Crystals and derived local moduli for ordinary K3 surfaces

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Abstract

It is shown that the K3 spectra which refine the local rings of the moduli stack of ordinary p-primitively polarized K3 surfaces in characteristic p allow for an E_∞ structure which is unique up to equivalence. This uses the E_∞ obstruction theory of Goerss and Hopkins and the description of the deformation theory of such K3 surfaces in terms of their Hodge F-crystals due to Deligne and Illusie. Furthermore, all automorphisms of such K3 surfaces can be realized by E_∞ maps which are unique up to homotopy, and this can be rigidified to an action if the automorphism group is tame.

14F30, 14J28, 55N22, 55P43, 55S12, 55S25.

Introduction

In recent years, progress has been made to enrich some classical moduli stacks of arithmetic origin to objects of stable homotopy theory, most notably in the case of elliptic curves, see [Hop95], [Hop02], [HMb], [Lur09], but also for abelian varieties, see [Law09], [BL10], and the Lubin-Tate moduli of formal groups, see [HMa], [Rez98], and [GH04]. To get an overview, there are the very useful surveys [Goe09] and [Goe]. This paper pursues that program in the context of K3 surfaces.

For each odd prime p there is a Deligne-Mumford moduli stack of p-primitively polarized K3 surfaces. See [Riz06], for example. It is smooth of dimension 19. This stack will be formally completed at p, and the resulting p-adic moduli stack of p-primitively polarized K3 surfaces will be denoted by $\mathcal{M}_{\mathrm{K3},p}$ in the following. Further decorations will become necessary in due course. The generic part $\mathcal{M}^{\mathrm{ord}}_{\mathrm{K3},p}$ of the moduli stack $\mathcal{M}_{\mathrm{K3},p}$ consists of the ordinary K3 surfaces: those whose associated formal Brauer group, see [AM77], is multiplicative. It is this part we will be dealing with here.

A general idea behind the enrichment of moduli stacks to objects of stable homotopy theory is to replace the structure sheaves of rings on the moduli stacks by sheaves of ring spectra, objects which represent multiplicative cohomology theories. Ring spectra nowadays come in two kinds of precision: the older 'up to homotopy' versions, and the more recent 'highly structured' versions. See [MQRT77] for the classic text on the latter, and [MMSS01] for a comparison of many of the more recent models. A *K3 spectrum* is a triple (E,X,ι), where E is an even periodic ring spectrum 'up to homotopy', X is a K3 surface over $\pi_0 E$, and ι is an isomorphism of the formal Brauer group $\hat{\mathrm{Br}}_X$ of X with the formal group associated with such an E, see [Szy10], where it is also proven that all local rings of $\mathcal{M}_{\mathrm{K3},p}$ at K3 surfaces X of finite height and their formal completions can be realized as underlying rings $\pi_0 E$ for suitable K3 spectra (E,X,ι).

The aim of this manuscript is to enhance the multiplications on these K3 spectra from less rigid 'up to homotopy' to 'highly structured' versions in the ordinary case. Although this is still less than having a sheaf of E_∞ ring spectra on the ordinary locus, this already tells us what the stalks will be. (This is what the term 'local' in the title refers to.) I will return to the construction of a sheaf of E_∞ ring spectra on the ordinary locus (and beyond) somewhere else. Briefly, the rather explicit canonical coordinates may be replaced by yet another obstruction theory.

There is a good reason why the local question in the K3 case should be thought of as the essential step: In contrast to elliptic curves and abelian varieties, where the local deformation theory is reduced to the deformation theory of the associated Barsotti-Tate groups by means of the Serre-Tate theorem, see [LST64], [Mes72], [Dri76], [Kat81b], and [Ill85], this does not hold for polarized K3 surfaces in general, and not obviously so in the ordinary case, although [N83b] proves a result along these lines for K3 surfaces without polarizations. Instead, it seems that K3 surfaces will have to be dealt with by means of their crystalline invariants, and this is the optic in which they will be viewed here. The formal Brauer group associated with a K3 surface will sometimes be mentioned for the benefit of the traditionalists, but the mindset of algebraic topologists has changed: from formal group laws to formal groups to Barsotti-Tate groups. The next step towards crystals now seems inevitable, and this may well be considered as the primary novelty introduced here.

Here is an outline of the following text: In Sections 1 and 2, we will discuss some structure present on the local moduli spaces of ordinary and trivialized K3 surfaces with polarization, respectively. As it turns out, this is exactly the structure observed on the p-adic K-homology of $\mathrm{K}(1)$-local E_∞ ring spectra. Thus, it may serve as an input for the obstruction theory of Goerss and Hopkins, which is reviewed in Section 3. In Section 4, this will be applied to prove the existence and uniqueness of an E_∞ structure on said ring spectra. The final Section 5 exploits the symmetries of ordinary K3 surfaces, in other words, the stackiness of $\mathcal{M}_{\mathrm{K3},p}$.

I would like to thank Mike Hopkins and Haynes Miller for their generous hospitality in different stages of this project, Mark Behrens and Jacob Lurie for explanations of various topics related to it, and Thomas Zink for listening to and commenting on my crystalline problems.

1 Ordinary K3 surfaces

In this section, we will review the local deformation theory of polarized K3 surfaces, especially of the ordinary ones, from the point of view of their crystals. The main references are [Ogu79], [DI81a], [DI81b], and [Kat81a].

1.1 K3 surfaces and their deformations

Let k be an algebraically closed field. A *K3 surface* X over k is a smooth projective surface over k such that its canonical bundle $\Omega^2_{X/k}$ is trivial and such that X is not abelian. Examples are the *Fermat quartic* defined by $T_1^4+T_2^4+T_3^4+T_4^4$ in $\mathbb{P}^3_k$, more generally any smooth hypersurface of degree 4, and the *Kummer surfaces*, which are obtained by extending the inversion on an abelian surface over the blowup at the 16 fixed points and passing to the quotient. In these examples and from now on it will be assumed that k is of odd characteristic p.

The Hodge diamond of a K3 surface X, which symmetrically displays its Hodge numbers $\dim_k \mathrm{H}^j(X,\Omega^i_{X/k})$, looks as follows.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

This implies that the Hodge-to-de Rham spectral sequence has to degenerate at E_1. In particular, there are no obstructions to extending deformations, the tangent space to the deformation functor has dimension 20, and there are no infinitesimal automorphisms. This gives the following result, where W denotes the ring of p-typical Witt vectors of k.

Theorem 1.1.1. ([DI81a], 1.2) *The formal deformation space S of X over W is formally smooth of dimension* 20*, so that there is a non-canonical isomorphism*

$$S \cong \mathbb{A}^{20}_W,$$

and there is a universal formal deformation $\mathfrak{X}$ over S.

Further down, see Theorem 1.3.1, we will see that there is a particularly useful set of coordinates for S in the ordinary case.

A *polarized K3 surface* is a pair (X,L), where X is a K3 surface and L is an ample line bundle on X. We shall always assume that the polarization is *p-primitive* for the chosen prime p in the sense that L is not isomorphic to the p-th power of another line bundle. This implies that p does not divide the degree of L.

Theorem 1.1.2. ([DI81a], 1.5 and 1.6) *Let L be a polarization on X as above. The formal deformation space of (X,L) is representable by a closed formal subscheme $S_L \subset S$, defined by a single equation. It is flat over W of relative dimension* 19.

Note that flatness implies that p does not divide an equation defining S_L. Further down, see Theorem 1.3.2, a more precise formula for such an equation will be given in the ordinary case, and this will show that S_L is in fact formally smooth.

