Preludes to the Eilenberg–Moore and the Leray–Serre spectral sequences

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The Leray–Serre and the Eilenberg–Moore spectral sequence are fundamental tools for computing the cohomology of a group or, more generally, of a space. We describe the relationship between these two spectral sequences when both of them share the same abutment. There exists a joint tri-graded refinement of the Leray–Serre and the Eilenberg–Moore spectral sequence. This refinement involves two more spectral sequences, the preludes from the title, which abut to the initial terms of the Leray–Serre and the Eilenberg–Moore spectral sequence, respectively. We show that one of these always degenerates from its second page on and that the other one satisfies a local-to-global property: It degenerates for all possible base spaces if and only if it does so when the base space is contractible. When the preludes degenerate early enough, they appear to echo Deligne's *décalage*, but in general, this is an illusion. We discuss several principal fibrations to illustrate the possible cases and give applications, in particular, to Lie groups, torus bundles, and generalizations.

1 Introduction

We consider principal fibration sequences $\Omega Z \to X \to Y \to Z$ for connected spaces X, Y, and Z, with the latter also simply-connected or at least nilpotent. For instance, these could be classifying spaces for a central extension of discrete groups, with $Z \simeq \mathrm{K}(A,2)$ an Eilenberg–Mac Lane space for the abelian kernel A. In general, there are two spectral sequences that converge to the cohomology of X with coefficients in a field K, say: a Leray–Serre spectral sequence for $\Omega Z \to X \to Y$ in the first quadrant and an Eilenberg–Moore spectral sequence for $X \to Y \to Z$ in the second. It has long been recognized that these two spectral sequences often process the same information in different ways. For example, for circle bundles, with $\Omega Z \simeq \mathrm{S}^1$ and $Z \simeq \mathbb{C}\mathrm{P}^\infty$, both spectral sequences are just alternative ways to view the Gysin sequence [Lea91]. On the other hand, the idea suggests itself that the two spectral sequences are always related by a simple transformation of indices, Deligne's *décalage* functor (see Section 2.2), but Example 4.5 shows that this is not always the case. The main goal of this paper is to replace intuition with certainty. We provide systematic tools and give examples that show that our assumptions are reasonable. It turns out that a comparison *is* always possible, and that, in favorable cases, it leads to computational effects, as our applications further down show. Our starting point is the following result.

Theorem 1.1. *For every principal fibration sequence $\Omega Z \to X \to Y \to Z$ there are two spectral sequences starting with the same tri-graded groups $\mathrm{E}_1^{s,t,u}$, and converging to E_1 pages of the Leray–Serre spectral sequence for $\Omega Z \to X \to Y$ and the Eilenberg–Moore spectral sequence for $X \to Y \to Z$, respectively, in turn abutting to $\mathrm{H}^{s+t+u}(X)$.*

These two spectral sequences, which are introduced at the beginning of Section 3, are the *preludes* from the title. Existence results for spectral sequences are nowadays mostly a formality, and Theorem 1.1 follows a pattern introduced by Deligne [Del71] (see also Miller [Mil81]). The crucial point of the above result is the agreement of the initial terms of the preludes, and its main thrust comes from our two accompanying degeneracy results. For the Eilenberg–Moore spectral sequence, we have the following:

Theorem 1.2. *The preludes to the Eilenberg–Moore spectral sequences always degenerate from their* E_2 *page on. They degenerate from their* E_1 *page on if and only if the space* Y *is K–minimal.*

This is proven as Theorem 3.5 below. It requires the notion of *K–minimality*, which refers to the existence of a cellular structure such that the cellular chain complex with coefficients in the field K is minimal in the sense that the differential is trivial (see Definition 2.4). Needless to say, many important classes of spaces have this property (see Examples 2.5 for a start). The pendant of Theorem 3.5 for the Leray–Serre spectral sequence is Theorem 3.12, a local-to-global principle for degeneracy:

Theorem 1.3. *Given a space Z, the preludes to the Leray–Serre spectral sequences degenerate for all Y from their* E_2 *page on if and only if this holds for a single point* $Y = \star$.

We will say that a space Z to which the result applies is *K–unbarred* (see Definition 2.14). Once again, many important classes of spaces have this property (see Examples 2.15). In particular, spaces with polynomial cohomology (Proposition 2.16) and suspensions (Proposition 2.17) are unbarred.

An abundance of examples demonstrates that neither the Eilenberg–Moore nor the Leray–Serre spectral sequence is more efficient (i.e., closer to the abutment) than the other. Sometimes, one is broken in half, with either algebraic or geometric differentials prepended in one of the preludes, but in general both of them are necessary, and our results suggest to consider always the full quartet of spectral sequences featuring in Theorem 1.1. We demonstrate the usefulness of this technique by deriving general degeneracy criteria and applying them to classical Lie group fibrations. The first criterion is for the Eilenberg–Moore spectral sequence for the homotopy fiber X of a map $f\colon Y \to Z$.

Theorem 1.4. *If the kernel of* $f^\bullet = \mathrm{H}^\bullet(f)\colon \mathrm{H}^\bullet(Z) \to \mathrm{H}^\bullet(Y)$ *is generated by a regular sequence, and if the cohomology* $\mathrm{H}^\bullet(Y)$ *is a free module over the image, then the Eilenberg–Moore spectral sequence for* $\mathrm{H}^\bullet(X)$ *degenerates from its* E_2 *page on.*

This result is proven as Theorem 5.2, which also describes the E_∞ page. The discussion following the proof that we give shows that it generalizes several earlier degeneracy criteria. Our comparison results allow us to derive our second degeneracy criterion from it, this time for the Leray–Serre spectral sequence.

Theorem 1.5. *Let Z be a K-minimal space with a polynomial K–cohomology ring on classes* z_j *such that the kernel of the induced morphism* $f^\bullet = \mathrm{H}^\bullet(f)\colon \mathrm{H}^\bullet(Z) \to \mathrm{H}^\bullet(Y)$ *is generated by a subset of the* z_j*'s. All differentials in the Leray–Serre spectral sequence converging to* $\mathrm{H}^\bullet(X)$ *are induced by the transgressions hitting the images* $f^\bullet(z_j)$*, and the spectral sequence degenerates from the* E_r *page on, where* $r = \max\{\deg(z_j) \mid f^\bullet(z_j) \neq 0\} + 1$.

We prove this result as Theorem 5.8, again together with the information about the various E_r pages, including E_∞. It applies, for instance, to Quillen's computation [Qui71] (see also [BC92]) of the cohomology of the extraspecial 2–groups (Example 5.9). We note that the Eilenberg–Moore spectral sequence for group extensions has been described by Rusin [Rus87, Rus89], and he attributed the idea to Lambe. For principal fibrations involving Lie groups, we have the following result:

Theorem 1.6. *For any connected compact Lie group G with maximal torus T, consider the principal fibration sequence* $T \to G \to G/T \to \mathrm{B}T$*. The Eilenberg–Moore spectral sequence*

$$\mathrm{E}_2^{p,q} \cong \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(\mathrm{B}T)}(K, \mathrm{H}^\bullet(G/T))^q \Longrightarrow \mathrm{H}^{p+q}(G)$$

always degenerates from its E_2 *page on, and the Leray–Serre spectral sequence*

$$\mathrm{E}_2^{p,q} \cong \mathrm{H}^p(G/T) \otimes_K \mathrm{H}^q(T) \Longrightarrow \mathrm{H}^{p+q}(G)$$

always degenerates from its E_3 *page on.*

This result generalizes all previous degeneracy results for the Eilenberg–Moore spectral sequence in the given situation, such as the one in [NNS99]. Similar results hold for finite loop spaces and p–compact groups (Remark 5.15) as well as Kac–Moody groups (see [Kac86, Kit14] and Remark 5.16).

Here is an outline of this article. In the following Section 2, we briefly review the construction of the spectral sequence of a filtered complex and how the Leray–Serre and the Eilenberg–Moore spectral sequences are particular cases of that. We also discuss the classes of K–minimal and K–unbarred spaces, the two conditions that appear naturally as assumptions in the following Section 3, where we prove our main comparison results. The short Section 4 includes some elementary but illustrative examples of degenerate cases and the principal fibrations in which the Hopf fibration features. The final Section 5 contains a new degeneracy theorem for Eilenberg–Moore spectral sequences that leads, together with our comparison theorems, to a structure theorem for Leray–Serre spectral sequences, ending in the most general degeneracy results for Lie groups.

2 Spectral sequences

In this section, we provide the necessary definitions and notations used later on. We start with the briefest review of the spectral sequence defined by a cochain complex that comes with a decreasing filtration, including the definition of Deligne's *décalage* functor. We also explain how Leray–Serre and Eilenberg–Moore spectral sequences are both instances of this general construction. The reader may want to skip this section and refer back to it when needed, but we point out the two non-standard Definitions 2.4 and 2.14. As a standard reference for spectral sequences we refer to McCleary's textbook [McC01].

2.1 The spectral sequence of a filtered cochain complex

We consider cochain complexes C with differentials $\mathrm{d}\colon C^n \to C^{n+1}$ that increase the cohomological degree. Occasionally, we will have to use the shifted complex $C[d]$ that satisfies $C[d]^n = C^{d+n}$.

When have a descending filtration F on such a cochain complex C by subcomplexes $F^pC \geqslant F^{p+1}C$, the associated graded is defined as $\mathrm{Gr}_F^p C = F^pC/F^{p+1}C$. We set

$$\mathrm{Z}_r^{p,q}(C,F) = \{\, x \in F^pC^{p+q} \,|\, \mathrm{d}x \in F^{p+r}C^{p+q+1} \,\}, \tag{2.1}$$

and this leads to a spectral sequence $(\mathrm{E}_r, \mathrm{d}_r)$ which starts with

$$\mathrm{E}_0^{p,q}(C,F) = \mathrm{Gr}_F^p C^{p+q},$$

and the differential is of the form $\mathrm{d}_r\colon \mathrm{E}_r^{p,q}(C,F) \to \mathrm{E}_r^{p+r,q-r+1}(C,F)$ induced by the differential d of the original cochain complex C. We have

$$\mathrm{E}_1^{p,q}(C,F) = \mathrm{H}^{p+q}(\mathrm{Gr}_F^p C).$$

The spectral sequence abuts to the associated graded of the corresponding filtration on the cohomology $\mathrm{H}^\bullet(C)$. Suitable references for the theory are Godement [God58], [Gro61, III.0 §11], and Deligne [Del71].

