K3 spectra

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Abstract

We introduce the notion of a K3 spectrum in analogy with that of an elliptic spectrum and show that there are "enough" K3 spectra in the sense that for all K3 surfaces X in a suitable moduli stack of K3 surfaces there is a K3 spectrum whose underlying ring is isomorphic to the local ring of the moduli stack in X with respect to the etale topology, and similarly for the ring of formal functions on the formal deformation space.

1 Introduction

Stable homotopy theory may be defined as the theory of the sphere spectrum S and its homotopy groups $\pi_* S$, the stable homotopy groups of spheres. But, just as number theory may be defined as the theory of the integers, in practice that entails the study of other structures such as fields, both global and local, Galois groups, their representations, deformations, and many other things which – typically – are harder to construct, but easier to understand.

In stable homotopy theory, some of the auxiliary spectra encountered are the Eilenberg-MacLane spectra such as $\mathrm{H}\mathbb{Q}$, leading to rational homotopy theory, topological K-theory spectra KU and KO, elliptic spectra and TMF, and many more. The "chromatic" hierarchy of these is organised by the complex bordism spectrum MU, which is relevant for the fact that $\pi_*\mathrm{MU}$ represents graded formal group laws rather than for its relationship with manifolds. For example, $\mathrm{H}\mathbb{Q}$ corresponds to the additive formal group law, KU to the multiplicative formal group law, and elliptic spectra to formal group laws coming from elliptic curves.

There have been efforts to extend the connection between elliptic curves and spectra to other geometric objects such as curves of higher genus, see [10] and [23], and abelian varieties of higher dimension, see [3]. The aim of this report is to present some other part of arithmetic geometry which may be lurking behind the next layers of the chromatic hierarchy: K3 surfaces and their corresponding K3 spectra.

It is not a new idea to use K3 surfaces, and more generally Calabi-Yau varieties, as generalisations of elliptic curves, not even in topology. I first read about the idea of cohomology theories related to K3 surfaces in Thomas' writings [28] and [27], where he refers to Morava. The latter was so kind to share his notes [19] based then again on a lecture of Hopkins. See also [6] for a recent contribution to the problem of finding a differential geometric (rather than homotopical) description of K3 cohomology.

In recent years a lot of work has been done, on the side of algebraic topology just as well as on the side of arithmetic geometry, and suggests to have a fresh look at this connection. For example, the papers [7] and [22] give detailed accounts of the height stratification on the moduli stacks of K3 surfaces in prime characteristic. And there are now various different methods to produce new species in the "brave new world" of highly structured ring spectra and localisations thereof, see [24], [9], and [17] to name a few. How far these apply to K3 spectra is the author's work in progress and will be addressed elsewhere.

The purpose here is to show that there are "enough" K3 spectra in the sense that for all K3 surfaces X in a suitable moduli stack of K3 surfaces there is a K3 spectrum whose underlying ring is isomorphic to the local ring of the moduli stack in X with respect to the etale topology, and similarly for the ring of formal functions on the formal deformation space. The precise statements are given in Propositions 8 and 9, which follow from the main Theorem 1. The proofs for these are in Section 5. Sections 2 and 3 review the geometry and arithmetic of K3 surfaces, respectively, and Section 4 gives the definition of K3 spectra and examples over the rationals.

2 K3 surfaces

The aim of this section is to review the geometry of K3 surfaces as far as needed in the rest of the text. All fields will be assumed perfect from now on.

2.1 Definition of K3 surfaces

An elliptic curve over a field k is a smooth proper curve X such that the canonical bundle $\omega_X = \bigwedge^1(\Omega_X)$ is trivial, with a chosen base point. The base point can be used for various purposes. It can be used to impose an abelian group structure on the curve. And it can be used to define a Weierstrass embedding into the projective plane.