1.2 Crystals associated with K3 surfaces

Here it will be explained, following [DI81b], 2.2, how to associate a Hodge F-crystal to a K3 surface, and what it means for that crystal, and therefore by definition for the K3 surface, to be ordinary.

As before, let $\mathcal{X}$ be a universal formal deformation over S of a K3 surface X. Then the $\mathcal{O}(S)$-module

$$H = \mathrm{H}^2_{\mathrm{dR}}(\mathcal{X}/S),$$

together with the Gauss-Manin connection $\nabla = \nabla_{\mathrm{GM}}$ is a crystal.

If φ is a lift of Frobenius to S which is compatible with the canonical lift of Frobenius to W, there is an induced φ-linear map

$$F_\varphi \colon H \longrightarrow H.$$

This would follow immediately from the existence of an S-morphism $\mathcal{X} \to \varphi^*\mathcal{X}$ which lifts the relative Frobenius of X, as F_φ could be defined as the composition

$$\varphi^*\mathrm{H}^2_{\mathrm{dR}}(\mathcal{X}/S) \cong \mathrm{H}^2_{\mathrm{dR}}(\varphi^*\mathcal{X}/S) \longrightarrow \mathrm{H}^2_{\mathrm{dR}}(\mathcal{X}/S).$$

However, such an arrow need not exist. But its mod p reduction, the relative Frobenius, always exists. Thus, one may use (a) the canonical isomorphism between the de Rham cohomology of $\mathcal{X}$ and the crystalline cohomology of its reduction, and (b) the functoriality of crystalline cohomology to obtain the desired maps. Summing up, this means that $(H, \nabla, F_\bullet)$ is an F-crystal. Note that some such lift φ of Frobenius always exists by the formal smoothness of S. Later on, see Section 1.3, a particular lift will be distinguished in the ordinary case.

The Hodge filtration

$$H = F^0 \supset F^1 \supset F^2 \supset F^3 = 0$$

lifts the Hodge filtration on the reduction and satisfies the so-called Griffiths transversality condition. In other words, $(H, \nabla, F_\bullet, F^\bullet)$ is a Hodge F-crystal.

A K3 surface is *ordinary* if the Hodge and Newton polygons of its associated Hodge F-crystal agree, see [Maz72], Section 3. The Hodge polygon codifies the Hodge numbers, as is illustrated in the following figure.

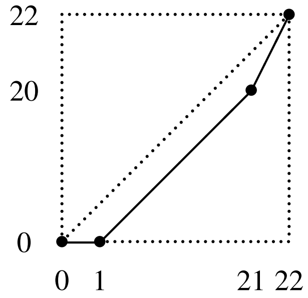


Figure: The Newton/Hodge polygon of an ordinary K3 surface (not drawn to scale)

The Newton polygon codifies the multiplicities and p-adic valuations of the eigenvalues of Frobenius. In the case of a K3 surface, this suggests correctly that the first slope of the Newton polygon is zero if and only if Frobenius acts on $\mathrm{H}^2(X, \mathcal{O}_X)$ bijectively. It is also known that the first slope is $1-(1/h)$, where h is the height of the formal Brauer group. See [Ill79], 7.2, for example. Thus, a K3 surface is ordinary if and only if its formal Brauer group is multiplicative.

An ordinary Hodge F-crystal $(H, \nabla, F_\bullet, F^\bullet)$ of level n, where $n = 2$ for K3 surfaces, has a filtration

$$0 \subset U_0 \subset U_1 \subset \cdots \subset U_n = H$$

such that Frobenius acts on U_j/U_{j-1} as the p^j-th multiple of a bijection, and this filtration is opposite to the Hodge filtration in the sense that

$$H = \bigoplus_j (U_j \cap F^j).$$

This again characterizes ordinary Hodge F-crystals, see [DI81b], 1.3.2.

1.3 Canonical coordinates and the Katz lift

The associated Hodge F-crystal of an ordinary K3 surface, as described in Section 1.2, has a particularly nice structure, and this can be used to find particularly nice coordinates on the base S of its universal formal deformation.

Theorem 1.3.1. ([DI81b], 2.1.7) *Let X be an ordinary K3 surface with universal formal deformation $\mathcal{X}$ over S. Then there is a basis $(a, b_1, \ldots, b_{20}, c)$ for the associated crystal as well as coordinates $t_1, \ldots, t_{20}$ on S such that the following properties* (1.3.1.1) – (1.3.1.4) *hold.*

(1.3.1.1) The basis is adapted to the decomposition

$$H = U_0 \oplus (U_1 \cap F^1) \oplus F^2$$

and satisfies $\langle a, b_j \rangle = 0$, $\langle b_j, c \rangle = 0$, $\langle a, a \rangle = 0$, $\langle c, c \rangle = 0$, and $\langle a, c \rangle = 1$, where $\langle \cdot, \cdot \rangle$ denotes the cup-product pairing on middle cohomology.

(1.3.1.2) If multiplicative notation $q_j = t_j + 1$ and $\omega_j = \mathrm{dlog}(q_j)$ is used, then (ω_j) is a W-basis of $\Omega_{S/W}$.

(1.3.1.3) The Gauss-Manin connection acts via

$$\nabla_{\mathrm{GM}}(a) = 0\,, \quad \nabla_{\mathrm{GM}}(b_j) = \omega_j \otimes a\,, \quad \nabla_{\mathrm{GM}}(c) = -\sum_j \omega_j \otimes b_j^{\vee}\,,$$

where $(b_j^{\vee})$ is the cup-dual basis to (b_j).

(1.3.1.3) If ψ_{can} is the lift of Frobenius given by $\psi_{\mathrm{can}}(q_j) = q_j^p$, then the induced ψ_{can}-linear map $F_{\psi_{\mathrm{can}}}$ on $H = \mathrm{H}^2_{\mathrm{dR}}(\mathcal{X}/S)$ is given by

$$F_{\psi_{\mathrm{can}}}(a) = a\,, \quad F_{\psi_{\mathrm{can}}}(b_j) = pb_i\,, \quad F_{\psi_{\mathrm{can}}}(c) = p^2 c\,.$$

A system (a, b, c, t) as in the preceding theorem is called a *system of canonical coordinates* on S, and ψ_{can} will be referred to as the *canonical lift* or *Katz lift* (after [Kat81a]) of Frobenius. The term *Deligne-Tate mapping* is also in use.

While a system of canonical coordinates is not unique, there is only a rather restricted choice involved: If (a', b', c', t') is another system, there is a scalar $\alpha \in \mathbb{Z}_p^{\times}$ and a matrix $\beta = (\beta_{ij}) \in \mathrm{GL}_{20}(\mathbb{Z}_p)$ such that

$$a' = \alpha a, \quad b_i' = \sum_j \beta_{ji} b_j, \quad c' = c/\alpha,$$

and

$$q'_i = \prod_j q_j^{\beta_{ji}/\alpha}.$$

See [DI81b], 2.1.13. In particular, this shows that the Katz lift does not depend on the canonical coordinates. It is intrinsic to the situation. As the notation $q_j = t_j + 1$ and $\omega_j = \mathrm{dlog}(q_j)$ already indicates, these coordinates can be used to identify S with a formal torus

$$S \cong \hat{\mathbb{G}}_{\mathrm{m}}^{20},$$

as in [DI81b]. (A description of this group structure on S without the use of the canonical coordinates has been given in [N83b].) The Katz lift is a formal group homomorphism and the unique lift of Frobenius whose associated F preserves the Hodge filtration, see [Kat81a], A4.1.