2.2 Décalage

Deligne [Del71, 1.3.3, 1.3.4] has introduced *décalage* as a way of changing a filtration on a cochain complex so that the spectral sequence does not change but for a re-indexing. More precisely:

Definition 2.1. If F is a descending filtration on a cochain complex C as above, then the subcomplex $(\mathrm{Dec}\,F)^p C^n \leqslant C^n$ is defined using (2.1) as

$$(\mathrm{Dec}\,F)^p C^n = \mathrm{Z}_1^{p+n,-p}(C,F) = \{x \in F^{p+n}C^n \,|\, \mathrm{d}x \in F^{p+n+1}C^{n+1}\}.$$

There is a canonical homomorphism $\mathrm{E}_0^{p,n-p}(C,\mathrm{Dec}\,F) \to \mathrm{E}_1^{p+n,-p}(C,F)$ which is compatible with the differentials. This homomorphism induces isomorphisms

$$\mathrm{E}_r^{p,n-p}(C,\mathrm{Dec}\,F) \cong \mathrm{E}_{r+1}^{p+n,-p}(C,F)$$

from $r \geqslant 1$ on.

Remark 2.2. When we have $n = p+q$ (see also Notation 3.1), the formula becomes

$$\mathrm{E}_r^{p,q}(C,\mathrm{Dec}\,F) \cong \mathrm{E}_{r+1}^{2p+q,-p}(C,F).$$

This index transformation identifies the regions where the Eilenberg–Moore and the Leray–Serre spectral sequences live. As we shall see later, in Section 3.2, both spectral sequences can be realized by two filtrations *on the same cochain complex*, and one might wonder if the filtration W for the Eilenberg–Moore spectral sequence might be the *décalage* of the filtration F which produces the Leray–Serre spectral sequence. However, as Example 4.5 shows, this is not the case in general. We shall describe situations where it is true in Proposition 3.2.

2.3 Filtrations on spaces

Let X be a space with an increasing filtration by subspaces $X^s \leqslant X^{s+1}$. Any such increasing filtration on X gives a decreasing filtration of the cochain complex $\mathrm{C}(X)$ by setting

$$F^s\mathrm{C}(X) = \mathrm{Ker}(\mathrm{C}(X) \to \mathrm{C}(X^{s-1})) = \mathrm{C}(X, X^{s-1}),$$

and this results in a spectral sequence starting from

$$\mathrm{E}_0^{s,t} = \mathrm{Gr}_F^s\,\mathrm{C}^{s+t}(X) = \frac{\mathrm{C}(X,X^{s-1})^{s+t}}{\mathrm{C}(X,X^s)^{s+t}} = \mathrm{C}(X^s,X^{s-1})^{s+t}.$$

Passing to cohomology, we get

$$\mathrm{E}_1^{s,t} = \mathrm{H}^{s+t}(X^s,X^{s-1}) \Longrightarrow \mathrm{H}^{s+t}(X). \tag{2.2}$$

The following turns out to be a conceptually important special case.

Example 2.3. For instance, the filtration on the space X could be the skeletal filtration of a CW-structure on it. We shall write

$$\mathrm{Cell}^s(X;V) = \mathrm{Hom}(\mathrm{H}_\bullet(X^s,X^{s-1}),V) = \mathrm{H}^\bullet(X^s,X^{s-1}) \otimes V$$

for the cellular cochains of X with coefficients in a K–vector space V. The E_1 page of the spectral sequence (2.2) is concentrated in the row with $t = 0$, and

$$\mathrm{E}_1^{s,0} = \mathrm{Cell}^s(X;K)$$

is the vector space of K–valued functions on the s–cells of X. The d_1 differential turns the groups above into the cellular cochain complex of X with coefficients in the field K, and from its E_2 page on, this spectral sequence necessarily degenerates. This is one argument to show that the cohomology $\mathrm{H}^\bullet(X)$ can be computed as cellular cohomology, and up to isomorphism it does not depend on the cellular structure.

2.4 Minimality

We often need to refer to the situation when a spectral sequence as in Example 2.3 degenerates from its E_1 page on. There does not seem to be established terminology for the following.

Definition 2.4. Given a field K, a space is called *K–minimal* if there exists a CW-structure on it such that the cellular chain complex with respect to this CW structure and coefficients in the field K has trivial differential.

Examples 2.5. The circle $S^1 \simeq B\mathbb{Z}$, and more generally the spheres S^r and tori $(S^1)^r$ are K–minimal for any field K. The real projective space $\mathbb{RP}^\infty \simeq B\mathbb{Z}/2$ is K–minimal with respect to the field $\mathbb{F}_2$, but *not* with respect to the field $\mathbb{Q}$.

Remark 2.6. Spaces with polynomial K–cohomology are *not* automatically K–minimal. For instance, any K–acyclic space automatically has polynomial K–cohomology for trivial reasons, but it is K–minimal if and only if it is contractible.

2.5 Leray–Serre spectral sequences

The filtrations on spaces X that we mostly care about come, more generally, from maps $X \to Y$ to another space Y. These maps allow us to pull back a skeletal filtration of Y, for instance. Then (2.2) is what we will refer to as the *Leray–Serre spectral sequence* for the map $X \to Y$ with respect to the given filtration on Y.

Remark 2.7. The skeletal filtration of a space is not homotopy invariant, and the E_1 page of a Leray–Serre spectral sequence usually depends on the filtration. As a consequence, we should not speak of *the* spectral sequence, because there are many.

The E_1 page of a Leray–Serre spectral sequence is particularly easy to describe when the map $X \to Y$ is a fibration with a simply-connected base Y, in terms of the fiber. More generally, if the map in question might not be a fibration, we have to use the homotopy fiber, and then the E_2 page is

$$E_2^{s,t} \cong H^s(Y; H^t(\mathrm{hofib}(X \to Y)))$$

This does *not* depend on the filtration on Y any longer.

In our standard situation

$$\Omega Z \longrightarrow X \longrightarrow Y \longrightarrow Z, \tag{2.3}$$

the homotopy fiber of the map $X \to Y$ is a loop space ΩZ, and the Leray–Serre spectral sequence looks as follows.

$$E_2^{s,t} \cong H^s(Y; H^t(\Omega Z)) \Longrightarrow H^{s+t}(X) \tag{2.4}$$

We refer to Remark 3.15 for an argument showing that this description is valid even in our specific situation (2.3), even if Y is not simply-connected.

Remark 2.8. Serre's original construction [Ser51] of the spectral sequence used filtered complexes, too, but those complexes were based on cubical chains. Later, variants of this idea appeared (see [Bro59], [Dre67], [Sin73], [MS93], [Bro94], [Tur98] for some of them, and Barnes [Bar85] for a systematic approach). We refer to [FH58] and [Shi62] for useful information on higher differentials.

2.6 The bar construction

Let A be an augmented differential graded K–algebra, for the ground field K, with augmentation ideal $\overline{A}$. Given a differential graded right A–module M and a differential graded left A–module N, let

$$B(M, A, N) \tag{2.5}$$

denote the *(normalized) bar complex* [EM53, Ch. II] of Eilenberg and Mac Lane. In our conventions, the object $\mathrm{B}(M,A,N)$ is a cochain complex of K–vector spaces, and its homogeneous elements are expressions

$$m\,[\,a_1\,|\ldots|\,a_k\,]\,n.$$

The grading is such that this element sits in degree

$$\deg(m\,[\,a_1\,|\ldots|\,a_k\,]\,n) = \deg(m) + \sum_j \deg(a_j) + \deg(n) - k. \tag{2.6}$$

The differential on the bar complex is the sum of an internal and an external differential. The internal differential

$$\mathrm{d}^{\mathrm{int}}(m\,[\,a_1\,|\ldots|\,a_k\,]\,n) = \sum_j \pm m\,[\,a_1\,|\ldots|\,\mathrm{d}a_j\,|\ldots|\,a_k\,]\,n$$

is induced from those on M, A, and N, whereas the external one

$$\mathrm{d}^{\mathrm{ext}}(m\,[\,a_1\,|\ldots|\,a_k\,]\,n) = \sum_j \pm m\,[\,a_1\,|\ldots|\,a_j a_{j+1}\,|\ldots|\,a_k\,]\,n$$

involves the multiplications and actions. These formulas for the differentials will play no role in the following, and there is no need to nail down the signs here. The complex $\mathrm{B}(M,A,N)$ serves as a specific model for the derived tensor product, the differential graded K–vector space $M \otimes_A^{\mathbb{L}} N$.

The bar complex comes with a descending filtration W defined by

$$W^p\mathrm{B}(M,A,N) = \{\,m\,[\,a_1\,|\ldots|\,a_k\,]\,n\,|\,k \leqslant -p\,\}, \tag{2.7}$$

so that $W^{-\infty}\mathrm{B}(M,A,N) = \mathrm{B}(M,A,N)$ and $W^1\mathrm{B}(M,A,N) = 0$. This filtration of the bar complex leads to a spectral sequence with

$$\mathrm{E}_0^{p,q} = \mathrm{Gr}_W^p\,\mathrm{B}(M,A,N)^{p+q} = (M \otimes \overline{A}^{\otimes -p} \otimes N)^q, \tag{2.8}$$

as follows from (2.6). The abutment is $\mathrm{H}^{p+q}(\mathrm{B}(M,A,N))$. The d_0 differential is the internal one, up to sign, so that

$$\mathrm{E}_1^{p,q} = (\mathrm{H}^\bullet(M) \otimes \overline{\mathrm{H}}^\bullet(A)^{\otimes -p} \otimes \mathrm{H}^\bullet(N))^q \tag{2.9}$$

by the Künneth theorem. Together with the d_1 differential, which is induced from the external one, the E_1 page is a complex that can be used to compute the graded Tor over the cohomology algebra $\mathrm{H}^\bullet(A)$, and we get

$$\mathrm{E}_2^{p,q} = \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(A)}(\mathrm{H}^\bullet(M),\mathrm{H}^\bullet(N))^q \Longrightarrow \mathrm{H}^{p+q}(\mathrm{B}(M,A,N)). \tag{2.10}$$

We will refer to this spectral sequence as the *bar spectral sequence*. It is also referred to sometimes as the *Moore spectral sequence* [Moo59, Cla65].

Remark 2.9. We have chosen the bar construction for convenience and explicitness. But, just as the choice of a cell-structure for the Leray–Serre spectral sequence influences its starting page, we can often achieve smaller E_1–terms for the Eilenberg–Moore spectral sequence when replacing the bar construction by smaller resolutions as follows. Recall the proper terminology of Moore [Smi67b]: A sequence of homomorphisms of differential graded modules over a differential graded K–algebra A is *proper exact* if it is exact in the usual sense and also on the level of cocycles and cohomology. *Proper epimorphisms* are a special case, and these in turn are used in the definition of *proper projectives*. A *proper projective resolution* of a differential graded A–module M is a proper exact complex

$$\cdots \longrightarrow M^{-1} \longrightarrow M^0 \longrightarrow M \longrightarrow 0 \longrightarrow \ldots \tag{2.11}$$

of proper projective differential graded A–modules. Each M^n is a differential graded A–module, so combined with the differentials in (2.11), this gives a bicomplex $\mathbb{M}^\bullet$ with totalization $\mathrm{Tot}(\mathbb{M}^\bullet)$, another differential graded A–module. We have

$$\mathrm{B}(M,A,N) = \mathrm{Tot}(\mathbb{B}^\bullet(M,A,A)) \otimes_A N,$$

where $\mathbb{B}^\bullet(M,A,A)$ is the bar resolution bicomplex of M over A before totalization. In the following, we could always replace $\mathrm{B}(M,A,N)$ by $\mathrm{Tot}(\mathbb{M}^\bullet) \otimes_A N$ for any proper projective resolution of M.

