) \longleftarrow \Omega_X(\mathrm{F}\mathbf{T}(R))$$

that displays $\Omega_X(Y)$ as the cokernel of the difference of the induced maps between free X-modules.

Proof. If the object $Y = \mathrm{F}\mathbf{T}(S)$ over X is free on a set S, then we want an X-module $\Omega_X(\mathrm{F}\mathbf{T}(S))$ together with natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Ab}(\mathbf{T}_X)}(\Omega_X(\mathrm{F}\mathbf{T}(S)),N) \cong \mathbf{T}_X(\mathrm{F}\mathbf{T}(S),N) \cong \mathbf{S}_X(S,N)$$

for all X-modules N. For any given set $S \to X$ over X, there exists a free X-module over it. It is again clear that such a free X-module over the composition $S \to \mathrm{F}\mathbf{T}(S) = Y \to X$ solves our problem.

In general, we can write the object Y as a colimit of free objects, and use that the functor Ω_X, as a left adjoint, has to preserve these. Specifically, if

$$Y \longleftarrow \mathrm{F}\mathbf{T}(S) \leftleftarrows \mathrm{F}\mathbf{T}(R)$$

displays Y as a coequalizer of free objects $\mathrm{F}\mathbf{T}(S)$ and $\mathrm{F}\mathbf{T}(R)$, then we have a coequalizer diagram

$$\Omega_X(Y) \longleftarrow \Omega_X(\mathrm{F}\mathbf{T}(S)) \leftleftarrows \Omega_X(\mathrm{F}\mathbf{T}(R))$$

of X-modules, so that the module $\Omega_X(Y)$ is the cokernel of the difference of the parallel induced maps. This presents $\Omega_X(Y)$ in terms of free X-modules, as desired. □

Remark 4.7. One of the most confusing aspects of the theory might be the interplay between absolute and relative abelianizations. Let us expand on this a bit. If $f\colon X \to Y$ is a morphism in $\mathbf{T}$, pullback defines a functor

$$f^*\colon \mathbf{T}_Y \longrightarrow \mathbf{T}_X$$

that preserves limits. It maps abelian group objects to abelian group objects, so that we also have a functor

$$\mathrm{Ab}(f^*)\colon \mathrm{Ab}(\mathbf{T}_Y) \longrightarrow \mathrm{Ab}(\mathbf{T}_X)$$

that commutes with the forgetful functors. We deserve a diagram.

$$\begin{array}{ccc} \mathbf{T}_X & \xleftarrow{\;f^*\;} & \mathbf{T}_Y \\ \uparrow & & \uparrow \\ \mathrm{Ab}(\mathbf{T}_X) & \xleftarrow[\mathrm{Ab}(f^*)]{} & \mathrm{Ab}(\mathbf{T}_Y) \end{array}$$

Both of the functors f^* and $\mathrm{Ab}(f^*)$ have left adjoints, say f_* and $\mathrm{Ab}(f_*)$. Typically, only the first is given by composition with f, the second is rarely. In any event, the left adjoints always commute with the abelianization functors as indicated in the following diagram.

$$\begin{array}{ccc} \mathbf{T}_X & \xrightarrow{\;f_*\;} & \mathbf{T}_Y \\ {\scriptstyle \Omega_X}\downarrow & & \downarrow{\scriptstyle \Omega_Y} \\ \mathrm{Ab}(\mathbf{T}_X) & \xrightarrow[\mathrm{Ab}(f_*)]{} & \mathrm{Ab}(\mathbf{T}_Y) \end{array}$$