If X is ordinary, and L is p-primitive, the base S_L of the universal formal deformation of (X,L) is not only flat but even formally smooth, see [Ogu79], 2.2. More can be said using a system (a,b,c,t) of canonical coordinates as above, see [DI81b], 2.2. Let

$$e\colon \mathcal{O}(S) \longrightarrow W$$

be the co-unit given by $q_j \mapsto 1$. This can be used to restrict the basis (a,b,c) to give a basis (e^*a, e^*b, e^*c) of the crystalline cohomology of (X,L). Then the first crystalline Chern class of L can be written

$$\sum_j \lambda_j e^* b_j$$

with p-adic integers λ_j. As the first crystalline Chern class of a p-primitive line bundle is not divisible by p, some λ_j will in fact be a p-adic unit.

Theorem 1.3.2. ([DI81b], 2.2.2) *In the notation as before,*

$$\prod_{j=1}^{20} q_j^{\lambda_j} = 1$$

is an equation for S_L in S.

In other words, we can interpret the first crystalline Chern class as a character of the formal torus S, and S_L is its kernel.

Proposition 1.3.3. *The Katz lift* ψ_{can} *on* S *maps* S_L *into itself.*

Proof. As S_L is defined in S by $\prod_j q_j^{\lambda_j} = 1$, the computation

$$\psi_{\mathrm{can}}\big(\prod_j q_j^{\lambda_j} - 1\big) = \prod_j \psi_{\mathrm{can}}(q_j)^{\lambda_j} - 1 = \prod_j (q_j^p)^{\lambda_j} - 1 = \big(\prod_j q_j^{\lambda_j}\big)^p - 1 = 0$$

shows that ψ_{can} preserves the equation. □

2 Trivialized K3 surfaces

In the section, an analogue for K3 surfaces of Katz' notion of trivialized elliptic curves will be discussed. See [Kat75a], [Kat75b], [Kat77] for the latter.

2.1 The rigidified moduli stack $\mathcal{M}^{\mathrm{triv}}_{\mathrm{K3},p}$

Recall that $\mathcal{M}_{\mathrm{K3},p}$ is the p-adic moduli stack of p-primitively polarized K3 surfaces. Now let $\mathcal{M}^{\mathrm{ord}}_{\mathrm{K3},p}$ denote the open substack of $\mathcal{M}_{\mathrm{K3},p}$ consisting of the ordinary surfaces.

Definition 2.1.1. The rigidified moduli stack

$$\mathcal{M}^{\mathrm{triv}}_{\mathrm{K3},p}$$

classifies *trivialized K3 surfaces*: triples (X,L,a) of ordinary K3 surfaces X together with a p-primitive polarization L, and an element a of H which is the a-part of a system of canonical coordinates: it is annihilated by the Gauss-Manin connection and left invariant by the Katz lift of Frobenius.

The choice of a corresponds to the choice of an isomorphism $\hat{\mathrm{Br}}_X \cong \hat{\mathbb{G}}_{\mathrm{m}}$, and it is in this way how Katz introduced trivializations. However, for our purposes, the definition given above corresponds to the crystalline mindset taken here, and it turns out to be easier to work with, too.

It follows from the discussion in the previous Section 1.3 that there is a free and transitive action of the group $\mathbb{Z}_p^\times$ of the p-adic units on the fibers of the forgetful morphism

$$\mathcal{M}_{\mathrm{K3},p}^{\mathrm{triv}} \longrightarrow \mathcal{M}_{\mathrm{K3},p}^{\mathrm{ord}}.$$

This yields a (pro-)Galois covering with group $\mathbb{Z}_p^\times$.

Let us now fix a p-primitively polarized K3 surface (X,L), and let S_L be the base of a universal formal deformation of it. As $\mathcal{M}_{\mathrm{K3},p}^{\mathrm{triv}}$ is Galois over $\mathcal{M}_{\mathrm{K3},p}^{\mathrm{ord}}$, so is the pullback T_L along the classifying morphism for the universal family.

$$\begin{array}{ccc} T_L & \longrightarrow & \mathcal{M}_{\mathrm{K3},p}^{\mathrm{triv}} \\ \Big\downarrow & & \Big\downarrow \\ S_L & \longrightarrow & \mathcal{M}_{\mathrm{K3},p}^{\mathrm{ord}} \end{array}$$

As S_L is affine, and T_L is Galois over it, so is T_L. We are now going to see some structure on its ring $\mathcal{O}(T_L)$ of formal functions.

2.2 The Adams operations

Let T_L be as in the end of the previous Section 2.1. The Galois action of $\mathrm{Aut}(\hat{\mathbb{G}}_{\mathrm{m}}) \cong \mathbb{Z}_p^\times$ on $\mathcal{M}_{\mathrm{K3},p}^{\mathrm{triv}}$ restricts to T_L, and the corresponding operations on the ring $\mathcal{O}(T_L)$ of formal functions will be denoted by ψ^k for p-adic units k. These will be referred to as the *Adams operations*, a terminology which will be justified in the following Section 3.

Let us denote by ω the Hodge line bundle over $\mathcal{M}_{\mathrm{K3},p}$ whose fiber over (X,L) is the line $\mathrm{H}^0(X,\Omega_X^2)$ of regular 2-forms on X: this is the direct image of the determinant of the relative cotangent bundle on the universal K3 surface over $\mathcal{M}_{\mathrm{K3},p}$ along the smooth projection to $\mathcal{M}_{\mathrm{K3},p}$. The same notation ω will also be used for the restriction of this line bundle to $\mathcal{M}_{\mathrm{K3},p}^{\mathrm{ord}}$, $\mathcal{M}_{\mathrm{K3},p}^{\mathrm{triv}}$, S_L, and T_L as needed. As the operators ψ^k on $\mathcal{O}(T_L)$ are induced by an action on T_L, it is clear, that we have a similar action on $\mathrm{H}^0(T_L,\omega^{\otimes n})$ for each integer n.

2.3 The operator θ

In addition to the lift of the action of the p-adic units to T_L, we will now explain how the Katz lift ψ_{can} on $\mathcal{O}(S_L)$ can be extended to $\mathcal{O}(T_L)$ as well. To do so, we need to produce, from the given (universal) a over T_L, another such element for $\psi^*_{\mathrm{can}}\mathcal{X}$ in place of $\mathcal{X}$. This is easy to do from the crystalline point of view on trivializations: The morphism $\psi^*_{\mathrm{can}}\mathcal{X} \to \mathcal{X}$ induces a morphism

$$\psi^*_{\mathrm{can}} \colon \mathrm{H}^2_{\mathrm{dR}}(\mathcal{X}/S_L) \longrightarrow \mathrm{H}^2_{\mathrm{dR}}(\psi^*_{\mathrm{can}}\mathcal{X}/S_L)$$

which sends a to some element $\psi^*_{\mathrm{can}}a$ which is annihilated by the Gauss-Manin connection and left invariant by the Frobenius operator. The resulting self-map of T_L which classifies $(\psi^*_{\mathrm{can}}\mathcal{X}, \psi^*_{\mathrm{can}}L, \psi^*_{\mathrm{can}}a)$ will be denoted by ψ_{can} as well.

Proposition 2.3.1. *The Katz lift ψ_{can} of Frobenius determines a unique operation θ on $\mathcal{O}(T_L)$ such that*

$$\psi_{\mathrm{can}}(f) = f^p + p\theta(f)$$

holds for each f in $\mathcal{O}(T_L)$.

Proof. As ψ_{can} is a lift of Frobenius, there is always one such $\theta(f)$ which satisfies the equation. This shows that θ exists.