2.7 The Eilenberg–Moore spectral sequence

Let $X = P \times_Z Y$ be a homotopy pullback of spaces (with Z nilpotent). If we again write $\mathrm{C}(X)$ for the cochains on X, and similarly for P, Z, and Y, then there is an equivalence

$$\mathrm{C}(X) \simeq \mathrm{B}(\mathrm{C}(P), \mathrm{C}(Z), \mathrm{C}(Y)) \tag{2.12}$$

of cochain complexes [EM66] (see also [Smi67b]). The right hand side is the bar complex (2.5). The equivalence becomes particularly evident when X itself is described as the totalization of a geometric bar construction [Rec70].

We shall later need a relative version of the Eilenberg–Moore equivalence (2.12):

Proposition 2.10. *Assume that $X \to Y$ is a fibration. If $B \subseteq Y$ is a subspace, and $A \subseteq X$ the restriction to X, then the bar complex $\mathrm{B}(\mathrm{C}(P), \mathrm{C}(Z), \mathrm{C}(Y,B))$, with its total differential, is equivalent to $\mathrm{C}(X,A)$.*

Proof. We have a short exact sequence

$$0 \longrightarrow \mathrm{C}(Y,B) \longrightarrow \mathrm{C}(Y) \longrightarrow \mathrm{C}(B) \longrightarrow 0 \tag{2.13}$$

of cochain complexes. Application of the functor $? \mapsto \mathrm{B}(\mathrm{C}(P), \mathrm{C}(Z), ?)$ leads to another short exact sequence

$$0 \to \mathrm{B}(\mathrm{C}(P), \mathrm{C}(Z), \mathrm{C}(Y,B)) \to \mathrm{B}(\mathrm{C}(P), \mathrm{C}(Z), \mathrm{C}(Y)) \to \mathrm{B}(\mathrm{C}(P), \mathrm{C}(Z), \mathrm{C}(B)) \to 0.$$

We compare this short exact sequence with the short exact sequence

$$0 \longrightarrow \mathrm{C}(X,A) \longrightarrow \mathrm{C}(X) \longrightarrow \mathrm{C}(A) \longrightarrow 0,$$

which is analogous to (2.13). Two out of three comparison maps are equivalences by the absolute Eilenberg–Moore theorem, and the relative one follows. □

The equivalence (2.12) implies that we have a spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(Z)}(\mathrm{H}^\bullet(P), \mathrm{H}^\bullet(Y))^q \Longrightarrow \mathrm{H}^{p+q}(\mathrm{B}(\mathrm{C}(P), \mathrm{C}(Z), \mathrm{C}(Y))) \cong \mathrm{H}^{p+q}(X).$$

This is the *Eilenberg–Moore spectral sequence*.

Only special cases will be relevant for us:

Example 2.11. If $P = \star$ is a point, the space X is the homotopy fiber of the map $Y \to Z$. We have $\mathrm{H}^\bullet(P) = K$, the ground field, and we get a spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(Z)}(K, \mathrm{H}^\bullet(Y))^q \Longrightarrow \mathrm{H}^{p+q}(X). \tag{2.14}$$

This is the usual description of the Eilenberg–Moore spectral sequence for our standard situation (2.3). A version of this Eilenberg–Moore spectral sequence, for homology with integral coefficients, has been implemented recently using functional programming [RRSS20].

Example 2.12. If, in addition to $P = \star$, the space $Y = \star$ is also a point, then $X \simeq \Omega Z$ is the loop space of Z and we get a spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(Z)}(K, K)^q \Longrightarrow \mathrm{H}^{p+q}(\Omega Z). \tag{2.15}$$

Remark 2.13. There are many interesting situations in which an Eilenberg–Moore spectral sequence does not degenerate from its E_2 page on (see Example 4.2 and Example 5.10 below and [Sch71, KS72, Rus89], for instance). Schochet's example is the 'double' of the dihedral extension: the homotopy fiber of the product class $\mathrm{K}(\mathbb{Z}/2,1) \times \mathrm{K}(\mathbb{Z}/2,1) \to \mathrm{K}(\mathbb{Z}/2,2)$ is a $\mathrm{K}(\mathrm{D}_8,1)$, and the homotopy fiber of the product class $\mathrm{K}(\mathbb{Z}/2,2) \times \mathrm{K}(\mathbb{Z}/2,2) \to \mathrm{K}(\mathbb{Z}/2,4)$ is his example. It might be interesting to find a general pattern in this direction; we have not attempted that.

2.8 Unbarred spaces

We need to single out a class of spaces that is well-adapted to the situations we are interested in.

Definition 2.14. Given a field K, we say that a nilpotent space Z is *K–unbarred* with respect to the field K if the Eilenberg–Moore spectral sequence (2.15) for the loop space ΩZ degenerates from its E_2 page on.

We will see later, in Theorem 3.12, that the condition in Definition 2.14 implies also the degeneration of all other Eilenberg–Moore spectral sequences for all other fibrations with base Z.

Examples 2.15. The circle $\mathrm{S}^1 \simeq \mathrm{B}\mathbb{Z}$, and more generally the spheres S^r and tori $(\mathrm{S}^1)^r$ are K–unbarred for any field K. The real projective space $\mathbb{R}\mathrm{P}^\infty \simeq \mathrm{B}\mathbb{Z}/2$ is $\mathbb{F}_2$–unbarred, but the analog for odd primes is wrong (see Example 4.2): the classifying space $\mathrm{B}\mathbb{Z}/\ell$ is barred for the field $\mathbb{F}_\ell$ if ℓ is an odd prime.

Proposition 2.16. *Spaces with polynomial K–cohomology are K–unbarred.*

Proof. If the cohomology $\mathrm{H}^\bullet(Z)$ of a space Z is a polynomial algebra over the ground field K, the Koszul complex shows that $\mathrm{Tor}^{\mathrm{H}^\bullet(Z)}_{\bullet,\bullet}(K,K)$ is an exterior algebra. This is the E_2 page of the Eilenberg–Moore spectral sequence. The exterior algebra generators sit in the column $p=-1$. Therefore, all differentials d_r for $r \geqslant 2$ vanish on the generators, and the spectral sequence degenerates. □

Proposition 2.17. *Suspensions are K–unbarred for any field K.*

Proof. For a suspension $Z = \Sigma Z'$ of a space Z', we have to show that the Eilenberg–Moore spectral sequence for the loop space fibration $\Omega\Sigma Z' \to \star \to \Sigma Z'$ degenerates. The cohomology algebra $\mathrm{H}^\bullet(\Sigma Z')$ has trivial multiplication and is Koszul. The Koszul dual

$$\mathrm{H}^\bullet(\Sigma Z')^! = \mathrm{Tor}^{\mathrm{H}^\bullet(\Sigma Z')}_{\bullet,\bullet}(K,K)$$

is the tensor algebra of the reduced cohomology $\overline{\mathrm{H}}^\bullet(\Sigma Z')$, and it is concentrated on the diagonal of the E_2 page. This agrees with the Bott–Samelson theorem: the cohomology of $\mathrm{H}^\bullet(\Omega\Sigma Z')$ is (additively) the tensor algebra on the reduced cohomology, and this is the E_∞ page. Therefore, there can be no differentials in the Eilenberg–Moore spectral sequence from E_2 on. □

3 Comparison

In this section, we first explain a general procedure to compare the two spectral sequences we have when given two filtrations on the same cochain complex B, following Deligne [Del71] (see Section 3.1). We then return to the situation of a principal fibration (2.3) and apply this procedure to the cochain complex $B = \mathrm{B}(K, \mathrm{C}(Z), \mathrm{C}(Y)) \simeq \mathrm{C}(X)$ from the bar construction which supports two filtrations F and W that lead to a Leray–Serre spectral sequence (Section 3.2) and an Eilenberg–Moore spectral sequence (Section 3.3), respectively, both with abutment $\mathrm{H}^\bullet(X)$. As a result, we obtain two more spectral sequences, both supported on the tri-graded vector space

$$\mathrm{E}_1^{s,t,u} = \mathrm{H}^\bullet(Y^s, Y^{s-1}) \otimes (\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t})^u = \mathrm{Cell}^s(Y; (\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t})^u),$$

and which abut to the E_1 page of an Eilenberg–Moore spectral sequence (as we show in Section 3.4) and the E_1 page of a Leray–Serre spectral sequence (as we show in Section 3.5), respectively, both for $\mathrm{H}^\bullet(X)$. We then prove that these spectral sequences very often degenerate early and we spell out the consequences, with Theorems 3.5 and 3.12 being the main results.