In particular, by evaluation at the identity id_X, thought of as the terminal object in the category $\mathbf{T}_X$, we get isomorphisms

$$\Omega_Y(X) \cong \mathrm{Ab}(f_*)\Omega_X(X). \tag{4.1}$$

For this reason, it is common to concentrate on the X-module $\Omega_X(X)$ and refer to the other X-modules $\Omega_Y(X)$ and $\mathrm{Ab}(f_*)$ only when needed. For instance, in the extreme case, if $Y = \star$ is the terminal object in the category $\mathbf{T}$, the relation (4.1) reads

$$\Omega(X) \cong \mathrm{Ab}(f_*)\Omega_X(X). \tag{4.2}$$

This correctly suggests that $\Omega_X(X)$ is the better invariant than $\Omega(X)$, and our focus will be on it from now on. We can also think of f as a morphism over Y, and then it induces a homomorphism $\Omega_Y(X) \to \Omega_Y(Y)$ of Y-modules that can be translated into a homomorphism

$$\mathrm{Ab}(f_*)\Omega_X(X) \longrightarrow \Omega_Y(Y)$$

using (4.1). These are useful to have around once one commits to working with the $\Omega_X(X)$ only.

Example 4.8. If $\mathbf{T} = \mathbf{G}$ is the category of groups, and G is any group, then the G-module $\Omega_G(G)$ corresponds to the $\mathbb{Z}G$-module $\mathrm{I}G$, the augmentation ideal of the group ring. For a group $\varphi\colon U \to G$ over G, the base change $\mathrm{Ab}(\varphi_*)$ from U-modules to G-modules is given by $M \mapsto \mathbb{Z}G \otimes_{\mathbb{Z}U} M$, so that the $\mathbb{Z}G$-module $\mathbb{Z}G \otimes_{\mathbb{Z}U} \mathrm{I}U$ corresponds to the G-module $\Omega_G(U)$. In particular, we recover the isomorphisms $G^{\mathrm{ab}} \cong \Omega(G) \cong \mathbb{Z} \otimes_{\mathbb{Z}G} \mathrm{I}G$. Compare with Quillen's notes [15, II.5] or Frankland's exposition [6, Sec. 5.1], for instance.

We will now turn our attention to Beck modules for the theories of racks and quandles, where it is, in general, no longer possible to describe these objects as modules over a ring.

5 Rack and quandle modules

Beck modules in the categories of racks and quandles have been studied by Jackson [8]. The following result rephrases Theorem 2.2 in *loc.cit.*.

Proposition 5.1. *If X is a rack, a rack module M over X is a family $(\,M(x)\,|\,x \in X\,)$ of abelian groups together with homomorphisms*

$$M(x) \xrightarrow{\varepsilon(x,y)} M(x \rhd y) \xleftarrow{\alpha(x,y)} M(y)$$

for each pair x, y of elements, such that the following conditions (M1)*,* (M2)*, and* (M3) *are satisfied.*

(M1) The homomorphisms $\alpha(x,y)$ are invertible and satisfy

$$\alpha(x, y \rhd z)\alpha(y,z) = \alpha(x \rhd y, x \rhd z)\alpha(x,z)$$

for all x, y, and z.

(M2) The αs and εs commute whenever it makes sense.

(M3) We have $\varepsilon^2 = (\mathrm{id} - \alpha)\varepsilon$ whenever it makes sense.

Remark 5.2. It follows from condition (M1) that a rack module M comes with (non-canonical) isomorphisms $M(y) \cong M(z)$ whenever the elements y and z are in the same orbit.

Remark 5.3. One might wonder if the situation can be described more concisely using the compositions $\alpha^{-1}\varepsilon$.

The following result rephrases [8, Thm. 2.6].

Proposition 5.4. *If X is a quandle, a quandle module M over X is a rack module M such that, in addition to* (M1)*,* (M2)*, and* (M3)*, also the following condition* (M4) *is satisfied.*

(M4) We have

$$\varepsilon(x,x) = \mathrm{id}_{M(x)} - \alpha(x,x)$$

as endomorphisms of $M(x)$ for all x in X.