As T_L is flat (even formally smooth) over W, multiplication by p is injective on $\mathcal{O}(T_L)$. This shows that θ is unique. □

Using terminology explained in the following Section 3.1, the same argument provides for the structure of a graded θ-algebra with Adams operations on

$$\left(\mathrm{H}^0(T_L, \omega^{\otimes n}) \,|\, n \in \mathbb{Z}\right).$$

This object can and will serve as a blueprint from which the E_∞ structures on the local K3 spectra mentioned in the introduction can be (re)constructed, using the obstruction theory described in the following section.

3 Goerss-Hopkins obstruction theory

In this section, we review the work of Goerss and Hopkins on $K(1)$-local E_∞ ring spectra and spaces of E_∞ maps between them. An odd prime p is fixed throughout. References are [Hop], [GH00], [GH04], and [GH].

3.1 The theory of θ-algebras

Let K denote the p-adic completion of the topological complex K-theory spectrum. In a broader context, see [HMa], [Rez98], and [GH04], this is also known as the first Lubin-Tate spectrum $\mathrm{E}_1 = \mathrm{E}(\mathbb{F}_p, \hat{\mathbb{G}}_\mathrm{m})$ It has an E_∞ structure such that the (stable) Adams operations $\psi^k\colon \mathrm{K} \to \mathrm{K}$ (for p-adic units k) are E_∞ maps. Therefore, if X is any spectrum, the K-homology $\mathrm{K}_0 X = \pi_0(\mathrm{K} \wedge X)$ also has these operations. As everything needs to be $\mathrm{K}(1)$-local here, smash products (such as $\mathrm{K} \wedge X$ above) will implicitly be $\mathrm{K}(1)$-localized. In other words, whenever $X \wedge Y$ is written, this really should read $\mathrm{L}_{\mathrm{K}(1)}(X \wedge Y)$.

If E is a $\mathrm{K}(1)$-local E_∞ ring spectrum, the underlying ring $\pi_0 E$ is a so-called θ-*algebra*. This means that there are two operations ψ^p and θ on $\pi_0 E$ which come about as follows. Given a class x in $\pi_0 E$, the E_∞ structure on E produces a morphism

$$P(x)\colon \mathrm{B}\Sigma_{p+} \longrightarrow E$$

which restricts to x^p along the inclusion $e\colon \mathrm{S}^0 = \mathrm{B}1_+ \to \mathrm{B}\Sigma_{p+}$. In the $\mathrm{K}(1)$-local category there are two other distinguished morphisms

$$\psi^p, \theta\colon \mathrm{S}^0 \to \mathrm{B}\Sigma_{p+},$$

and the 'restriction' of $P(x)$ along these will be denoted by $\psi^p(x)$ and $\theta(x)$. For example, if X is a space, the function spectrum K^X is a $\mathrm{K}(1)$-local E_∞ ring spectrum with $\pi_0(\mathrm{K}^X) = \mathrm{K}^0(X)$, and ψ^p is the p-th (unstable) Adams operation, whereas θ is Atiyah's operation [Ati66]. In general, the equation $e = \psi^p - p\theta$ implies the relation

$$\psi^p(x) = x^p + p\theta(x)$$

for all x in $\pi_0 E$ so that ψ^p is a lift of Frobenius on $(\pi_0 E)/p$ and θ is the error term. This also means that the operation θ determines the operation ψ^p. The converse holds if the ring is p-torsion free.

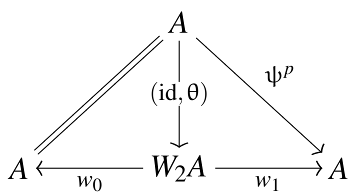
While the operation ψ^p is a ring map, the map θ satisfies the following equations.

$$\begin{aligned}\theta(x+y) &= \theta(x)+\theta(y)-\sum_{j=1}^{p-1}\binom{p}{j}x^j y^{p-j}\\ \theta(x\cdot y) &= x^p\theta(y)+y^p\theta(x)+p\theta(x)\theta(y)\end{aligned}$$

These are best phrased saying that $s=(\mathrm{id},\theta)$ is a ring map to the ring of Witt vectors of length 2 which defines a section of the first Witt component w_0. As the other Witt component is given by

$$w_1(a_0,a_1)=a_0^p+pa_1,$$

composition of the latter with $s=(\mathrm{id},\theta)$ then gives ψ^p.



Putting the two structures together, if E is a $\mathrm{K}(1)$-local E_∞ ring spectrum, the underlying ring $\mathrm{K}_0E=\pi_0(\mathrm{K}\wedge E)$ is a θ-algebra with Adams operations. This is the primary algebraic invariant of the $\mathrm{K}(1)$-local E_∞ ring spectrum E, and the obstruction theory laid out in the following describes how good this invariant is.

There is a graded version of the previous notions which is modeled to capture the structure present on K_*E rather than just K_0E, see Definition 2.2.3 in [GH]. In the case at hand, where we are dealing with even periodic E, these contain essentially the same information as their degree zero part, and we need not go into further detail here.

3.2 Existence and uniqueness of E_∞ structures

Goerss and Hopkins address the following question: Given a graded θ-algebra B_* with Adams operations, when is there a $\mathrm{K}(1)$-local E_∞ ring spectrum E such that

$$\mathrm{K}_*E\cong B_* \tag{3.1}$$

as θ-algebras with Adams operations? Their answer is as follows.

Proposition 3.2.1. ([GH04], 5.9, and [GH], 3.3.7) *Given a graded* θ*-algebra* B_* *with Adams operations, there exists a* $\mathrm{K}(1)$*-local* E_∞ *ring spectrum* E *such that* $\mathrm{K}_*E \cong B_*$ *as* θ*-algebras with Adams operations if certain obstruction groups*

$$\mathrm{D}^{t+2,t}_{\theta\mathrm{Alg}/\mathrm{K}_*}(B_*/\mathrm{K}_*, B_*)$$

vanish for all $t \geqslant 1$*. Furthermore, the* E_∞ *structure is unique up to equivalence if the groups*

$$\mathrm{D}^{t+1,t}_{\theta\mathrm{Alg}/\mathrm{K}_*}(B_*/\mathrm{K}_*, B_*)$$

vanish for all $t \geqslant 1$*.*

Uniqueness here does not mean that there are no non-trivial automorphisms; in fact there usually will be many. It only says that two such objects will be equivalent, in possibly many different ways.

The theorem can be thought of as the obstruction theory for a spectral sequence with E_2 term

$$E_2^{s,t} = \mathrm{D}^{s,t}_{\theta\mathrm{Alg}/\mathrm{K}_*}(B_*/\mathrm{K}_*, B_*),$$

trying to converge to the homotopy groups π_{t-s} of an appropriate space of all such realizations.

Rather than defining the obstruction groups, we will only describe – in Section 3.4 – methods to compute them, as this will be what is needed for the applications. It should be mentioned, however, that the letter D stands for 'derivations', and the obstruction groups come about as a topological version of the André-Quillen theory of non-abelian derived derivations. However, see the next subsection for a hint why derivations come in.

The coefficients M_* of the obstruction groups $\mathrm{D}^s_{\theta\mathrm{Alg}/\mathrm{K}_*}(B_*/\mathrm{K}_*, M_*)$ for a θ-algebra B_* are θ*-modules* in the sense of Definition 2.2.7 in [GH]. These are B_*-modules with the structure of a θ-algebra on $B_* \oplus M_*$, which is essentially given by a map θ on M_* that satisfies $\theta(bm) = \psi(b)\theta(m)$ if both have even degree.