3.1 Zassenhaus squares

Let B be a cochain complex with two filtrations F and W. From the two filtrations, we get two spectral sequences with abutment $\mathrm{H}^n(B)$:

$$\mathrm{E}_1^{s,n-s}(B,F) \cong \mathrm{H}^n(\mathrm{Gr}_F^s B) \Longrightarrow \mathrm{H}^n(B) \tag{3.1}$$

and

$$\mathrm{E}_1^{t,n-t}(B,W) \cong \mathrm{H}^n(\mathrm{Gr}_W^t B) \Longrightarrow \mathrm{H}^n(B). \tag{3.2}$$

The Zassenhaus Lemma gives us canonical isomorphisms

$$\mathrm{Gr}_F^s \mathrm{Gr}_W^t B \cong \mathrm{Gr}_W^t \mathrm{Gr}_F^s B \tag{3.3}$$

for all $s,t \in \mathbb{Z}$, and these allow us, following Deligne's thesis [Del71, 1.4.9], to relate the two spectral sequences (3.1) and (3.2). The chain complexes (3.3) come equipped with a differential which is induced from B, and their cohomology is the tri-graded vector space

$$\mathrm{E}_1^{s,t,u} \cong \mathrm{H}^n(\mathrm{Gr}_F^s \mathrm{Gr}_W^t B),$$

with

$$n = s+t+u.$$

These vector spaces support two d_1 differentials, say d_1^F and d_1^W, as parts of spectral sequences with abutments $\mathrm{H}^n(\mathrm{Gr}_W^t B)$ and $\mathrm{H}^n(\mathrm{Gr}_F^s B)$. These abutments are the vector spaces on the E_1 pages of the spectral sequences (3.1) and (3.2), respectively. It is worth spelling out the indices in detail: the differential $\mathrm{d}_1^F \colon \mathrm{E}_1^{s,t,u} \to \mathrm{E}_1^{s+1,t,u}$ is part of a spectral sequence

$$\mathrm{E}_1^{s,t,u} \cong \mathrm{H}^{s+t+u}(\mathrm{Gr}_F^s \mathrm{Gr}_W^t B) \Longrightarrow \mathrm{E}_1^{t,s+u}(B,W) \cong \mathrm{H}^{s+t+u}(\mathrm{Gr}_W^t B), \tag{3.4}$$

whereas the differential $\mathrm{d}_1^W \colon \mathrm{E}_1^{s,t,u} \to \mathrm{E}_1^{s,t+1,u}$ is part of a spectral sequence

$$\mathrm{E}_1^{s,t,u} \cong \mathrm{H}^{s+t+u}(\mathrm{Gr}_F^s \mathrm{Gr}_W^t B) \Longrightarrow \mathrm{E}_1^{s,t+u}(B,F) \cong \mathrm{H}^{s+t+u}(\mathrm{Gr}_F^s B). \tag{3.5}$$

We summarize the present situation concisely in the following *Zassenhaus square* (compare with [Del71, (1.4.9.2)] and [Mil81, (4.1)]).

$$\begin{array}{ccccc} & & \mathrm{E}_1^{s,t,u} & & \\ & \overset{(3.4)}{\swarrow\!\!\swarrow} & & \overset{(3.5)}{\searrow\!\!\searrow} & \\ \mathrm{E}_1^{t,s+u}(B,W) & & & & \mathrm{E}_1^{s,t+u}(B,F) \\ & \underset{(3.2)}{\searrow\!\!\searrow} & & \underset{(3.1)}{\swarrow\!\!\swarrow} & \\ & & \mathrm{H}^{s+t+u}(B) & & \end{array}$$

Notation 3.1. Sometimes the spectral sequences call for indices with $n = p+q$. We shall use

$$s = p+n, \quad t = p, \quad u = -2p$$

on those occasions.

Even though an Eilenberg–Moore and the corresponding Leray–Serre spectral sequence need not be related by *décalage*, the following general result is the blueprint to show that the index transformation is nevertheless exactly as expected.

Proposition 3.2. *If the spectral sequences* (3.4) *and* (3.5) *both degenerate, so that we have isomorphisms*

$$\mathrm{E}_1^{t,n-t}(W) \cong \bigoplus_{s+u=n-t} \mathrm{E}_1^{s,t,u} \tag{3.6}$$

and

$$\mathrm{E}_1^{s,n-s}(F) \cong \bigoplus_{t+u=n-s} \mathrm{E}_1^{s,t,u}, \tag{3.7}$$

then the spectral sequences (3.1) *and* (3.2) *are related by*

$$\mathrm{E}_1^{p,n-p}(W) \cong \mathrm{E}_1^{p+n,p}(F) \tag{3.8}$$

under the index transformation in Notation 3.1.

Proof. Setting $t = p$ from Notation 3.1 into (3.6), we get

$$\mathrm{E}_1^{p,n-p}(W) \cong \bigoplus_{s+u=n-p} \mathrm{E}_1^{s,p,u}$$

With $s = p+n$, this gives

$$\bigoplus_{s+u=n-p} \mathrm{E}_1^{s,p,u} \cong \bigoplus_{p+n+u=n-p} \mathrm{E}_1^{p+n,p,u} \cong \bigoplus_{p+u=-p} \mathrm{E}_1^{p+n,p,u},$$

cancelling the n in the summation index. With $t = p$ and $s = p+n$ in (3.7), we come back to

$$\bigoplus_{t+u=-p} \mathrm{E}_1^{p+n,t,u} \cong \mathrm{E}_1^{p+n,-p}(F),$$

and this finishes the proof. □

In the rest of this section, we develop the four spectral sequences (3.1), (3.2), (3.4), and (3.5) for the bifiltered cochain complex $B = \mathrm{B}(K,\mathrm{C}(Z),\mathrm{C}(Y)) \simeq \mathrm{C}(X)$, where the two filtrations come from a chosen cellular filtration F on Y and the bar filtration W.

3.2 Identifying the Leray–Serre spectral sequence

We now consider the spectral sequence (3.1) for the cochain complex $B = \mathrm{B}(K,\mathrm{C}(Z),\mathrm{C}(Y)) \simeq \mathrm{C}(X)$. The filtration F is induced from a chosen cellular filtration on Y.

Proposition 3.3. *For the F–filtered cochain complex $B = \mathrm{B}(K,\mathrm{C}(Z),\mathrm{C}(Y))$, the spectral sequence* (3.1) *agrees from its* E_1 *page on with the Leray–Serre spectral sequence for the principal fibration sequence $\Omega Z \to X \to Y$.*

Proof. This follows by noting that the Eilenberg–Moore equivalence (2.12) is natural and, therefore, preserves the F–filtrations on both sides. The induced map between the associated graded cochain complexes is an equivalence of cochain complexes by the relative Eilenberg–Moore theorem (Proposition 2.10 above), and it induces an isomorphism from the next page, their E_1 pages, on. □

3.3 Identifying the Eilenberg–Moore spectral sequence

We now consider the spectral sequence (3.2) for the cochain complex $B = \mathrm{B}(K, \mathrm{C}(Z), \mathrm{C}(Y)) \simeq \mathrm{C}(X)$.

Proposition 3.4. *For the cochain complex $B = \mathrm{B}(K, \mathrm{C}(Z), \mathrm{C}(Y))$ of a space X with its W–filtration, the spectral sequence* (3.2) *agrees with the Eilenberg–Moore spectral sequence for the fibration sequence $X \to Y \to Z$.*

Proof. In contrast to Proposition 3.3, which required some identifications, this is basically the definition of the Eilenberg–Moore spectral sequence. The E_0 page is given by $\mathrm{Gr}^t_W B$, and by (2.8) this is isomorphic to

$$\mathrm{E}_0^{t,n-t}(B,W) = \mathrm{Gr}^t_W B^n = (\overline{\mathrm{C}}(Z)^{\otimes -t} \otimes \mathrm{C}(Y))^{n-t}. \tag{3.9}$$

The d_0 differential on this is induced from B, but on the associated graded for the W–filtration we only see the internal differential from the cochain complexes $\mathrm{C}(?)$. Therefore, passing to cohomology, we get

$$\mathrm{E}_1^{t,n-t}(B,W) = \mathrm{H}^n(\mathrm{Gr}^t_W B) = (\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t} \otimes \mathrm{H}^\bullet(Y))^{n-t} \tag{3.10}$$

from (2.9). The d_1 differential comes from the bar resolution, which can be used to compute Tor. By (2.10), this gives

$$\mathrm{E}_2^{t,n-t}(B,W) = \mathrm{Tor}_{-t}^{\mathrm{H}^\bullet(Z)}(K, \mathrm{H}^\bullet(Y))^{n-t},$$

and

$$\mathrm{H}^n(B) = \mathrm{H}^n(\mathrm{B}(K, \mathrm{C}(Z), \mathrm{C}(Y))) = \mathrm{H}^n(\mathrm{C}(X)) = \mathrm{H}^n(X)$$

is the abutment. □

3.4 Prelude to the Eilenberg–Moore spectral sequence

We now consider the spectral sequence (3.4) for the particular situation (2.3) of a principal fibration sequence $\Omega Z \to X \to Y \to Z$ and the cochain complex $B = \mathrm{B}(K, \mathrm{C}(Z), \mathrm{C}(Y)) \simeq \mathrm{C}(X)$ from the bar construction. The F–filtration is given by a chosen cellular structure on Y, and the W–filtration is given by the bar filtration (2.7). The spectral sequence (3.4) comes from the filtration on the associated graded cochain complex $\mathrm{Gr}^t_W B$ that is induced by the F–filtration. The main result here is the following.

Theorem 3.5. *For the complex B with the two filtrations F and W as above, the spectral sequence* (3.4) *always degenerates from its* E_2 *page on. It degenerates from its* E_1 *page on if and only if the space Y is K–minimal (see Definition 2.4).*

We start by describing the spectral sequence (3.4) in our situation.

Proposition 3.6. *For the cochain complex B with the two filtrations F and W as above, the spectral sequence* (3.4) *has*

$$\mathrm{E}_1^{s,t,u} = (\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t} \otimes \mathrm{H}^\bullet(Y^s, Y^{s-1}))^q$$

and

$$\mathrm{E}_2^{s,t,u} = (\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t} \otimes \mathrm{H}^s(Y))^q,$$

with $q = s+u = n-t$ as in Notation 3.1.

Proof. We start by noting that we look at the spectral sequence for the filtered complex

$$\mathrm{Gr}^t_W B = \overline{\mathrm{C}}(Z)^{\otimes -t} \otimes \mathrm{C}(Y)[-t], \tag{3.11}$$

with $\mathrm{Gr}^t_W B^{t+u} = (\overline{\mathrm{C}}(Z)^{\otimes -t} \otimes \mathrm{C}(Y))^u$ and the shift as in (3.9). The filtration is induced by the F–filtration, the cellular filtration on Y. Therefore, we have

$$\mathrm{E}_0^{s,t,u} = \mathrm{Gr}^s_F \mathrm{Gr}^t_W B^{s+t+u} = (\overline{\mathrm{C}}(Z)^{\otimes -t} \otimes \mathrm{C}(Y^s, Y^{s-1}))^{s+u}, \tag{3.12}$$

with t fixed throughout the spectral sequence. In other words, we can think of this as a family of spectral sequences, indexed by t. The differential on the E_0 page is induced from B. Since we have passed to the associated graded for the W–filtration, we do not see the external component of the bar differential, but only the internal one, which the cochain complexes $\mathrm{C}(?)$ bring with them. Passing to cohomology, we then get

$$\mathrm{E}_1^{s,t,u} = \mathrm{H}^n(\mathrm{Gr}^s_F \mathrm{Gr}^t_W B) = (\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t} \otimes \mathrm{H}^\bullet(Y^s, Y^{s-1}))^{s+u}. \tag{3.13}$$

The d_1 differential is induced from the F–filtration, in other words, from the cells of Y. The vector spaces $\mathrm{H}^\bullet(Y^s, Y^{s-1})$, for varying s, form the cellular cochain complex for the space Y with coefficients in the field K. Because the functor $? \longmapsto \overline{\mathrm{H}}^\bullet(Z)^{\otimes -t} \otimes ?$ is exact, we get an isomorphism $\mathrm{E}_2^{s,t,u} \cong (\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t} \otimes \mathrm{H}^s(Y))^{s+u}$, as claimed. □

Remark 3.7. Note how passing from B to $\mathrm{Gr}^t_W B$ has eliminated the external direction from the bar differential. As this is the only place where the $\mathrm{H}^\bullet(Z)$–module structure on $\mathrm{H}^\bullet(Y)$ would have been relevant, there was no need for us to take it into account, and we were instead allowed to treat $\mathrm{H}^\bullet(Y)$ as a graded vector space. As such, it decomposes into its homogeneous pieces, of course: $\mathrm{H}^\bullet(Y) = \bigoplus_s \mathrm{H}^s(Y)$. It is clear that it is, in general, impossible to recover the $\mathrm{H}^\bullet(Z)$–module structure on $\mathrm{H}^\bullet(Y)$ from the trivial one on the $\mathrm{H}^s(Y)$'s. We shall come back to this point in Remark 3.11.