One dimension higher, there are two kinds of smooth proper surfaces X such that the canonical bundle $\omega_X = \bigwedge^2(\Omega_X)$ is trivial: abelian surfaces and another class of surfaces which satisfy the additional condition $\mathrm{H}^1(X, \mathcal{O}_X) = 0$: the *K3 surfaces*. One generalisation of this to higher dimensions would be Calabi-Yau n-folds, which are defined by the triviality of the canonical bundle $\omega_X = \bigwedge^n(\Omega_X)$ and the vanishing of $\mathrm{H}^i(X, \mathcal{O}_X)$ for all $1 \leqslant i \leqslant n-1$. Apart from a few side remarks, these will play no rôle in the following.

The Euler characteristic of K3 surfaces is 24, so that these will never be abelian groups. However, every K3 surface X can be embedded in some projective space by means of an ample line bundle L on X; the choice of such an L is called a *polarisation* of X. This corresponds to the choice of the base point for elliptic curves. The self-intersection of L is called the *degree* of the polarisation; this will be always be an even integer $2d$ for some $d \geqslant 1$.

2.2 Examples of K3 surfaces

It will be useful to bear the following two classes of examples in mind. For these, the base field k needs to be of characteristic $\mathrm{char}(k) \neq 2$.

Example. The most famous example for a K3 surface is the *Fermat quartic* defined by the equation

$$T_0^4 + T_1^4 + T_2^4 + T_3^4 = 0$$

in projective 3-space $\mathbb{P}^3$. More generally, all smooth quartics in $\mathbb{P}^3$ define K3 surfaces. This is analogous to the fact that smooth cubics in $\mathbb{P}^2$ define elliptic curves. However, not every K3 surface can be embedded into $\mathbb{P}^3$.

Example. If A is an abelian surface, the inversion $[-1]\colon A \to A$ is an involution, and it extends to the blow-up of A along the sixteen 2-torsion points. The quotient of the resulting free action on the blow-up by this involution turns out to be a K3 surface, called the *Kummer surface* of A.

2.3 Moduli of K3 surfaces

Given the two classes of examples above, one might want to get an overview of all K3 surfaces and how they vary in families, leading to the question of their moduli. While the moduli stack of elliptic curves is only 1-dimensional, the moduli stack of Kummer surfaces is 3-dimensional, and that of quartics in projective space is 19-dimensional. In general, one may consider the moduli stack $\mathcal{M}_{2d}$ of polarised K3 surfaces of fixed degree $2d$. That is a separated Deligne-Mumford stack of finite type, which is smooth of dimension 19 over $\mathbb{Z}[1/2d]$, see [25] for example.

Example. The case $d = 2$ corresponds to the quartics in $\mathbb{P}^3$. (See [20] for other small d.) The space of quartics in $\mathbb{P}^3$ is a projective space of dimension

$$\dim \mathrm{H}^0(\mathbb{P}^3; \mathcal{O}_{\mathbb{P}^3}(4)) - 1 = \binom{7}{4} - 1 = 34.$$

The discriminant locus $\Delta \subset \mathbb{P}^{34}$ corresponding to the singular quartics is a hypersurface. The Veronese embedding shows that the open complement $\mathbb{P}^{34} \setminus \Delta$ is affine. The universal K3 surface over it has a canonical polarisation given by the restriction of $\mathcal{O}_{\mathbb{P}^3}(1)$. There results a morphism

$$\mathbb{P}^{34} \setminus \Delta \longrightarrow \mathcal{M}_4.$$

This is not surjective, as ample line bundles on K3 surfaces need not be very ample; the canonical image in $\mathbb{P}^3$ may acquire ordinary double points. However, the map factors over the quotient stack

$$\big(\mathbb{P}^{34} \setminus \Delta\big) /\!\!/ \mathrm{PGL}(4)$$

of the affine scheme $\mathbb{P}^{34} \setminus \Delta$ by the action of the affine group $\mathrm{PGL}(4)$ which acts by change of co-ordinates, and links $\mathcal{M}_4$ to a stack associated to a Hopf algebroid.