3.3 Spaces of E_∞ maps

We will also have occasion to employ the obstruction theory for spaces of E_∞ maps between $\mathrm{K}(1)$-local E_∞ spectra E and F. In fact, this may be easier to grasp than the obstruction theory for E_∞ structures, which can be thought of as a theory to realize the identity as an E_∞ map. In particular, the obstruction groups will be the same as before, so that the same computational methods will apply. The reference for the material here is [GH04], Section 4, and [GH], Section 2.4.4.

Proposition 3.3.1. ([GH04], 4.4, [GH], 2.4.15) *Let E and F be $\mathrm{K}(1)$-local E_∞ ring spectra, and let $d_*\colon \mathrm{K}_*E \to \mathrm{K}_*F$ be a map of θ-algebras over K_*. The obstructions to realizing d_* as the* K*-homology of an E_∞ map $g\colon E \to F$ lie in groups*

$$\mathrm{D}^{t+1,t}_{\theta\mathrm{Alg}/\mathrm{K}_*}(\mathrm{K}_*E/\mathrm{K}_*, \mathrm{K}_*F)$$

for $t \geqslant 1$. Assuming that such a g exists, the obstructions for uniqueness lie in groups

$$\mathrm{D}^{t,t}_{\theta\mathrm{Alg}/\mathrm{K}_*}(\mathrm{K}_*E/\mathrm{K}_*, \mathrm{K}_*F)$$

for $t \geqslant 1$.

Again this is just the beginning of a spectral sequence which computes the homotopy groups of the component $\mathcal{E}_\infty(E,F)_g$ of g in the space of E_∞ maps. The idea behind the construction of this spectral sequence is to use the cosimplicial resolution of the source E by the triple of the standard adjunction between E_∞ ring spectra and K-algebras. The precise statement is as follows.

Proposition 3.3.2. ([GH04], 4.3, [GH], 2.4.14) *Given an E_∞ map $g\colon E \to F$ of $\mathrm{K}(1)$-local E_∞ ring spectra, there is a spectral sequence*

$$\mathrm{D}^{s,t}_{\theta\mathrm{Alg}/\mathrm{K}_*}(\mathrm{K}_*E/\mathrm{K}_*, \mathrm{K}_*F) \Longrightarrow \pi_{t-s}\mathcal{E}_\infty(E,F)_g$$

converging to the homotopy groups of the component of g in the space of E_∞ maps from E to F.

It it easy to see where derivations come into the picture here. If $g\colon E \to F$ is an E_∞ map, and

$$\mathrm{S}^n \longrightarrow \mathcal{E}_\infty(E,F)$$

is a map based at g for some $n \geqslant 0$, its adjoint is an E_∞ map

$$E \longrightarrow F^{S^n}. \tag{3.2}$$

The K-homology of F^{S^n} takes the form

$$K_*F^{S^n} \cong K_*F \oplus \Sigma^{-n}K_*F$$

for some θ-module $\Sigma^{-n}K_*F$ over K_*F, a shifted and twisted copy of K_*F itself, see [GH], Example 2.2.9, where the notation Ω is used instead of Σ^{-1}. The map induced by (3.2) in K-homology is K_*g in the first factor, and a derivation $K_*E \to K_*F$ in the second factor. In fact, it is a θ-derivation, see [GH], Section 2.4.3. The obstruction groups are derived functors of these.

3.4 Techniques for computing the obstruction groups

If B_* is an even periodic θ-algebra, the (cohomology of the) cotangent complex L_{B_*/K_*} inherits the structure of a θ-module over B_*. This is easy to see for the cotangent module itself: consider the isomorphism between derivations $B_* \to M_*$ and algebra maps $B_* \to B_* \oplus M_*$ over B_*, where θ acts on the right hand side by

$$\begin{aligned}
\theta(b,m) &= \theta((b,0)+(0,m)) \\
&= \theta(b,0)+\theta(0,m)-\frac{1}{p}\sum_{j=1}^{p-1}\binom{p}{j}(b,0)^j(0,m)^{p-j} \\
&= (\theta(b),0)+(0,\theta(m))-(b^{p-1},0)(0,m) \\
&= (\theta(b),\theta(m)-b^{p-1}m)
\end{aligned}$$

Writing $m = D(b)$, this shows that θ acts on a derivation D as

$$(\theta D)b = D(\theta b)+b^{p-1}Db,$$

see [GH], Section 2.4.3.

This observation allows us to treat the two problems separately: that of deforming the algebra first and that of deforming the θ-action later. For practical purposes, this manifests in a composite functor spectral sequence which takes the following form.

Proposition 3.4.1. ([GH], (2.4.7)) *There is a spectral sequence*

$$\mathrm{Ext}^m_{\theta\mathrm{Mod}/B_*}(\mathrm{H}^n(\mathrm{L}_{B_*/\mathrm{K}_*}),M_*) \Longrightarrow \mathrm{D}^{m+n}_{\theta\mathrm{Alg}/\mathrm{K}_*}(B_*/\mathrm{K}_*,M_*).$$

In our cases of interest, the algebra B_* will always be smooth over K_*, and this implies that the spectral sequence degenerates to give the isomorphism

$$\mathrm{D}^s_{\theta\mathrm{Alg}/\mathrm{K}_*}(B_*/\mathrm{K}_*,M_*) \cong \mathrm{Ext}^s_{\theta\mathrm{Mod}/B_*}(\Omega_{B_*/\mathrm{K}_*},M_*) \tag{3.3}$$

from [GH] (2.4.9).

In the same vein, the action of the p-adic units through the Adams operations ψ^k on M_* can be separated from the action of θ: As morphisms have to be compatible with both, there is a Grothendieck spectral sequence as follows.

Proposition 3.4.2. *There is a spectral sequence*

$$\mathrm{Ext}^m_{A_*[\theta]}(\Omega_{A_*/\mathrm{K}_*},\mathrm{H}^n(\mathbb{Z}_p^\times,M_*)) \Longrightarrow \mathrm{Ext}^{m+n}_{\theta\mathrm{Mod}/A_*}(\Omega_{A_*/\mathrm{K}_*},M_*).$$

Thus, if the action of the p-adic units through the Adams operations ψ^k on M_* is induced, we may also eliminate this action from the picture to obtain

$$\mathrm{Ext}^s_{\theta\mathrm{Mod}/A_*}(\Omega_{A_*/\mathrm{K}_*},M_*) \cong \mathrm{Ext}^s_{A_*[\theta]}(\Omega_{A_*/\mathrm{K}_*},M_*^{\mathbb{Z}_p^\times}) \tag{3.4}$$

as in [GH] (2.4.10). The right hand side turns out to be manageable in the cases relevant here.

4 Applications to E_∞ structures on K3 spectra

Let (X,L) be a p-primitively polarized K3 surface as before. In this section we will see that there is a unique E_∞ structure on the K3 spectrum $\mathrm{E}(X,L)$ over the formal completion $\mathcal{O}(S_L)$ of the local ring of $\mathcal{M}^{\mathrm{ord}}_{\mathrm{K3},p}$ at (X,L). It should be emphasized that, while the existence of $\mathrm{E}(X,L)$ as ring spectrum up to homotopy is known from [Szy10], this information will not be needed: the spectrum with an E_∞ structure is shown to exist here.