Examples 5.5. Given any abelian rack or quandle A in the sense of Section 2, thought of as a module over the terminal object, its pullback $X \times A$ is a 'constant' Beck module over X. These modules are 'trivial' if, in addition, we have $\alpha = \mathrm{id}$ and $\varepsilon = 0$. More generally, the condition $\alpha = \mathrm{id}$ only forces $\varepsilon^2 = 0$, and these modules may be called 'differential.' At the opposite extreme, if we have $\varepsilon = 0$, then we are left with the automorphism α, and an 'automorphic' module.

Remark 5.6. If M is a rack module over a rack X, then the disjoint union of the family $(\, M(x) \,|\, x \in X \,)$ can be turned into a rack such that the canonical map to X is a rack morphism. A formula for the operation is given by

$$m \rhd n = \varepsilon(x,y)m + \alpha(x,y)n, \tag{5.1}$$

when $m \in M(x)$ and $n \in M(y)$, generalizing (2.1). If X is a quandle, and M is a quandle module over it, then this construction will give a quandle. We will denote the resulting object over X by M again.

Remark 5.7. If X is a rack (or a quandle), and N is an X-module (in the appropriate sense), then an X-derivation from X into N is just a section $\nu\colon X \to N$ of the morphisms $N \to X$. In other words, it picks out a family $(\, \nu(x) \in N(x) \,|\, x \in X \,)$ of elements such that

$$\nu(x \rhd y) = \varepsilon(x,y)\nu(x) + \alpha(x,y)\nu(y)$$

holds. The right hand side equals $\nu(x) \rhd \nu(y)$ by (5.1). Of course, if N is constant, this formula simplifies to $\nu(x \rhd y) = \varepsilon\nu(x) + \alpha\nu(y)$. More generally we have X-derivations from Y into N, where Y is any rack (or quandle) over X.

6 Alexander–Beck modules of knots

We are now ready to apply the general theory of quandle modules to the fundamental quandles of knots.

Definition 6.1. Let K be a knot with knot quandle $\mathrm{Q}K$. The *Alexander–Beck module* of K is the $\mathrm{Q}K$-module $\Omega_{\mathrm{Q}K}(\mathrm{Q}K)$.

We can find a presentation of the Alexander–Beck module of a knot from any diagram of the knot.

Proposition 6.2. *Given any diagram of any knot K, its Alexander–Beck $\mathrm{Q}K$-module $\Omega_{\mathrm{Q}K}(\mathrm{Q}K)$ is a cokernel of the homomorphism*

$$\Omega_{\mathrm{Q}K}(\mathrm{F}\mathbf{Q}\{\mathrm{arcs}\}) \longleftarrow \Omega_{\mathrm{Q}K}(\mathrm{F}\mathbf{Q}\{\mathrm{crossings}\})$$

between free $\mathrm{Q}K$-modules that sends a crossing as in Figure 3.1 to the element

$$\varepsilon(x,y)x + \alpha(x,y)y - z \tag{6.1}$$

in $\Omega_{\mathrm{Q}K}(\mathrm{F}\mathbf{Q}\{\mathrm{arcs}\})(z)$.

Using the rack structure on $\mathrm{Q}K$-modules from Remark 5.6, the relation (6.1) can be written $x \rhd y = z$, of course.

Proof. Recall that the knot quandle $\mathrm{Q}K$ has a presentation as a coequalizer

$$\mathrm{Q}K \longleftarrow \mathrm{F}\mathbf{Q}\{\mathrm{arcs}\} \leftleftarrows \mathrm{F}\mathbf{Q}\{\mathrm{crossings}\}.$$

The (relative) abelianization functor $\Omega_{\mathrm{Q}K}$ is a left-adjoint, and it therefore preserves colimits such as coequalizers. □

Proposition 6.3. *The Alexander–Beck module of a knot determines its classical Alexander module.*