Finally, it should be pointed out that the moduli problem of (polarised) Calabi-Yau n-folds is much more complicated if $n \geqslant 3$. There are 3-folds with different Hodge numbers, so that – globally – there will be many components. The Hodge numbers also show that – locally – the obstruction space for deformations need not be zero (as it is for K3 surfaces). And in fact there are examples of Calabi-Yau 3-folds which do not lift to characteristic zero, see [12].

3 Formal Brauer groups

The aim of this section is to review the arithmetic of K3 surfaces as far as needed in the rest of the text.

3.1 Formal groups

Let R be a local ring with residue field k. A functor Γ from the category of artinian local R-algebras with residue field k to the category of abelian groups is a (1-dimensional, commutative) *formal group* if the underlying functor to the category of sets is (pro-)representable, formally smooth, and 1-dimensional. However, the actual representation is not part of the data. The choice of a representation (necessarily by a power series ring $R[\![T]\!]$) yields a *co-ordinate* for Γ. With respect to this co-ordinate the formal group is described by the *formal group law*, which is a power series in $R[\![T_1, T_2]\!]$. There are ways to globalise the notion of a formal group in order to work over arbitrary rings or even base schemes. In that case, co-ordinates need not exist globally, only locally. This generality will not be needed here.

3.2 The theorem of Artin and Mazur

Let X be an elliptic curve over a field k. Its Picard group $\mathrm{Pic}_X(k)$ is the group of isomorphism classes of line bundles on X. As it stands, this is just a set with an abelian group structure, but with some work it can be made into an algebraic group Pic_X, whose component of the identity is isomorphic to X by a map which sends the base point to the unit. The formal completion $\widehat{\mathrm{Pic}}_X$ is a 1-dimensional formal group. It has been those formal groups that arose the interest of algebraic topologists in elliptic curves, see [15] for example.

For a K3 surface X, the Picard group is 0-dimensional, so that its formal Picard group is trivial, but the isomorphism

$$\mathrm{Pic}_X(k) \cong \mathrm{H}^1_{\mathrm{et}}(X; \mathbb{G}_m)$$

suggests that one should consider $\mathrm{H}^2_{\mathrm{et}}(X; \mathbb{G}_m)$ instead. As for a geometric interpretation, there is an isomorphism

$$\mathrm{H}^2_{\mathrm{et}}(X; \mathbb{G}_m) \cong \mathrm{Br}_X(k)$$

with the Brauer group $\mathrm{Br}_X(k)$ of X. Although the notation may suggest that, it turns out that these are not the k-valued points of an algebraic group, let alone one whose component of the identity is isomorphic to X. However, Artin and Mazur [2] have shown that the functor

$$A \longmapsto \mathrm{Ker}\left(\mathrm{H}^2_{\mathrm{et}}(X \times \mathrm{Spec}(A); \mathbb{G}_m) \to \mathrm{H}^2_{\mathrm{et}}(X; \mathbb{G}_m)\right)$$

on artinian k-algebras A with residue field k is (pro-)representable and formally smooth. This is referred to as the *formal Brauer group* $\widehat{\mathrm{Br}}_X$ of X. Its dimension is 1; in fact

$$\mathrm{Lie}(\widehat{\mathrm{Br}}_X) \cong \mathrm{H}^2_{\mathrm{et}}(X; \mathbb{G}_a) \cong \mathrm{H}^2(X, \mathcal{O}_X) \tag{1}$$

gives the Lie algebra. These formal groups are the reason for the topologists' interest in K3 surfaces.

As a remark, the results of Artin and Mazur are general enough to show that every Calabi-Yau n-fold gives rise to a 1-dimensional formal group, using the same construction based on $\mathrm{H}^n_{\mathrm{et}}(X; \mathbb{G}_m)$.

3.3 Heights

Over rings of prime characteristic, formal groups have an associated height, which can be a positive integer or infinite. It counts the maximal number of Frobenius morphisms over which multiplication by p factors. It is the most basic invariant of formal groups in prime characteristic, and over separably closed fields it is in fact their only invariant. Over local rings, the height is dominated by the height over the residue field.