4.1 A calculation of the obstruction groups

We would like to have an even periodic E_∞ ring spectrum $\mathrm{E}(X,L)$ with

$$\pi_{2n}\mathrm{E}(X,L) \cong \mathrm{H}^0(S_L, \omega^{\otimes n})$$

and

$$\mathrm{K}_{2n}\mathrm{E}(X,L) \cong \mathrm{H}^0(T_L, \omega^{\otimes n}). \tag{4.1}$$

Let us write A_* for the even graded ring with $\mathrm{H}^0(S_L, \omega^{\otimes n})$ in degree $2n$, and B_* for the even graded ring with $\mathrm{H}^0(T_L, \omega^{\otimes n})$ in degree $2n$. As has been shown in Section 2.3, the latter is a θ-algebra with Adams operations over K_* and can therefore serve as an input for the Goerss-Hopkins obstruction theory. We shall now study the obstruction groups in the range of interest.

Proposition 4.1.1. *For the graded* θ*-algebra B_* as above, the obstruction groups*

$$\mathrm{D}^{s,t}_{\theta\mathrm{Alg}/\mathrm{K}_*}(B_*/\mathrm{K}_*, B_*)$$

vanish for $s \geqslant 2$.

Proof. Using the techniques from Section 2.4.3 in [GH], as recalled here in Section 3.4, this can be seen as follows.

First, as T_L is Galois over S_L, and S_L smooth over $\mathbb{Z}_p$, the cotangent complex $\mathrm{L}_{B_*/\mathrm{K}_*}$ is discrete, equivalent to $\Omega_{B_*/\mathrm{K}_*}$ concentrated in degree 0. Therefore,

$$\mathrm{D}^s_{\theta\mathrm{Alg}/\mathrm{K}_*}(B_*/\mathrm{K}_*, B_*) \cong \mathrm{Ext}^s_{\theta\mathrm{Mod}/B_*}(\Omega_{B_*/\mathrm{K}_*}, B_*) \tag{4.2}$$

as in (3.3).

Second, again since T_L is Galois over S_L, we have $\Omega_{B_*/\mathrm{K}_*} \cong B_* \otimes_{A_*} \Omega_{A_*/\mathrm{K}_*}$ by change-of-rings, and obtain

$$\mathrm{Ext}^s_{\theta\mathrm{Mod}/B_*}(\Omega_{B_*/\mathrm{K}_*}, B_*) \cong \mathrm{Ext}^s_{\theta\mathrm{Mod}/B_*}(B_* \otimes_{A_*} \Omega_{A_*/\mathrm{K}_*}, B_*) \tag{4.3}$$

and the latter is $\mathrm{Ext}^s_{\theta\mathrm{Mod}/A_*}(\Omega_{A_*/\mathrm{K}_*}, B_*)$ by adjunction.

Third, we may use (3.4) to get

$$\mathrm{Ext}^s_{\theta\mathrm{Mod}/A_*}(\Omega_{A_*/\mathrm{K}_*}, B_*) \cong \mathrm{Ext}^s_{A_*[\theta]}(\Omega_{A_*/\mathrm{K}_*}, A_*). \tag{4.4}$$

Putting (4.2), (4.3), and (4.4) together yields an isomorphism

$$\mathrm{D}^s_{\theta\mathrm{Alg}/B_*}(B_*/\mathrm{K}_*, B_*) \cong \mathrm{Ext}^s_{A_*[\theta]}(\Omega_{A_*/\mathrm{K}_*}, A_*).$$

The Ext-groups into any module M_* can be calculated by the resolution

$$0 \longrightarrow A_*[\theta] \otimes_{A_*} \Omega_{A_*/\mathrm{K}_*} \xrightarrow{\theta} A_*[\theta] \otimes_{A_*} \Omega_{A_*/\mathrm{K}_*} \longrightarrow \Omega_{A_*/\mathrm{K}_*} \longrightarrow 0$$

of $\Omega_{A_*/\mathrm{K}_*}$. As S_L is smooth over $\mathbb{Z}_p$, the module $\Omega_{A_*/\mathrm{K}_*}$ is projective, so that

$$\mathrm{Ext}^s_{A_*[\theta]}(A_*[\theta] \otimes_{A_*} \Omega_{A_*/\mathrm{K}_*}, M_*) \cong \mathrm{Ext}^s_{A_*}(\Omega_{A_*/\mathrm{K}_*}, M_*) = 0$$

for all $s \geqslant 1$. It follows that $\mathrm{Ext}^s_{A_*[\theta]}(\Omega_{A_*/\mathrm{K}_*}, A_*)$ is zero for all $s \geqslant 2$. □

It should be noted that the vanishing of the obstruction groups for $s \geqslant 2$ implies that the spectral sequences mentioned in the previous section degenerate at E_2. As we will see, for the even periodic spectra we will be dealing with, there will neither be extension problems, so that the homotopy groups of the target can always be identified with certain obstruction groups in the present situation.

4.2 Existence and uniqueness of E_∞ structures

The vanishing of the obstruction groups has the following consequence.

Theorem 4.2.1. *For each p-primitively polarized K3 surface (X, L) as before, there is an even periodic* $\mathrm{K}(1)$*-local E_∞ ring spectrum* $\mathrm{E}(X, L)$ *such that*

$$\begin{aligned} \pi_{2n}\mathrm{E}(X,L) &\cong \mathrm{H}^0(S_L, \omega^{\otimes n}), \\ \mathrm{K}_{2n}\mathrm{E}(X,L) &\cong \mathrm{H}^0(T_L, \omega^{\otimes n}). \end{aligned}$$

The E_∞ structure is unique up to equivalence.

Proof. By Proposition 3.2.1, the obstructions for existence lie in the groups

$$\mathrm{D}^{t+2,t}_{\theta\mathrm{Alg}/\mathrm{K}_*}(B_*/\mathrm{K}_*, B_*)$$

for $t \geqslant 1$. These vanish by Proposition 4.1.1.

Similarly, the obstructions for uniqueness lie in the groups

$$\mathrm{D}^{t+1,t}_{\theta\mathrm{Alg}/\mathrm{K}_*}(B_*/\mathrm{K}_*, B_*)$$

for $t \geqslant 1$. These vanish by Proposition 4.1.1 as well.

The K-homology together with the Adams operations determine the homotopy groups of $\mathrm{E}(X,L)$ in the sense that there is a spectral sequence

$$\pi_{t-s}\mathrm{E}(X,L) \Longleftarrow \mathrm{H}^s(\mathbb{Z}_p^\times, B_t)$$

converging to them. As p is odd, the cohomological dimension of $\mathbb{Z}_p^\times$ is 1, and the spectral sequence degenerates to the long exact sequence induced by the $\mathrm{K}(1)$-local fibration

$$\mathrm{S}^0 \longrightarrow \mathrm{K} \stackrel{\psi^g-\mathrm{id}}{\longrightarrow} \mathrm{K},$$

where g is a topological generator of $\mathbb{Z}_p^\times$. As B_* is concentrated in even degrees, this implies

$$\pi_{2n}\mathrm{E}(X,L) \cong \mathrm{H}^0(\mathbb{Z}_p^\times, B_{2n}) \quad \text{and} \quad \pi_{2n-1}\mathrm{E}(X,L) \cong \mathrm{H}^1(\mathbb{Z}_p^\times, B_{2n}).$$

And as B_* is Galois over A_*, this implies

$$\pi_{2n}\mathrm{E}(X,L) \cong (B_{2n})^{\mathbb{Z}_p^\times} \cong A_{2n} = \mathrm{H}^0(S_L, \omega^{\otimes n})$$

as well as $\pi_{2n-1}\mathrm{E}(X,L) = 0$. □

For rather formal reasons, the $\mathrm{K}(1)$-local E_∞ ring spectra $\mathrm{E}(X,L)$ admit K-algebra structures: We are assuming that the base field is algebraically closed, and the ring $\mathcal{O}(S_L)$ is strictly Henselian. In particular, there is only one form of the multiplicative group, and the $\mathbb{Z}_p^\times$-Galois covering $T_L \to S_L$ must be trivial. The choice of a trivialization yields a map $T_L \to \mathbb{Z}_p^\times$, which induces a morphism

$$\mathrm{K}_*\mathrm{K} \longrightarrow \mathrm{K}_*\mathrm{E}(X,L)$$

of θ-algebras. The Goerss-Hopkins spectral sequence shows that this can be realized as the K-homology of an E_∞ ring map $\mathrm{K} \to \mathrm{E}(X,L)$.