Proof. According to (4.2), the absolute abelianization $\Omega(Q)$ of a quandle Q is determined by the relative abelianization $\Omega_Q(Q)$ as the pushforward along the unique morphism $Q \to \star$ to the terminal object. For $Q = \mathrm{Q}K$ this means that the Alexander–Beck module $\Omega_{\mathrm{Q}K}(\mathrm{Q}K)$ determines the $\mathbb{Z}[\mathrm{A}^{\pm 1}]$-module $\Omega(\mathrm{Q}K)$. According to Proposition 3.2, the latter is isomorphic to the classical Alexander module of the knot. □

In Example 6.5 we will see that the converse to the statement in Proposition 6.3 does not hold: The Alexander–Beck module of a knot is a better invariant than its classical Alexander module. This is based on the following result.

Theorem 6.4. *A knot is trivial if and only its Alexander–Beck module is a free module over its knot quandle.*

Proof. One direction is easy: If $K = U$ is the unknot, then $\mathrm{Q}U = \mathrm{F}\mathbf{Q}_1$ is a free quandle on one generator. This is the singleton with the unique quandle structure, which is also the terminal object in the category $\mathbf{Q}$ of quandles. It follows that there is no difference between the absolute and the relative abelianization, and we get

$$\Omega_{\mathrm{Q}U}(\mathrm{Q}U) = \Omega(\mathrm{Q}U) = \Omega(\mathrm{F}\mathbf{Q}_1),$$

and this corresponds to the free $\mathbb{Z}\mathbf{Q}$-module of rank 1 as we have seen in Section 4.

As for the other direction: Let K be a knot such that the $\mathrm{Q}K$-module $\Omega_{\mathrm{Q}U}(\mathrm{Q}U)$ is free. The left adjoint Φ_* in the adjunction

$$\Phi_* \colon \mathbf{Q} \rightleftarrows \mathbf{G} \colon \Phi^* \tag{6.2}$$

between the category $\mathbf{Q}$ of quandles and the category $\mathbf{G}$ of groups sends the knot quandle $\mathrm{Q}K$ to the knot group $\pi K = \Phi_* \mathrm{Q}K$. This adjunction induces an adjunction

$$\Phi_* \colon \mathbf{Q}_{\mathrm{Q}K} \rightleftarrows \mathbf{G}_{\pi K} \colon \Psi^*$$

between the slice categories. See [6, Prop. 4.1]. The left adjoint Φ_* sends $P \to \mathrm{Q}K$ to the arrow $\Phi_* P \to \Phi_* \mathrm{Q}K = \pi K$ induced by Φ_*, justifying the notation. The right adjoint Ψ^* is the composition of Φ^* with the pullback functor along the unit

$$u\colon \mathrm{Q}K \longrightarrow \Phi^*\Phi_*\mathrm{Q}K = \Phi^*\pi K$$

of the adjunction (6.2) at $\mathrm{Q}K$.

$$\begin{array}{ccc} \Psi^*G & \longrightarrow & \Phi^*G \\ \downarrow & & \downarrow \\ \mathrm{Q}K & \longrightarrow & \Phi^*\pi K \end{array}$$

This right adjoint Ψ^* restricts to a functor between abelian group objects, and that functor has a left adjoint Ψ_*, say.

$$\Psi_*\colon \mathrm{Ab}(\mathbf{Q}_{\mathrm{Q}K}) \rightleftarrows \mathrm{Ab}(\mathbf{G}_{\pi K})\colon \Psi^* \tag{6.3}$$