The multiplicative formal group $\hat{\mathbb{G}}_m$ has height 1, whereas the additive formal group $\hat{\mathbb{G}}_a$ has infinite height. The height of the formal Picard group of an elliptic curve is either 1 or 2, in which case the elliptic curve is called ordinary or supersingular, respectively.

Example. It is known that the height of the formal Brauer group of the Fermat surface is 1 in the case when $p \equiv 1$ modulo 4 and infinite if $p \equiv 3$ modulo 4. The inclined reader may want to see this by means of Stienstra's formula

$$\log'(T) = \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} T^{4n}$$

for the derivative of the logarithm [26]. This example shows that the height need not vary nicely from prime to prime in a family of mixed characteristic.

Example. The height of formal Brauer groups of Kummer surfaces are 1, 2 or infinite, see [8]. Specifically, to get height 2, take the Kummer surface of a product of an ordinary and a supersingular elliptic curve.

In general, the height of the formal Brauer group of a K3 surface will be at most 10 or infinite, and all these possibilities actually occur. There are explicit examples known for all heights except for 7, see [31] and [11]. A K3 surface is called *ordinary* or *supersingular* (in the sense of Artin, see [1]) if the height is 1 or infinite, respectively.

The pattern continues: there is a bound on the height of the Artin-Mazur formal group associated to a Calabi-Yau n-fold, see [8] again, but note that the height given there is not sharp even in the case of K3 surfaces.

Now let p be a prime that does not divide $2d$, and let

$$\mathcal{M}_{2d,p}$$

be the base change of $\mathcal{M}_{2d}$ from $\mathbb{Z}[1/2d]$ to $\mathbb{Z}/p$. For all $h \geqslant 1$, there is a closed substack

$$\mathcal{M}_{2d,p,h}$$

of $\mathcal{M}_{2d,p}$ defined by those K3 surfaces which have a formal Brauer group of height at least h. These define the *height stratification*

$$\mathcal{M}_{2d,p} = \mathcal{M}_{2d,p,1} \supseteq \mathcal{M}_{2d,p,2} \supseteq \mathcal{M}_{2d,p,3} \supseteq \dots \tag{2}$$

of $\mathcal{M}_{2d,p}$, see [7] and [22]. By what has been said above, the chain stabilises at $h = 11$, with $\mathcal{M}_{2d,p,11}$ being the supersingular locus, at least set-theoretically. Its open complement

$$\mathcal{M}^{\text{fin}}_{2d,p,h}$$

in $\mathcal{M}_{2d,p,h}$ is the moduli stack of polarised K3 surfaces of finite height at least h in characteristic p. These are known to be smooth of dimension $20-h$ for $h = 1, \dots, 10$, and empty for $h = 11$. The substack $\mathcal{M}^{\text{fin}}_{2d,p,h+1}$ of $\mathcal{M}^{\text{fin}}_{2d,p,h}$ is defined by the vanishing of a section of an invertible sheaf of ideals, see again [7] and [22]. It follows that these sections define a regular sequence locally on the moduli stack.

4 K3 spectra

The aim of this section is to define the notion of a K3 spectrum in analogy with elliptic spectra.

4.1 Even periodic ring spectra

Let E be a ring spectrum in the weak "up to homotopy" sense. It is called *even* if its homotopy groups are trivial in odd degrees, and *periodic* if the multiplication induces isomorphisms

$$\pi_{2m}E \otimes_{\pi_0 E} \pi_{2n}E \xrightarrow{\cong} \pi_{2(m+n)}E$$

for all integers m and n. An even periodic ring spectrum has an associated formal group Γ_E, represented by the ring $\pi_0 E^{\mathrm{B}\mathbb{T}} = E^0\mathrm{B}\mathbb{T}$. This ring will be interpreted as the ring of functions on the formal group over $\pi_0 E$.