Using $s + u = n - t$, the abutment of the spectral sequence (3.4) is

$$\mathrm{H}^n(\mathrm{Gr}^t_W B) = \mathrm{H}^n(\overline{\mathrm{C}}(Z)^{\otimes -t} \otimes \mathrm{C}(Y)[-t]) = (\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t} \otimes \mathrm{H}^\bullet(Y))^{n-t}$$

by definition and the Künneth theorem. We have seen in (3.10) that this abutment agrees with the E_1 page $\mathrm{E}_1^{t,s+u}(B,W) = \mathrm{E}_1^{t,n-t}(B,W)$ of the Eilenberg–Moore spectral sequence, and also with

$$\bigoplus_{s+u=n-t} \mathrm{E}_2^{s,t,u} = \bigoplus_{s+u=n-t} (\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t})^u \otimes \mathrm{H}^s(Y).$$

As we shall see, the spectral sequence degenerates from its E_2 page on.

Proposition 3.8. *For the complex B with the two filtrations F and W as above, the spectral sequence* (3.4) *is isomorphic, from its* E_1 *page on, to the spectral sequence obtained by applying the exact functor $? \mapsto \overline{\mathrm{H}}^\bullet(Z)^{\otimes -t} \otimes ?$ to the spectral sequence constructed from the chosen cellular filtration F on the cochain complex* $\mathrm{C}(Y)$.

Proof. There is essentially only one reasonable way to prove this: we show that the two spectral sequences are induced by two equivalent cochain complexes with two equivalent filtrations. The spectral sequence (3.4) originates in the filtered complex (3.11) with the F–filtration from the chosen cellular structure on the space Y. Since we are working over a field, we can choose an equivalence

$$\mathrm{H}^\bullet(Y) \xrightarrow{\sim} \mathrm{C}(Y) \tag{3.14}$$

of cochain complexes, where the left hand side has zero differential. This induces an equivalence

$$\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t} \otimes \mathrm{C}(Y) \xrightarrow{\sim} \overline{\mathrm{C}}(Z)^{\otimes -t} \otimes \mathrm{C}(Y) \tag{3.15}$$

of cochain complexes. The multiplicative structures which we have on both sides of (3.14) play no role because the tensor products are over the ground field K. The equivalence (3.15) respects the F–filtration, and therefore induces an isomorphism of spectral sequences from their E_1 pages on. The right hand side of (3.15) gives the spectral sequence (3.4). The left hand side of (3.15) is obtained from the filtered cochain complex $\mathrm{C}(Y)$, together with its cellular filtration F, by applying the exact functor $? \mapsto \overline{\mathrm{H}}^\bullet(Z)^{\otimes -t} \otimes ?$. The spectral sequence of the left hand side is, therefore, obtained from that of the cochain complex $\mathrm{C}(Y)$ with respect to the F–filtration by applying the same functor, as claimed. □

Proof of Theorem 3.5. The spectral sequence for the cochain complex $\mathrm{C}(Y)$ with respect to any cellular filtration F degenerates from its E_2 page on because the E_1 page is concentrated in its zeroth row. Applying any exact functor to it yields another spectral sequence that also degenerates from its E_2 page on. As we have seen in Proposition 3.8, when we apply the exact functor $? \mapsto \overline{\mathrm{H}}^\bullet(Z)^{\otimes -t} \otimes ?$, we get the spectral sequence (3.4). Therefore, this spectral sequence has to degenerate from its E_2 page on, too.

If the space Y in question is K–minimal, then we have a cellular structure on it such that the spectral sequence for the cochain complex $\mathrm{C}(Y)$ with respect to that cellular filtration F degenerates already from its E_1 page on, and the same argument as in the first part of the proof implies that the spectral sequence (3.4) does so, too. □

Remark 3.9. We can think of (3.4) as a cochain complex that computes the E_1 page of an Eilenberg–Moore spectral sequence. Note the difference to the E_0 term of the latter: there we take the bar construction on the cochain complexes, whereas (3.4) involves only the cohomology of those.

Let us spell out the consequence when the stronger hypothesis in Theorem 3.5 is satisfied:

Corollary 3.10. *When the space Y is K–minimal, the Zassenhaus square factors the Eilenberg–Moore spectral sequence as a Leray–Serre spectral sequence after the bar spectral sequences for the cells of Y.*

One consequence of this is that, for a K–minimal space Y, the Leray–Serre spectral sequence has at most as many non-trivial differentials as the Eilenberg–Moore spectral sequence: some terms might already have been killed by the bar differentials. The following Section 3.5 contains our discussion of these preliminary bar differentials.

Remark 3.11. One might wonder if it is possible to somehow commute the external and the cell differentials to obtain a relation between the complex $\mathrm{Tor}^{\mathrm{H}^\bullet(Z)}(K, \mathrm{H}^\bullet(Y^s, Y^{s-1}))$ with respect to the differential induced by the cellular differential on the $\mathrm{H}^\bullet(Y^s, Y^{s-1})$, and the E_2 page $\mathrm{Tor}^{\mathrm{H}^\bullet(Z)}(K, \mathrm{H}^\bullet(Y))$ of the Eilenberg–Moore spectral sequence. But this is hopeless, as for example the case $Y = Z$ shows. One of the problems is that there is a canonical isomorphism $\mathrm{H}^\bullet(Y) = \bigoplus_s \mathrm{H}^s(Y)$ as graded K–vector spaces, but not as graded $\mathrm{H}^\bullet(Z)$–modules, in general, as already emphasized in Remark 3.7.

3.5 Prelude to the Leray–Serre spectral sequence

We now consider the spectral sequence (3.5) for the particular situation (2.3) of a principal fibration sequence $\Omega Z \to X \to Y \to Z$ and the cochain complex $B = \mathrm{B}(K, \mathrm{C}(Z), \mathrm{C}(Y)) \simeq \mathrm{C}(X)$ from the bar construction. The F–filtration is given by a chosen cellular structure on Y, and the W–filtration is given by the bar filtration (2.7). The spectral sequence (3.5) comes from the filtration on $\mathrm{Gr}^s_F B$ that is induced by the W–filtration. The main result here is the following.

Theorem 3.12. *For the complex B with the two filtrations F and W as above, the spectral sequence* (3.5) *degenerates from its* E_2 *page on for all Y if and only if this holds for $Y \simeq \star$, that is if the Eilenberg–Moore spectral sequence* (2.15) *for the loop space ΩZ of Z degenerates from its* E_2 *page on.*

Recall from Definition 2.14 that we have called those spaces Z that lead to degeneracy in Theorem 3.12 K–unbarred. Therefore, we can refer to Example 2.15 for examples of spaces Z where Theorem 3.12 applies (see also Corollary 3.16 below for the resulting formula).

Recall our conventions for the cellular cochains on the space Y from Section 2.3: for all coefficients V we have set

$$\mathrm{Cell}^s(Y; V) = \mathrm{Hom}(\mathrm{H}_\bullet(Y^s, Y^{s-1}), V) = \mathrm{H}^\bullet(Y^s, Y^{s-1}) \otimes V,$$

and this is exact as a functor in V, because we are working over a field K. We can use this to give a description of the spectral sequence (3.5) in our situation.

Proposition 3.13. *For the cochain complex B with the two filtrations F and W as above, the spectral sequence* (3.5) *has*

$$\mathrm{E}_1^{s,t,u} = \mathrm{Cell}^s(Y;(\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t})^u)$$

and

$$\mathrm{E}_2^{s,t,u} = \mathrm{Cell}^s(Y;\mathrm{Tor}_{-t}^{\mathrm{H}^\bullet(Z)}(K,K)^u).$$

Its abutment is $\mathrm{H}^n(X^s,X^{s-1})$.

Note that, in this description, we recognize that the abutment of (3.5) is indeed the E_1 page (2.2) of the Leray–Serre spectral sequence for $\mathrm{H}^\bullet(X)$ with respect to the chosen cellular structure on the space Y.

Proof. We are looking at the spectral sequence

$$\mathrm{E}_1^{s,t,u} = \mathrm{H}^n(\mathrm{Gr}_F^s \mathrm{Gr}_W^t B) \Longrightarrow \mathrm{E}_1^{s,t+u}(B,F) = \mathrm{H}^n(\mathrm{Gr}_F^s B).$$

As in (3.12), this spectral sequence starts with

$$\mathrm{E}_0^{s,t,u} = \mathrm{Gr}_F^s \mathrm{Gr}_W^t B^{s+t+u} = (\overline{\mathrm{C}}(Z)^{\otimes -t} \otimes \mathrm{C}(Y^s,Y^{s-1}))^{s+u}.$$

The s is always fixed. The d_0 differential is induced from B, the totalization of the bar construction, and since we have passed to Gr_W^t, we do not see the external differential, but only the internal differential. Therefore, as in (3.13), passing to cohomology, we get

$$\mathrm{E}_1^{s,t,u} = \mathrm{H}^n(\mathrm{Gr}_F^s \mathrm{Gr}_W^t B) = (\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t} \otimes \mathrm{H}^\bullet(Y^s,Y^{s-1}))^{s+u} \tag{3.16}$$

from the Künneth theorem. We note that $\mathrm{H}^\bullet(Y^s,Y^{s-1})$, as an $\mathrm{H}^\bullet(Z)$–module, is trivial and concentrated in degree s, so that we can pull it out, and we find

$$\mathrm{E}_1^{s,t,u} = (\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t})^u \otimes \mathrm{H}^\bullet(Y^s,Y^{s-1}) = \mathrm{Cell}^s(Y;(\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t})^u). \tag{3.17}$$

These identifications are compatible with the differential d_1 which is induced by the differential on the associated graded. This differential is the external part of the bar differential. This leads to

$$\mathrm{E}_2^{s,t,u} = \mathrm{Cell}^s(Y;\mathrm{Tor}_{-t}^{\mathrm{H}^\bullet(Z)}(K,K)^u)$$

in the same way, as claimed.