See [6, Prop. 4.2]. We get a commutative diagram

$$\begin{array}{ccc} \mathbf{Q}_{\mathrm{Q}K} & \xleftarrow{\Psi^*} & \mathbf{G}_{\pi K} \\ \subseteq\uparrow & & \uparrow\subseteq \\ \mathrm{Ab}(\mathbf{Q}_{\mathrm{Q}K}) & \xleftarrow[\Psi^*]{} & \mathrm{Ab}(\mathbf{G}_{\pi K}) \end{array}$$

of right adjoints, so that the diagram

$$\begin{array}{ccc} \mathbf{Q}_{\mathrm{Q}K} & \xrightarrow{\Psi_*} & \mathbf{G}_{\pi K} \\ \Omega_{\mathrm{Q}K}\downarrow & & \downarrow\Omega_{\pi K} \\ \mathrm{Ab}(\mathbf{Q}_{\mathrm{Q}K}) & \xrightarrow[\Psi_*]{} & \mathrm{Ab}(\mathbf{G}_{\pi K}) \end{array}$$

of left adjoints also commutes. Evaluated at the identity $\mathrm{Q}K = \mathrm{Q}K$, this proves

$$\Psi_*\Omega_{\mathrm{Q}K}(\mathrm{Q}K) = \Omega_{\pi K}(\pi K).$$

If the left adjoint Ψ_* preserves free objects, then our assumption implies that the πK-module $\Omega_{\pi K}(\pi K)$ is free. Let us assume for a moment that this is the case, to see that this allows us to finish the proof as follows: Under the equivalence between the category of πK-modules and the category of $\mathbb{Z}\pi K$-modules, this module corresponds to the $\mathbb{Z}\pi K$-module $\mathrm{I}\pi K$, the augmentation ideal in the group ring $\mathbb{Z}\pi K$. It follows that $\mathrm{I}\pi K$ is a free $\mathbb{Z}\pi K$-module. Then the defining exact sequence

$$0 \longleftarrow \mathbb{Z} \longleftarrow \mathbb{Z}\pi K \longleftarrow \mathrm{I}\pi K \longleftarrow 0$$

is a free resolution of the trivial $\mathbb{Z}\pi K$-module $\mathbb{Z}$. It follows that the knot group πK has cohomological dimension 1. By the characterization of groups of cohomological dimension 1, due to Stallings [17] and Swan [20], the knot group πK is free, and therefore necessarily cyclic. By [14] again, the knot K is trivial, as desired.

It remains to be seen that the left adjoint Ψ_* preserves free objects. To do so, let us choose a free module R over $Q = \mathrm{Q}K$ with basis a set $S \to Q$ over Q, so that there is an adjunction bijection

$$\mathrm{Hom}_{\mathrm{Ab}(\mathbf{Q}_Q)}(R,M) \cong \mathbf{S}_Q(S,M) \tag{6.4}$$

for all Q-modules M. We need to show that the image $\Psi_* R$ is a free module over the group $\pi = \pi K$. So we compute

$$\mathrm{Hom}_{\mathrm{Ab}(\mathbf{G}_\pi)}(\Psi_*(R),N) \cong \mathrm{Hom}_{\mathrm{Ab}(\mathbf{Q}_Q)}(R,\Psi^*(N)) \cong \mathbf{S}_Q(S,\Psi^*(N)),$$

the first by the adjunction (6.3), and the second by the adjunction (6.4). We are done if the right hand side is also naturally isomorphic to $\mathbf{S}_\pi(L(S),N)$ for some functor L that takes sets over Q to sets over π. In other words, we need to know that the composition

$$\mathrm{Ab}(\mathbf{G}_\pi) \xrightarrow{\Psi^*} \mathrm{Ab}(\mathbf{Q}_Q) \xrightarrow{\subseteq} \mathbf{S}_Q$$

has a left adjoint, and that is clear: It is a composition of right adjoints, so that the composition of their left adjoints is the left adjoint. □

Example 6.5. On the one hand, the Alexander polynomial of the Conway knot [5] is trivial, and therefore its Alexander module is free. On the other hand, the Conway knot is not trivial, and Theorem 6.4 ensures that its Alexander–Beck module is not free. It it therefore a stronger invariant.

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Department of Mathematical Sciences
NTNU Norwegian University of Science and Technology
7491 Trondheim
NORWAY

markus.szymik@math.ntnu.no
www.math.ntnu.no/∼markussz