The projection form $\mathrm{B}\mathbb{T}$ to a point induces the structure map $\pi_0 E \to \pi_0 E^{\mathrm{B}\mathbb{T}}$ of the $\pi_0 E$-algebra. The unit $* = \mathrm{B}1 \to \mathrm{B}\mathbb{T}$ of the group $\mathbb{T}$ induces a co-unit $\pi_0 E^{\mathrm{B}\mathbb{T}} \to \pi_0 E$. This is to be interpreted as the evaluation map at the origin. Its kernel I is the ideal of functions vanishing at the origin. The cotangent space I/I^2 at the origin can then be identified with $\pi_2 E$, so that its dual $\pi_{-2}E$ is the Lie algebra

$$\mathrm{Lie}(\Gamma_E) \cong \pi_{-2}E \tag{3}$$

of Γ_E.

4.2 Definition of K3 spectra

Recall, or see [13] for example, that an elliptic spectrum is a triple (E, X, φ) consisting of an even periodic ring spectrum E, an elliptic curve X over $\pi_0 E$, and an isomorphism φ of the formal Picard group of X with the formal group Γ_E

associated to E over $\pi_0 E$. There may be reasons to allow X to be some sort of generalised elliptic curve, but these will not matter here.

Definition. A *K3 spectrum* is a triple (E, X, φ) consisting of an even periodic ring spectrum E, a K3 surface X over $\pi_0 E$, and an isomorphism φ of the formal Brauer group of X with the formal group Γ_E of E.

The purpose of the rest of this text is to show how one obtains examples of K3 spectra.

4.3 Examples of K3 spectra

Let X be a K3 surface over a field of characteristic 0, so that the formal Brauer group is automatically additive. By (1), its Lie algebra is $\mathrm{Lie}(\widehat{\mathrm{Br}}_X) \cong \mathrm{H}^2(X; \mathbb{G}_m)$, and the usual logarithm $\mathbb{G}_m \cong \mathbb{G}_a$ induces an isomorphism $\mathrm{H}^2(X; \mathbb{G}_m) \cong \mathrm{H}^2(X; \mathbb{G}_a)$. The latter group is just $\mathrm{H}^2(X; \mathcal{O}_X)$ which calculates the Lie algebra of $\widehat{\mathrm{Br}}_X$:

$$\mathrm{Lie}(\widehat{\mathrm{Br}}_X) \cong \mathrm{H}^2(X; \mathcal{O}_X).$$

Let E be the Eilenberg-MacLane spectrum for the (graded) canonical ring of X, so that

$$\pi_{2n} E = \mathrm{H}^0(X; \omega_X^{\otimes n}).$$

As has been noted above, see (3), the Lie algebra of Γ_E is $\pi_{-2} E$, which is the dual of $\pi_2 E$:

$$\mathrm{Lie}(\Gamma_E) \cong \mathrm{H}^0(X; \omega_X)^\vee.$$

Let φ be the unique isomorphism between $\widehat{\mathrm{Br}}_X$ and Γ_E such that the induced isomorphism on Lie algebras is Serre duality

$$\mathrm{H}^2(X; \mathcal{O}_X) \cong \mathrm{H}^0(X; \omega_X)^\vee.$$

Then (E, X, φ) is a K3 spectrum.

The next section will explain how to obtain examples in non-trivial characteristic.

5 Landweber exactness

The aim of this section is to show that there are “enough” K3 spectra. Their existence will be a consequence of Landweber’s exact functor theorem.

5.1 The exact functor theorem

Let us first recall Landweber’s theorem from [14]. See also [18].