As for the abutment, we have $\mathrm{Gr}_F^s B = \mathrm{B}(K,\mathrm{C}(Z),\mathrm{C}(Y^s,Y^{s-1})) \simeq \mathrm{C}(X^s,X^{s-1})$, where the equivalence is given by the relative version of the Eilenberg–Moore equivalence (see our Proposition 2.10 above). Therefore, passing to cohomology, we find that the abutment is $\mathrm{H}^n(\mathrm{Gr}_F^s B) = \mathrm{H}^n(X^s,X^{s-1})$, as claimed. □

Proposition 3.14. *For the cochain complex B with the two filtrations F and W as above, the spectral sequence* (3.5) *is isomorphic, from* E_1 *on, to the result of applying the exact functor* $? \mapsto \mathrm{Cell}^s(Y;?)$ *to the Eilenberg–Moore spectral sequence*

$$\mathrm{E}_1^{t,u} = (\overline{\mathrm{H}}^\bullet(Z)^{\otimes -t})^u \Longrightarrow \mathrm{H}^{t+u}(\Omega Z)$$

for the loop space ΩZ.

Proof. Again, there is essentially only one reasonable way to prove this: we show that the two spectral sequences are induced by two equivalent cochain complexes with two equivalent filtrations. And we already know from Proposition 3.13 that the E_1 pages are isomorphic: The spectral sequence (3.5) originates in the filtered complex

$$\mathrm{Gr}_F^s B = \mathrm{Gr}_F^s \mathrm{B}(K,\mathrm{C}(Z),\mathrm{C}(Y)) \cong \mathrm{B}(K,\mathrm{C}(Z),\mathrm{C}(Y^s,Y^{s-1})),$$

with the W–filtration from the bar construction. Since we are working over a field, we can choose an equivalence

$$\mathrm{H}^\bullet(Y^s,Y^{s-1}) \xrightarrow{\sim} \mathrm{C}(Y^s,Y^{s-1})$$

from the zero-differential complex $\mathrm{H}^\bullet(Y^s,Y^{s-1})$, concentrated in degree s, to the cochain complex $\mathrm{C}(Y^s,Y^{s-1})$. This equivalence induces an equivalence

$$\mathrm{B}(K,\mathrm{C}(Z),\mathrm{H}^\bullet(Y^s,Y^{s-1})) \simeq \mathrm{B}(K,\mathrm{C}(Z),\mathrm{C}(Y^s,Y^{s-1})),$$

which, by construction, respects the bar filtrations that we have on both sides. The right hand side leads to the spectral sequence (3.5). The left hand side is isomorphic, again respecting the bar filtrations, to $\mathrm{Cell}^s(Y;\mathrm{B}(K,\mathrm{H}^\bullet(Z),K))$. Since the latter is the result of applying the functor $? \mapsto \mathrm{Cell}^s(Y;?)$ to the filtered cochain complex $\mathrm{B}(K,\mathrm{H}^\bullet(Z),K)$, the result follows. □

Remark 3.15. The preceding result implies an agreement

$$\mathrm{H}^n(X^s,X^{s-1}) \cong \mathrm{Cell}^s(Y;\mathrm{H}^{t+u}(\Omega Z))$$

of the abutments of the two spectral sequences, too. Such a statement is a standard part of the identification of the E_1 page of the Leray–Serre spectral sequence, at least in the case when the fundamental group of the base acts trivially on the cohomology of the fiber. In our situation, the fundamental group of the space Y need not be trivial, but the action on $\mathrm{H}^\bullet(\Omega Z)$ is, because it factors through the morphism $\pi_1(Y) = \pi_0(\Omega Y) \to \pi_0(\Omega Z) = \pi_1(Z)$, and this is trivial if the space Z is simply-connected.

Proof of Theorem 3.12. One direction is clear: if the spectral sequences degenerate for all spaces Y, then, in particular, the ones for contractible spaces do so.

For the other direction, assume that the Eilenberg–Moore spectral sequence for the loop space degenerates from its E_2 page on. Applying any exact functor to it yields another spectral sequence that also degenerates from its E_2 page on. As we have seen in Proposition 3.14, when we apply the exact functor $? \mapsto \mathrm{Cell}^s(Y;?)$, we get the spectral sequence (3.5). Therefore, this has to degenerate from its E_2 page on, too. □

Corollary 3.16. *If Theorem 3.12 applies, and the spectral sequence* (3.5) *does indeed degenerate from* E_2 *on, we get*

$$\mathrm{H}^n(X^s,X^{s-1}) \cong \bigoplus_{t+u=n-s} \mathrm{Tor}_{-t}^{\mathrm{H}^\bullet(Z)}(K,K)^u \otimes \mathrm{H}^\bullet(Y^s,Y^{s-1})$$

for the E_1 *page of the Leray–Serre spectral sequence. This holds whenever the space Z is K–unbarred.*

Proof. This follows immediately from what has been said before and the description of the E_2 term in Proposition 3.13. □

We can summarize our findings by saying that, if we want to compute the cohomology $\mathrm{H}^\bullet(X)$ of X by first running the spectral sequence (3.5) and then the Leray–Serre spectral sequence, then the cellular structure of Y does not interfere with the first spectral sequence (3.5) at all, only afterwards does it become relevant, in the Leray–Serre spectral sequence.

The following formulation applies our results to K–minimal spaces.

Theorem 3.17. *In our situation* (2.3)*, if the space Y is K–minimal, then there is a spectral sequence*

$$\mathrm{E}_2^{s,t,u} = \mathrm{H}^s(Y;\mathrm{Tor}_{-t}^{\mathrm{H}^\bullet(Z)}(K,K)^u) \Longrightarrow \mathrm{H}^s(Y;\mathrm{H}^{t+u}(\Omega Z))$$

that computes the E_2 *page of the Leray–Serre spectral sequence for the cohomology of X. It degenerates from its* E_2 *page on if, in addition, the space Z is K–unbarred.*

Proof. When Y is K–minimal, we have $\mathrm{Cell}^s(Y;?) \cong \mathrm{H}^s(Y;?)$. Then the spectral sequences in question are those from Proposition 3.13, and the abutment is identified in Remark 3.15. Theorem 3.12 gives the statement about the degeneration. □

4 Examples

In this section, we briefly discuss the most critical test cases of the situation (2.3) for our purposes. In particular, we discuss the spectral sequences when $X \simeq \star$, or $Y \simeq \star$, or $Z \simeq \star$, respectively, and the three principal fibrations that feature the Hopf fibration $\eta\colon \mathrm{S}^3 \to \mathrm{S}^2$. These examples are chosen to be easy to work out; their purpose here is to supply evidence for the applicability and the limits of our theory. In particular, Example 4.5 eliminates all hope that, even in the most favorable cases, the relation between the Leray–Serre and the Eilenberg–Moore spectral sequences is given by a simple *décalage* as in Section 2.2.

4.1 Examples involving contractible spaces

Example 4.1. First, we consider the fibration sequences $\Omega Z \to X \to Y \to Z$ where $Y \simeq X \times Z$ and the map $Y \to Z$ is equivalent to the projection. The Eilenberg–Moore spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(Z)}(K, \mathrm{H}^\bullet(X \times Z))^q \Longrightarrow \mathrm{H}^{p+q}(X)$$

always degenerates from E_2 on, because $\mathrm{H}^\bullet(X \times Z) \cong \mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(Z)$ is a free $\mathrm{H}^\bullet(Z)$–module, so that the E_2 page is just $\mathrm{H}^\bullet(X)$ concentrated in the $p = 0$ column. In contrast, the Leray–Serre spectral sequence can have arbitrarily long differentials, even in the simplest case of the situation, when the fiber

$$X \simeq \star \tag{4.1}$$

is contractible, so that the map $Y \to Z$ is an equivalence. For a specific example, take $Y = \mathrm{S}^r$ a sphere of dimension $r \geqslant 2$. Then the Leray–Serre spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(\mathrm{S}^r; \mathrm{H}^t(\Omega \mathrm{S}^r)) \Longrightarrow \mathrm{H}^{s+t}(\star)$$

has a nontrivial d_r. In contrast, the Eilenberg–Moore spectral sequence is concentrated in $\mathrm{E}^{0,0}$, independent of the spaces $Y \simeq Z$.

Example 4.2. We consider the fibration sequences $\Omega Z \to X \to Y \to Z$ for spaces Y such that the spectral sequence

$$\mathrm{E}_1^{s,t} = \mathrm{H}^{s+t}(Y^s, Y^{s-1}) \Longrightarrow \mathrm{H}^{s+t}(Y),$$

which is concentrated in the $t = 0$ row and which always degenerates from E_2 on, already degenerates from E_1 on. These are the spaces that we called K–minimal for the field K in Definition 2.4. The results from Section 3.4, especially Theorem 3.5 and its Corollary 3.10, apply. A fundamental example of the situation here is the case when the total space

$$Y \simeq \star \tag{4.2}$$

is contractible, so that the map $\Omega Z \to X$ is an equivalence. In this case, the Leray–Serre spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(\star; \mathrm{H}^t(\Omega Z)) \Longrightarrow \mathrm{H}^{s+t}(\Omega Z)$$

always degenerates from the beginning, because it is concentrated in the $s = 0$ column. This even includes the E_1 page, because we can assume the cellular filtration to be trivial. In contrast, the Eilenberg–Moore spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(Z)}(K, K)^q \Longrightarrow \mathrm{H}^{p+q}(\Omega Z)$$

can have arbitrarily long differentials. Specifically, take $Z = \mathrm{B}\mathbb{Z}/\ell$ for an odd prime ℓ. Then the Eilenberg–Moore spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(\mathrm{B}\mathbb{Z}/\ell)}(\mathbb{F}_\ell,\mathbb{F}_\ell)^q \Longrightarrow \mathrm{H}^{p+q}(\mathbb{Z}/\ell)$$

has a nonzero differential $\mathrm{d}_{\ell-1}$ (see Smith [Smi67a, Sec. 3] and Baker–Richter [BR11, Sec. 6]). Eventually, the ℓ–dimensional E_∞ page is spread out into 1–dimensional pieces for the Eilenberg–Moore spectral sequence, whereas it is concentrated in the $\mathrm{E}_\infty^{0,0}$ spot for the Leray–Serre spectral sequence.

Remark 4.3. The preceding example contradicts the statement of [McC01, Ex. 8.12]: the Eilenberg–Moore spectral sequence does not have to degenerate at E_2, even if the Leray–Serre spectral sequence has all differentials coming from transgressions. Besides, we learned from Schochet (private communication) that the attribution of that exercise to him is inaccurate.