While the E_∞ structure on the $\mathrm{E}(X,L)$ is unique up to equivalence, the equivalence will in general not be unique, due to the presence of automorphisms. Therefore, let us now turn our attention to the possible equivalences of $\mathrm{E}(X,L)$.

5 Symmetries

Let (X,L) be an ordinary p-primitively polarized K3 surface as before. In this section we will investigate how the action of the automorphism group of (X,L) on its universal deformation can be lifted into the world of highly structured ring spectra. This is the K3 analogue of the question settled by Hopkins-Miller in the Lubin-Tate context.

5.1 Symmetries of K3 surfaces

Although K3 surfaces have no infinitesimal automorphisms, the group of automorphisms may nevertheless be infinite. However, the subgroup preserving a chosen polarization is always finite. This is one reason to work with polarized K3 surfaces.

A glance into [Muk88] and [DK09] reveals that there are many simple groups (in the technical sense) which act on K3 surfaces. (See also Section 5.4.) These actions cannot be detected by means of the associated formal Brauer groups, as the finite subgroups of automorphism groups of formal groups are rather restricted, see [Hew95]. This is another argument to tackle K3 surfaces from a crystalline perspective, due to the following result.

Theorem 5.1.1. ([Ogu79], 2.5, [BO83], 3.23) *If p is odd, the map*

$$\mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(\mathrm{H}^2_{\mathrm{cris}}(X/W))$$

is injective.

While the classification of finite groups of symmetries of *complex* K3 surfaces has been worked out some time ago, see [Nik80] and [Muk88], the situation in positive characteristic is more complicated, partially due to the existence of wild automorphisms: an automorphism of a K3 surface in characteristic p is called *wild* if p divides its order. Similarly, a group of automorphisms is wild if it contains a wild automorphism; otherwise it is *tame*.

Theorem 5.1.2. ([DK09], 2.1) *If $p > 11$, the automorphism group of a K3 surface in characteristic p is tame.*

The authors also show by means of examples that their bound is sharp. It seems a remarkable coincidence that this bound is 1 larger than the bound $h \leqslant 10$ for the height of the formal Brauer group (if finite); the same happens in the case of elliptic curves.

Amongst the K3 surfaces with wild automorphisms, there are also ordinary ones. For example, if an elliptic K3 surface in characteristic p has a p-torsion section, translation by the section is a wild automorphism. Such surfaces do not exist in characteristic 11 and 7, but for $p = 5$ and $p = 3$, and the generic ones are ordinary. See [IL].

5.2 Existence and classification of E_∞ maps

Let (X,L) be an ordinary p-primitively polarized K3 surface as before. We have already seen, in Theorem 4.2.1, that there is an E_∞ structure on the K3 spectrum $\mathrm{E}(X,L)$ which is unique up to equivalence. While this means that two different models will be equivalent, there may be many different equivalences between them. We will now see that automorphisms of (X,L) give rise to E_∞ automorphisms of $\mathrm{E}(X,L)$.

Proposition 5.2.1. *The Hurewicz map*

$$\pi_0 \mathcal{E}_\infty(\mathrm{E}(X,L),\mathrm{E}(X,L)) \longrightarrow \mathrm{Hom}_{\theta\mathrm{Alg}/\mathrm{K}_*}(\mathrm{K}_*\mathrm{E}(X,L),\mathrm{K}_*\mathrm{E}(X,L))$$

is bijective.

Proof. Let us abbreviate $A_* = \pi_*\mathrm{E}(X,L)$ and $B_* = \mathrm{K}_*\mathrm{E}(X,L)$ as before.

By Proposition 3.3.1, the obstructions to surjectivity of the Hurewicz map lie in the groups

$$\mathrm{D}^{t+1,t}_{\theta\mathrm{Alg}/\mathrm{K}_*}(B_*/\mathrm{K}_*,B_*)$$

for $t \geqslant 1$. These vanish by Proposition 4.1.1.

Similarly, by Proposition 3.3.2, the obstructions for uniqueness lie in the groups

$$\mathrm{D}^{t,t}_{\theta\mathrm{Alg}/\mathrm{K}_*}(B_*/\mathrm{K}_*,B_*)$$

for $t \geqslant 1$. These vanish by Proposition 4.1.1 except possibly for the group

$$\mathrm{D}^{1,1}_{\theta\mathrm{Alg}/\mathrm{K}_*}(B_*/\mathrm{K}_*, B_*) \cong \mathrm{Ext}^1_{A_*[\theta]}(\Omega_{A_*/\mathrm{K}_*}, \Sigma^{-1}A_*).$$

The last steps in the proof of Proposition 4.1.1 have shown that this group can be computed as the cokernel of an endomorphism of

$$\mathrm{Hom}_{A_*[\theta]}(A_*[\theta] \otimes_{A_*} \Omega_{A_*/\mathrm{K}_*}, \Sigma^{-1}A_*) \cong \mathrm{Hom}_{A_*}(\Omega_{A_*/\mathrm{K}_*}, \Sigma^{-1}A_*). \tag{5.1}$$

However, as A_* is concentrated in even degrees, $\Sigma^{-1}A_*$ is concentrated in odd degrees. It follows that the group (5.1) itself is already zero. □

Concerning the components of the space $\mathcal{E}_\infty(\mathrm{E}(X,L), \mathrm{E}(X,L))$: arguments similar to those in the proof of the preceding result also show that the higher homotopy groups π_{2n-1} and π_{2n} for $n \geqslant 1$ can be identified with the cokernel and kernel of θ acting on $\mathrm{Hom}_{A_*}(\Omega_{A_*/\mathrm{K}_*}, \Sigma^{-2n}A_*)$.

As a consequence of the previous proposition, in order to define homotopy classes of E_∞ maps on $\mathrm{E}(X,L)$, we merely need to guess their effect in K-homology. As with the K-homology of $\mathrm{E}(X,L)$ itself, the geometry of (X,L) provides us with the required information. Here, we may use the fact that the automorphism group $\mathrm{Aut}(X,L)$ acts on the universal formal deformation by changing the identification of the special fiber with (X,L). As automorphisms of K3 surfaces are rigid, this can also be understood as follows: an automorphism g of (X,L) sends an A-point s of S_L to s' if g extends to an isomorphism between the deformations (X_s, L_s) and $(X_{s'}, L_{s'})$ corresponding to s and s'. Either way, the action of $\mathrm{Aut}(X,L)$ on S_L can be extended to T_L as follows. As the map $T_L \to S_L$ should be equivariant, we only need to consider two A-points t and t' over s and s' where $gs = s'$ as above. Then $gt = t'$ holds for the action on T_L if the extension of g is compatible with the chosen a-parts.