Let Γ be a formal group over a local ring R. Choose a co-ordinate T and let

$$[p](T) = a_0T + a_1T^2 + \cdots + a_{p-1}T^p + a_pT^{p+1} + \dots$$

be the p-series of Γ with respect to that co-ordinate. For an integer $n \geqslant 0$ let $I_{p,n}$ be the ideal generated by the first p^{n-1} co-efficients. It is known that this does not depend on the co-ordinate. There results an increasing sequence

$$0 = I_{p,0} \subseteq I_{p,1} \subseteq I_{p,2} \subseteq \dots \tag{4}$$

of ideals. Note that $I_{p,1}$ is the ideal (p) generate by $a_0 = p$. For $n \geqslant 1$, the surjection $R/p \to R/I_{p,n}$ corresponds to the closed subscheme of $\mathrm{Spec}(R/p)$ where the height of Γ is at least n. It is also known that there is a sequence $(v_n \mid n \geqslant 0)$ of elements in R such that $I_{p,n+1}$ is generated by $I_{n,p}$ and v_n. The formal group Γ is called *regular at* p if $(v_n \mid n \geqslant 0)$ is a regular sequence in R. This does not depend on the choice of the v_n. For example, if p is invertible in R, then Γ is automatically p-regular.

The graded formal group law over the graded ring $R[u^{\pm 1}]$ which is defined by Γ is classified by a graded morphism $\mathrm{MU}_* \to R[u^{\pm 1}]$. Landweber’s theorem states that the functor

$$X \mapsto \mathrm{MU}_*X \otimes^{\Gamma}_{\mathrm{MU}_*} R[u^{\pm 1}]$$

is a homology theory if Γ is p-regular for all primes p and the sequence (4) eventually stabilises. This homology theory is representable by an even periodic ring spectrum E which has $\pi_0E \cong R$ and $\Gamma_E \cong \Gamma$.

5.2 The statement

As before, let us fix an integer $d \geqslant 1$, and consider the moduli stack $\mathcal{M}_{2d}$ of polarized K3 surfaces of degree $2d$ over $\mathbb{Z}[1/2d]$. Suppose that X is a K3 surface over an affine scheme $\operatorname{Spec}(R)$ on which $2d$ is invertible. Then X is classified by a map

$$X\colon \operatorname{Spec}(R) \longrightarrow \mathcal{M}_{2d}.$$

The following theorem will assert that – under certain conditions – the associated formal Brauer group is Landweber exact. Before giving the precise statement, let me motivate the choice of hypotheses.

First of all, flatness of the map classifying X will be indispensable for the argument. This will imply that R is flat over $\mathbb{Z}[1/2d]$, so that R is torsion-free, as required by Landweber's theorem. The primes which divide $2d$ are automatically units in R. If R is a $\mathbb{Q}$-algebra, there is no need to invoke Landweber's theorem to get examples, see the previous Section 4.3. On the other hand, the height filtration does not vary nicely from prime to prime, see Section 3.3. Therefore, we will concentrate on one prime and assume that the ring R is a local $\mathbb{Z}_{(p)}$-algebra for some prime p which does not divide $2d$. Recall that "local" means that R has a unique maximal ideal m, and that this maximal ideal contains p, so that the residue characteristic of R is p. There may be other settings which make the following argument work, but this one has its merits, as will hopefully become clear in due course.

Theorem 1. *Let R be a noetherian local $\mathbb{Z}_{(p)}$-algebra for some prime p which does not divide $2d$. Let X be a polarised K3 surface of degree $2d$ over R such that the height of the closed fibre is finite. If the map*

$$X\colon \operatorname{Spec}(R) \longrightarrow \mathcal{M}_{2d} \tag{5}$$

classifying X is flat, then the formal Brauer group $\widehat{\operatorname{Br}}_X$ is Landweber exact, so that there is an even periodic ring spectrum E with $\pi_0 E \cong R$ and $\Gamma_E \cong \widehat{\operatorname{Br}}_X$.

Examples will be given after the proof, showing how this can be used to show that there are "enough" K3 spectra.

5.3 The proof

By assumption on the ring R, all primes different from p are invertible, so that the formal group will automatically be q-regular for all $q \neq p$. It remains to show that it is p-regular as well.