Example 4.4. We consider the fibration sequences $\Omega Z \to X \to Y \to Z$ for spaces Z such that the Eilenberg–Moore spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(Z)}(K,K)^q \Longrightarrow \mathrm{H}^{p+q}(\Omega Z)$$

degenerates from its E_2 page on. These are the spaces that we called K–unbarred in Definition 2.14. The results from Section 3.5, especially Theorem 3.12 and its Corollary 3.16, apply. A special case of this situation is when the base space

$$Z \simeq \star \tag{4.3}$$

is contractible, so that the map $X \to Y$ is an equivalence. This is another example of product fibrations as in Example 4.1. In this case, the Eilenberg–Moore spectral sequence has an E_2 page that is concentrated on the $p = 0$ column; it degenerates already from E_1 on. The Leray–Serre spectral sequence has an E_2 page that is concentrated on the $t = 0$ line; it degenerates from E_2 on, but on the E_1 page one might still see differentials from the cellular structure on Y. As we have already mentioned in Example 2.15, more interesting instances of the situation here are given in the case when the base space Z is a torus $(\mathrm{S}^1)^d$ or a sphere S^d where K is any field, but also $\mathbb{R}P^\infty \simeq \mathrm{B}\mathbb{Z}/2$ when the characteristic of the field K is 2.

4.2 Principal fibrations involving the Hopf map

Example 4.5. Consider the fibration sequence

$$\mathrm{S}^3 \xrightarrow{\eta} \mathrm{S}^2 \longrightarrow \mathbb{C}\mathrm{P}^\infty \longrightarrow \mathbb{H}\mathrm{P}^\infty.$$

The projective spaces are K–minimal and K–unbarred, so that we are in the best possible situation. Since the cohomology ring $\mathrm{H}^\bullet(\mathbb{C}\mathrm{P}^\infty)$ is free as a module over the algebra $\mathrm{H}^\bullet(\mathbb{H}\mathrm{P}^\infty)$, on two generators in degree 0 and 2, the Eilenberg–Moore spectral sequence degenerates at its E_2 page to compute $\mathrm{H}^\bullet(\mathrm{S}^2)$. On the other hand, the Leray–Serre spectral sequence needs infinitely many d_4's to achieve the same goal. Thus, there is no way that a reindexing of the pages (Deligne's *décalage*, Section 2.2) could explain the relationship between the spectral sequences.

Example 4.6. Consider the fibration sequence

$$\mathrm{S}^1 \longrightarrow \mathrm{S}^3 \xrightarrow{\eta} \mathrm{S}^2 \longrightarrow \mathbb{C}\mathrm{P}^\infty.$$

The spheres and the projective space are all K–minimal and K–unbarred, so that we are in the best possible situation. When computing $\mathrm{H}^\bullet(\mathrm{S}^3)$, the Eilenberg–Moore spectral sequence degenerates from its E_2 page on, but the Leray–Serre spectral sequence needs a nontrivial d_2 differential before it degenerates from its E_3 page on.

Example 4.7. Consider the fibration sequence

$$\Omega S^2 \longrightarrow S^1 \longrightarrow S^3 \xrightarrow{\eta} S^2.$$

The spheres are all K–minimal and K–unbarred, so that we are in the best possible situation. We look at the two spectral sequences that compute $H^\bullet(S^1)$. We see that the Eilenberg–Moore spectral sequence has non-trivial d_2 differentials, and the Leray–Serre spectral sequence has non-trivial d_3 differentials. The theory in Section 3 applies. Both preludes degenerate and show that the Eilenberg–Moore spectral sequence and the Leray–Serre spectral sequence are bigraded incarnations of the same tri-graded spectral sequence. In fact, we have

$$E_1^{s,t,u} = \mathrm{Cell}^s(S^3; (\overline{H}^\bullet(S^2)^{\otimes -t})^u) = \begin{cases} K & s \in \{0,3\} \text{ and } t \geqslant 0 \text{ and } u = 2t \\ 0 & \text{otherwise.} \end{cases}$$

from (3.17). Under the projections (3.6) and (3.7), these give rise to the usual E_2 pages, and the index transformation (3.8) shifts the Eilenberg–Moore d_2 to become a Leray–Serre d_3.

5 Applications

In this section, we consider a general principal fibration sequence (2.3) of spaces. As before, we assume that the spaces X, Y, and Z are path-connected and that, in addition, the space Z is simply-connected (or at least nilpotent): Dwyer has shown in [Dwy74, Dwy75] that it suffices that Z be nilpotent (see also Shipley's work [Shi95, Shi96] for later improvements).

Let $I \subseteq H^\bullet(Z)$ denote the kernel of the induced homomorphism $H^\bullet(Z) \to H^\bullet(Y)$, and let $R \subseteq H^\bullet(Y)$ denote the image, which is a subalgebra, isomorphic to

$$H^\bullet(Z)/I \cong R. \tag{5.1}$$

Note that the $H^\bullet(Z)$–module structure on $H^\bullet(Y)$ factors through an R–module structure. We also write $J \subseteq H^\bullet(Y)$ for the ideal generated by the elements of R of positive degree, so that the surjection $H^\bullet(Y) = R \otimes_R H^\bullet(Y) \to K \otimes_R H^\bullet(Y)$ induces an isomorphism

$$H^\bullet(Y)/J \cong K \otimes_R H^\bullet(Y). \tag{5.2}$$

We consider the following two assumptions:

The ideal $I \subseteq H^\bullet(Z)$ is generated by a regular sequence. (5.3)

The R–module $H^\bullet(Y)$ is free. (5.4)

Remark 5.1. For the second assumption (5.4), note that freeness, projectivity, and flatness are equivalent for a bounded-below module such as $H^\bullet(Y)$ over a connected algebra such as R. The second assumption is satisfied if $H^\bullet(Z)$ is polynomial and the images of its generators in $H^\bullet(Y)$ form a regular sequence; in this case, the morphism $H^\bullet(Z) \to H^\bullet(Y)$ is automatically injective. More often than that, however, the images of only some generators form a regular sequence, whereas the other generators are mapped to zero. Our results will apply in this more general situation as well.

5.1 Degeneracy of the Eilenberg–Moore spectral sequence

The assumptions (5.3) and (5.4) allow us to obtain the following degeneracy criterion. Afterward, we shall show that it generalizes all previous degeneracy results for the Eilenberg–Moore spectral sequence of Lie groups (see Theorem 5.11).

Theorem 5.2. *Under the assumptions* (5.3) *and* (5.4)*, the Eilenberg–Moore spectral sequence for* $\mathrm{H}^\bullet(X)$ *degenerates from its* E_2 *page on. If* $h_1, h_2, \ldots, h_r$ *is a regular sequence for I, then we have*

$$\mathrm{E}_\infty \cong \mathrm{E}_2 \cong \Lambda(z_1, z_2, \ldots, z_r) \otimes_K (\mathrm{H}^\bullet(Y)/J),$$

where $\mathrm{bideg}(z_i) = (-1, \deg(h_i))$ *for all* $i = 1, 2, \ldots, r$.

Proof. In general, the Eilenberg–Moore spectral sequence for $\mathrm{H}^\bullet(X)$ takes the form

$$\mathrm{E}_2^{p,q} \cong \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(Z)}(K, \mathrm{H}^\bullet(Y))^q \Longrightarrow \mathrm{H}^{p+q}(X).$$

We have already noted that the $\mathrm{H}^\bullet(Z)$–module structure on $\mathrm{H}^\bullet(Y)$ factors through the R–module structure. By assumption (5.4), the latter is free. Using the isomorphism (5.2), this allows us to write

$$\mathrm{H}^\bullet(Y) \cong R \otimes_K K \otimes_R \otimes \mathrm{H}^\bullet(Y) \cong R \otimes_K \mathrm{H}^\bullet(Y)/J$$

as R–modules, and we get

$$\begin{aligned} \mathrm{E}_2^{p,\bullet} &\cong \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(Z)}(K, \mathrm{H}^\bullet(Y)) \\ &\cong \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(Z)}(K, R \otimes_K \mathrm{H}^\bullet(Y)/J) \\ &\cong \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(Z)}(K, R) \otimes_K \mathrm{H}^\bullet(Y)/J. \end{aligned}$$

By (5.1) and the assumption (5.3), the Koszul resolution [Ser89, Ch. IV, Prop. 2] gives

$$\mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(Z)}(K, R) \cong \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(Z)}(K, \mathrm{H}^\bullet(Z)/(h_1, h_2, \ldots, h_r)) \cong \Lambda(z_1, z_2, \ldots, z_r)^p.$$

with $\deg(z_i) = \deg(h_i) - 1$ for all $i = 1, 2, \ldots, r$. Therefore, the E_2 page becomes

$$\mathrm{E}_2^{p,\bullet} \cong \Lambda(z_1, z_2, \ldots, z_r)^p \otimes_K (\mathrm{H}^\bullet(Y)/J).$$

We find that the bigraded algebra $\mathrm{E}_2^{\bullet,\bullet}$ is generated by the classes $z_1, z_2, \ldots, z_n \in \mathrm{E}_2^{-1,\bullet}$ and classes in $\mathrm{E}_2^{0,\bullet} \cong \mathrm{H}^\bullet(Y)/J$. Because $\mathrm{E}_2^{p,\bullet} = 0$ for $p > 0$ it follows that the differentials d_r vanish on all generators, for all $r \geqslant 2$. □

Corollary 5.3. *If* $\mathrm{H}^\bullet(Z) \to \mathrm{H}^\bullet(Y)$ *is an epimorphism with a kernel that is generated by a regular sequence, then* $\mathrm{H}^\bullet(X) \cong \Lambda(z_1, z_2, \ldots, z_r)$ *with the* z_j *as above.*

Proof. We have $R = \mathrm{H}^\bullet(Y)$ and (5.4) is trivially satisfied. We also have that $J \subseteq \mathrm{H}^\bullet(Y)$ is the irrelevant ideal of cohomology classes of positive degree, so that $\mathrm{H}^\bullet(Y)/J = K$. We see that the Eilenberg–Moore spectral sequence degenerates from its E_2 page on, which is simply the exterior algebra $\Lambda(z_1, z_2, \ldots, z_r)$. This graded commutative algebra is free, and there are, therefore, no extension problems. □

Remark 5.4. The particular case of Corollary 5.3, where the cohomology $\mathrm{H}^\bullet(Z)$ is a polynomial algebra on r generators in even degrees, was noted before by Adem and Reichstein [AR10, Thm. 3.2]: If $\mathrm{H}^\bullet(Z)$ is polynomial, and the induced homomorphism $\mathrm{H}^\bullet(Z) \to \mathrm{H}^\bullet(Y)$ is an epimorphism with a kernel generated by a regular sequence, then $\mathrm{H}^\bullet(X) \cong \Lambda(z_1, z_2, \ldots, z_r)$.