Proposition 5.2.2. *The action of* $\mathrm{Aut}(X,L)$ *on* T_L *respects the structure of a* θ-*algebra with Adams operations on* $\mathcal{O}(T_L)$ *defined in Section 2.3, so that there is a factorization*

$$\mathrm{Aut}(X,L) \longrightarrow \mathrm{Aut}_{\theta\mathrm{Alg}/\mathrm{K}_*}(\mathrm{K}_*\mathrm{E}(X,L)) \xrightarrow{\subseteq} \mathrm{Aut}(\mathcal{O}(T_L))$$

of this action through the corresponding subgroup.

Proof. The compatibility of the action of $\mathrm{Aut}(X,L)$ with the Adams operations follows immediately from the fact that $T_L \to S_L$ is equivariant. For the same reason we may check the compatibility with ψ^p (hence θ) on S_L. But for every automorphism g of (X,L), the conjugate $g\psi_{\mathrm{can}}g^{-1}$ satisfies the characterization of the Katz lift, so that $g\psi_{\mathrm{can}}g^{-1} = \psi_{\mathrm{can}}$. This shows that ψ_{can} is equivariant. □

Theorem 5.2.3. *For all ordinary p-primitively polarized K3 surfaces* (X,L)*, there is a unique homotopy action*

$$\mathrm{Aut}(X,L) \longrightarrow \mathrm{Aut}_{\mathrm{Ho}(\mathcal{E}_\infty)}(\mathrm{E}(X,L))$$

of its automorphism group through E_∞ *maps on the associated* E_∞ *ring spectrum.*

Proof. The theorem is a consequence of Proposition 5.2.1, which implies that the right hand side is isomorphic to $\mathrm{Aut}_{\theta\mathrm{Alg}/\mathrm{K}_*}(\mathrm{K}_*\mathrm{E}(X,L))$, and Proposition 5.2.2, which provides for the required action on $\mathrm{K}_*\mathrm{E}(X,L)$. □

5.3 Rigidification

There is another obstruction theory to decide when a homotopy action of a group can be rigidified to a topological action, see [Coo78] in the context of topological spaces. This can be used to prove the following result.

Theorem 5.3.1. *If the automorphism group of a p-primitively polarized K3 surface* (X,L) *is tame, it acts through* E_∞ *maps on the associated* E_∞ *ring spectrum* $\mathrm{E}(X,L)$.

Proof. The obstructions to rigidification lie in the groups

$$\mathrm{H}^n(\mathrm{Aut}(X,L), \pi_{n-2}\,\mathrm{Aut}_{\mathrm{id}}\mathrm{E}(X,L))$$

for $n \geqslant 3$, where $\mathrm{Aut}_{\mathrm{id}}\mathrm{E}(X,L)$ denotes the identity component of the derived space of E_∞ self-equivalences of $E = \mathrm{E}(X,L)$. As the order of the group $\mathrm{Aut}(X,L)$ is prime to p by assumption, it suffices to show that it is invertible in the group $\pi_{n-2}\mathcal{E}_\infty(E,E) \cong \pi_0\mathcal{E}_\infty(E,E^{\mathrm{S}^{n-2}})$. But for $E = \mathrm{E}(X,L)$, the groups $\pi_*(E^{\mathrm{S}^{n-2}}) \cong E^{-*}(\mathrm{S}^{n-2})$ are p-complete (Theorem 4.2.1). □

The proof shows that the rigidification is unique up to unique equivalence under the given hypothesis. Also note that the action is faithful by definition: it can be detected in K-homology.

Corollary 5.3.2. *If $p > 11$, the automorphism group of a p-primitively polarized K3 surface (X,L) acts through E_∞ maps on the associated E_∞ ring spectrum* $\mathrm{E}(X,L)$.

Proof. This follows immediately from Theorem 5.1.2 and the previous result. □

5.4 Examples: Fermat and Klein quartics

Let us consider the Fermat quartics X defined by $T_1^4+T_2^4+T_3^4+T_4^4$ in $\mathbb{P}^3_k$ with the polarization $L=\mathcal{O}(1)$ given by its projective embedding. The Fermat quartic over a field of odd characteristic p is known to be ordinary if and only if $p\equiv 1$ modulo 4, see [Art74] and [AM77]. The cases $p\geqslant 13$ can be dealt with using Corollary 5.3.2, and the case $p=5$ can be dealt with by hand: Since restriction induces an isomorphism $\mathrm{H}^0(\mathbb{P}^3_k,\mathcal{O}(1))\cong \mathrm{H}^0(X,L)$, every automorphism of X which preserves the polarization L extends uniquely over $\mathbb{P}^3_k$. The subgroup of $\mathrm{PGL}_4(k)$ which preserves the Fermat quartic has been determined by Oguiso, see [Shi88], in the case $p\neq 3$. This shows that $\mathrm{Aut}(X,L)$ is an extension of the symmetric group Σ_4, which acts by permutations of the coordinates, by the group of diagonal matrices with 4-th roots of unity as entries. As this group does not contain an element of order 5, Theorem 5.3.1 can be used to show that the automorphism group of the Fermat quartic (X,L) acts faithfully through E_∞ maps on the associated E_∞ ring spectrum $\mathrm{E}(X,L)$ for all primes p where the Fermat quartic is ordinary. The homotopy fixed point spectral sequence

$$\mathrm{H}^s(\mathrm{Aut}(X,L),\mathrm{H}^0(S_L,\omega^{\otimes t/2}))\Longrightarrow \pi_{t-s}\mathrm{E}(X,L)^{\mathrm{hAut}(X,L)}$$

collapses to give

$$\pi_t(\mathrm{E}(X,L)^{\mathrm{hAut}(X,L)})\cong \mathrm{H}^0(S_L,\omega^{\otimes t/2})^{\mathrm{Aut}(X,L)}$$

in this case.

It seems unrewarding to work this out explicitly here: For example, the underlying ring $\pi_0(\mathrm{E}(X,L)^{\mathrm{hAut}(X,L)})\cong \mathcal{O}(S_L)^{\mathrm{Aut}(X,L)}$ is the ring of invariant formal functions on the formal deformation space. As this is isomorphic to the ring of formal functions on the orbit space $S_L/\operatorname{Aut}(X,L)$, it will still be a local W-algebra of dimension 19. But it is no longer regular [Ser67]: The 19-dimensional family

$$T_1^4 + T_2^4 + T_3^4 + T_4^4 + \sum_{\substack{a+b+c+d=4\\ a,b,c,d\leqslant 2}} w_{a,b,c,d} T_1^a T_2^b T_3^c T_4^d$$

is a transversal to the action of PGL_4 on the space of quartics, and the diagonal matrices in $\operatorname{Aut}(X,L)$ act evidently not via pseudo-reflections. Notice that the fixed locus is the 1-dimensional Dwork family $T_1^4+T_2^4+T_3^4+T_4^4+wT_1T_2T_3T_4$, which consists of the quartics invariant under the action of $\operatorname{Aut}(X,L)$.

Similar arguments work in similar cases, such as the Klein quartics. These K3 surfaces X are defined by the equation $T_1^3T_2 + T_2^3T_3 + T_3^3T_1 + T_4^4$ in $\mathbb{P}^3_k$ with the polarization $L = \mathcal{O}(1)$ again given by its projective embedding. The Klein quartic over a field of odd characteristic p is known to be ordinary if and only if $p \equiv 1$ modulo 27, see [Got04]. Therefore, all these cases can be dealt with using Corollary 5.3.2 as above. Note that the automorphism groups of polarized Klein quartics contain a simple group of order 168: these have a 3-dimensional irreducible representation such that the polynomial $T_1^3T_2 + T_2^3T_3 + T_3^3T_1$ is an invariant.

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