Lemma 2. *The prime p is a nonzerodivisor on R.*

Proof. As has already been remarked before, this follows from the flatness of R as an algebra over $\mathbb{Z}[1/2d]$. □

Let us now reduce everything modulo p. The reduction X/p of X modulo p is classified by a morphism

$$X/p\colon \operatorname{Spec}(R/p) \longrightarrow \mathcal{M}_{2d,p}. \tag{6}$$

Lemma 3. *If the map* (5) *classifying X is flat, so is the map* (6) *classifying X/p.*

Proof. Consider the following diagram.

$$\begin{array}{ccc}
\operatorname{Spec}(R/p) & \longrightarrow & \operatorname{Spec}(R) \\
{\scriptstyle X/p}\downarrow & & \downarrow{\scriptstyle X} \\
\mathcal{M}_{2d,p} & \longrightarrow & \mathcal{M}_{2d} \\
\downarrow & & \downarrow \\
\operatorname{Spec}(\mathbb{Z}/p) & \longrightarrow & \operatorname{Spec}(\mathbb{Z}[1/2d])
\end{array}$$

The bottom square is a pullback by the definition of $\mathcal{M}_{2d,p}$. The outer rectangle is a pullback by elementary algebra. It follows that the upper square is a pullback as well. The result follows by base change for flat morphism. □

Lemma 4. *The map* (6) *classifying X/p factors over the open substack $\mathcal{M}^{\mathrm{fin}}_{2d,p}$.*

Proof. By assumption, the height of the closed fibre X/m of X/p is finite, and this height bounds the height of X/p. □

Let us now bring in the height filtration from Section 3.3. The pullbacks of the ideal sheaves defining $\mathcal{M}_{2d,p,n}$ in $\mathcal{M}_{2d,p} = \mathcal{M}_{2d,p}$ along the morphism (6) give rise to a sequence of ideals

$$0 = J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots \subseteq R/p,$$

such that $I_{p,n}$ is the pre-image of J_n along $R \to R/p$.

Lemma 5. *If the height of the closed fibre is h, one has $J_h \neq R/p$ and $J_{h+1} = R/p$, as well as $I_{p,h} \neq R$ and $I_{p,h+1} = R$.*

Proof. The results for the J imply those for the I, so we only need to prove the first ones.

The ideal J_h cuts out the locus in $\mathrm{Spec}(R/p)$ where the height is at least h, and we have to show that this locus is not empty. But it contains the closed point.

Similarly, the ideal J_{h+1} cuts out the locus in $\mathrm{Spec}(R/p)$ where the height is at least $h+1$, and we have to show that this locus is empty. But it is closed and does not contain the closed point $\mathrm{Spec}(R/m)$ by assumption. A geometric interpretation of Nakayama's Lemma gives the result: a closed subset of a spectrum of a noetherian local ring which does not contain the closed point is empty. □

The preceding lemma implies that v_h is a unit in R, and, unless the closed fibre is ordinary, we are left to deal with the $v_1, \ldots, v_{h-1}$.

Lemma 6. *The sequence $(v_1, \ldots, v_h)$ is regular on R/p.*

Proof. This requires the more delicate results about the height filtration described in Subsection 3.3: the height stratification on $\mathcal{M}_{2d,p}$ is defined locally by a regular sequence of sections of line bundles on the moduli stack. As flat morphisms preserve regularity, these sections pull back to a regular sequence on R/p. □

As p is a nonzerodivisor on R by Lemma 2, it follows that the sequence

$$(p, v_1, \ldots, v_h)$$

is regular on R, with v_h a unit. This finishes the proof of Theorem 1.

There are weaker versions of Theorem 1 based on corresponding versions of Landweber's result modulo p. (See [29] and [30] for the latter.) However, in order to impose "brave new rings" structures on the resulting spectra, it is desirable to work with torsion-free co-efficient rings. The extra effort it took to achieve this will pay off in the extra examples encompassed, to which we turn now.

5.4 Examples

The rest of this section serves the purpose to show that all geometric points of $\mathcal{M}_{2d}$ of finite height can be thickened to give rise to K3 spectra by means of the preceding Theorem 1. The problem does not lie so much in finding a lifting to characteristic 0, as there always is a lifting to the Witt ring, for example, see [21], [4]. It lies in finding such a lift with a flat map to the moduli stack. However, the algebraicity of the stack provides such, as will now be explained.

For every degree $2d$, there is a smooth surjection

$$H \longrightarrow \mathcal{M}_{2d}$$

where H is a suitable piece of a Hilbert scheme, see [25] for example. Let R be one of the local rings of H. As $\mathcal{M}_{2d}$ has finite type over $\mathbb{Z}[1/2d]$, this will be noetherian. And if the residue field k of R has characteristic prime to $2d$, the ring R will be local over $\mathbb{Z}_{(p)}$. The composition

$$\mathrm{Spec}(R) \longrightarrow H \longrightarrow \mathcal{M}_{2d}$$

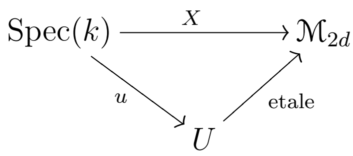
is flat as a composition of flat maps, and classifies a K3 surface X over R: the pullback of the universal family over the Hilbert scheme. If the closed fibre of X over k has finite height, Theorem 1 applies to give an even periodic ring spectrum E such that $\pi_0 E \cong R$ and $\Gamma_E \cong \widehat{\mathrm{Br}}_X$.

Proposition 7. *Let R be one of the local rings of the Hilbert scheme covering $\mathcal{M}_{2d}$ with residue field of characteristic prime to $2d$. Then there is an even periodic ring spectrum E such that $\pi_0 E \cong R$ and Γ_E is isomorphic to the formal Brauer group of the germ of the universal family over R.*

The reader may wonder why a smooth cover has been used in the preceding discussion, while an etale cover is available for the Deligne-Mumford stack $\mathcal{M}_{2d}$. The reason is that the smooth cover can be made fairly concrete using the Hilbert schemes above, whereas the existence of an etale cover is only guaranteed by means of an unramified diagonal, which implies for abstract reasons the existence of etale slices for the smooth cover, see (4.21) in [5] or (8.1) in [16]. This is in contrast to the case of elliptic curves, where etale covers can be written down explicitly.

Proposition 8. *If $X\colon \operatorname{Spec}(k) \to \mathcal{M}_{2d}$ is a geometric point of finite height and characteristic prime to $2d$, there is an even periodic ring spectrum E such that $\pi_0 E$ is isomorphic to the local ring of $\mathcal{M}_{2d}$ in X (with respect to the etale topology) and the reduction of Γ_E to k is the formal Brauer group of X.*

Proof. The local ring $\widetilde{\mathcal{O}}_{\mathcal{M}_{2d},X}$ (with a tilde to indicate the etale topology) is the colimit of the local rings $\mathcal{O}_{U,u}$, where (U,u) runs over the etale neighbourhoods



of X in $\mathcal{M}_{2d}$. As in the proof of the previous proposition, the hypotheses of Theorem 1 are satisfied for the local rings $\mathcal{O}_{U,u}$. As $\widetilde{\mathcal{O}}_{\mathcal{M}_{2d},X}$ is the (strict) henselisation of the $\mathcal{O}_{U,u}$, it is flat over them. Therefore, the hypotheses of Theorem 1 are satisfied for $\widetilde{\mathcal{O}}_{\mathcal{M}_{2d},X}$ as well. $\square$

Furthermore, passage from the local ring at X to its completion, which is flat, yields the following result.

Proposition 9. *If $X\colon \mathrm{Spec}(k) \to \mathcal{M}_{2d}$ is a geometric point of finite height and characteristic prime to $2d$, there is an even periodic ring spectrum E such that $\pi_0 E$ is isomorphic to the ring of formal functions on the formal deformation space of $\mathcal{M}_{2d}$ in X and such that the reduction of Γ_E is the formal Brauer group of X.*

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