Remark 5.5. Corollary 5.3 follows from our Theorem 5.2, but it is also a special case of Smith's result [Smi67b, Thm 3.1]: take $E_0 \simeq E$ in his notation.

We can also deduce a version of Corollary 5.3 for injective (instead of surjective) maps:

Corollary 5.6. *If* $\mathrm{H}^\bullet(Z)$ *is polynomial and* $\mathrm{H}^\bullet(Z) \to \mathrm{H}^\bullet(Y)$ *is injective, then the Eilenberg–Moore spectral sequence for* $\mathrm{H}^\bullet(X)$ *has*

$$\mathrm{E}_2 \cong \mathrm{H}^\bullet(Y) \otimes_{\mathrm{H}^\bullet(Z)} K$$

and degenerates from its E_2 *page on.*

Proof. Theorem 5.2 applies. The images form a regular sequence if and only if the morphism is injective, and Remark 5.1 shows that the assumption (5.4) is satisfied. The assumption (5.3) on the kernel is automatically satisfied for injective maps. □

Remark 5.7. Corollary 5.6 follows from our Theorem 5.2, but it is also a special case of Smith's result [Smi67b, Thm 3.1]: in his notation, take $E_0 \simeq \star$ (compare Remark 5.5).

5.2 Degeneracy of the Leray–Serre spectral sequence

In comparison, we now study the Leray–Serre spectral sequence for the principal fibration sequence (2.3) that computes the cohomology $\mathrm{H}^\bullet(X)$ of the homotopy fiber X of $f\colon Y \to Z$.

Theorem 5.8. *Let Z be a K–minimal space such that $\mathrm{H}^\bullet(Z)$ is polynomial on a set $\{\, z_j \mid j \in J\,\}$ of classes in degrees $|z_j| \geqslant 2$. If the ideal $I = \mathrm{Ker}(f^\bullet)$ is generated by some of the polynomial generators z_j, the Leray–Serre spectral sequence converging to $\mathrm{H}^\bullet(X)$ has*

$$\mathrm{E}_2 \cong \mathrm{H}^\bullet(Y) \otimes \Lambda(\,\omega(z_j) \mid j \in J\,)$$

with 'desuspended' classes $\omega(z_j) \in \mathrm{E}_2^{0,|z_j|-1}$. The differentials in this Leray–Serre spectral sequence are all induced by transgressions $\mathrm{d}_r(\omega(z_j)) = f^\bullet(z_j)$ with $r = |z_j|$ to give

$$\mathrm{E}_r \cong \mathrm{H}^\bullet(Y)/(\,f^\bullet(z_j) \mid f^\bullet(z_j) \neq 0, |z_j| < r\,) \otimes \Lambda(\,\omega(z_j) \mid f^\bullet(z_j) = 0, |z_j| \geqslant 0\,)$$

and

$$\mathrm{E}_\infty \cong \mathrm{H}^\bullet(Y)/(\,f^\bullet(z_j) \mid f^\bullet(z_j) \neq 0\,) \otimes \Lambda(\,\omega(z_j) \mid f^\bullet(z_j) = 0\,). \tag{5.5}$$

If $r = \max\{|z_j| \mid f^\bullet(z_j) \neq 0\} + 1$ is finite, the Leray–Serre spectral sequence degenerates from E_r on.

Proof. We first note that the assumptions here are slightly stronger than for Theorem 5.2. Indeed (5.3) is clearly satisfied because subsets of polynomial generators form regular sequences. The quotient $R = \mathrm{H}^\bullet(Z)/I$ is a polynomial algebra as well, and this is embedded into $\mathrm{H}^\bullet(Y)$ via $f^\bullet$. It follows that (5.4) is also satisfied. Theorem 5.2 shows that the Eilenberg–Moore spectral sequence for $\mathrm{H}^\bullet(X)$ degenerates from its E_2 page on and that the $\mathrm{E}_2 \cong \mathrm{E}_\infty$ page agrees with (5.5).

To use our comparison results, we need to understand an E_1 page of the Eilenberg–Moore spectral sequence and its d_1 differential. Of course, as suggested in Remark 2.9, we can replace the bar resolution by the Koszul resolution $K \simeq \mathrm{H}^\bullet(Z) \otimes \Lambda(\,\omega(z_j) \mid j \in J\,)$ to compute Tor over the polynomial ring $\mathrm{H}^\bullet(Z)$ in the variables z_j. The differential of the Koszul complex is given by $\mathrm{d}(\omega(z_j)) = z_j$, and the d_1 differential of the Eilenberg–Moore spectral sequence is therefore given by $\mathrm{d}_1(\omega(z_j)) = f^\bullet(z_j)$ because $\mathrm{H}^\bullet(Z)$ acts on $\mathrm{H}^\bullet(Y)$ via $f^\bullet$. By what we found out in the first part of the proof, this is the last differential in the Eilenberg–Moore spectral sequence before it degenerates.

Since the space Z is K–minimal by assumption, our Theorem 3.5 shows that the prelude to the Eilenberg–Moore spectral sequence degenerates. Since the space Z has polynomial cohomology, Proposition 2.16 gives that Z is K–unbarred. Then, our Theorem 3.12 shows that the prelude to the Leray–Serre spectral sequence also degenerates. Consequently, our tri-graded refinement gives a comparison between the E_1 page of the Eilenberg–Moore spectral sequence and the E_2 page of the Leray–Serre spectral sequence, and the E_2 page of the Leray–Serre spectral sequence can then be described, additively, as

$$\mathrm{E}_2 \cong \mathrm{H}^\bullet(Y) \otimes \Lambda(\,\omega(z_j) \mid j \in J\,).$$

We can make the index transformation explicit. In the tri-graded refinement, the class $f^\bullet(z_j)$ is in tri-degree $(s,t,u) = (|z_j|,0,0)$ and the class $\omega(z_j)$ is in tri-degree $(0,-1,|z_j|)$. The differential $\omega(z_j) \mapsto f^\bullet(z_j)$ changes the tri-degree (s,t,u) by $(|z_j|,1,-|z_j|)$. In the Eilenberg–Moore indexing, this is a d_1, whereas in the Leray–Serre indexing, it is d_r with $r = |z_j|$. We find that the d_1 differentials of the Eilenberg–Moore spectral sequence translate into the transgression differentials in the statement. The situation is illustrated in Figure 5.1. □

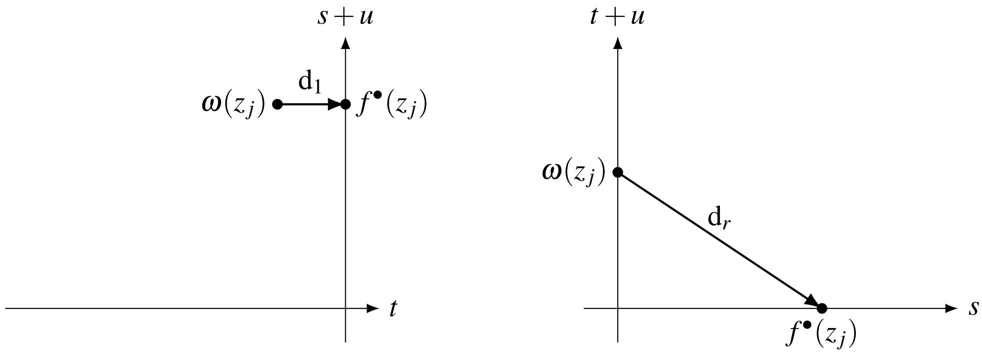

Figure 5.1: The indexing of the Eilenberg–Moore spectral sequence (left) and the Leray–Serre spectral sequence (right)

Example 5.9. The patterns in Theorems 5.2 and 5.8 are well-known from several examples. Best-known is perhaps Quillen's computation of the cohomology of the extra-special 2–groups [Qui71] (see also [BC92]). Since we have nothing new to add to that result except for exposition, we restrict to the cases of the dihedral group D_8 and the quaternion group Q_8 of order 8. Both are extensions $\mathbb{Z}/2 \to G \to V$ of the elementary abelian Klein group $V = \mathbb{Z}/2 \times \mathbb{Z}/2$ by a central $\mathbb{Z}/2$, classified by a class $f\colon \mathrm{B}V \to \mathrm{K}(\mathbb{Z}/2,2)$ in $\mathrm{H}^2(V)$. This extension class is $f^\bullet(z_2)$, where z_2 is the tautological class in the cohomology $\mathrm{H}^\bullet(\mathrm{K}(\mathbb{Z}/2,2))$. That cohomology ring is polynomial in the classes $z_2, \mathrm{Sq}^1(z_2), \mathrm{Sq}^2\,\mathrm{Sq}^1(z_2), \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1(z_2), \dots$. Quillen showed that the images of those generators that are mapped non-trivially form a regular sequence [Qui71, Cor. 2.6]. The other generators, of course, go to zero, and Theorem 5.2 implies that the Eilenberg–Moore spectral sequence degenerates from E_2 on in both cases. In the case $G = D_8$, the extension class is given by the product xy of the generators of $\mathrm{H}^\bullet(V)$, and $\mathrm{Sq}^1(xy) = x^2y + xy^2$ vanishes modulo (xy). In the case $G = Q_8$, the extension class is given by x^2+xy+y^2, and $\mathrm{Sq}^1(x^2+xy+y^2) = \mathrm{Sq}^1(xy)$ is the same as above, but it does not vanish modulo (x^2+xy+y^2). Only $\mathrm{Sq}^2\,\mathrm{Sq}^1(x^2+xy+y^2)$ vanishes modulo (x^2+xy+y^2, x^2y+xy^2). These calculations, together with the description of $\mathrm{E}_2 \cong \mathrm{E}_\infty$ in Theorem 5.2, immediately give the well-known cohomology rings of D_8 and Q_8. By the comparison in the proof of Theorem 5.8, the very same computations also show that the Leray–Serre spectral sequence for D_8 degenerates from E_3 on, and the one for Q_8 degenerates from E_5 on.

5.3 Lie groups and torus bundles

Let $G \simeq \Omega\mathrm{B}G$ be a connected Lie group. More generally, it suffices to assume only that G is a topological group such that the group $\pi_0 G$ of components is nilpotent. If P is a principal G–bundle, there is a fibration sequence

$$G \longrightarrow P \longrightarrow P/G \longrightarrow \mathrm{B}G \tag{5.6}$$

to which our results apply to compute the cohomology of P. Our Example 4.5 was an instance of that situation with $G = \mathrm{SU}(2)$. We now consider the particular situation where the connected Lie group G is compact and becomes the total space P, and the Lie group that is acting on it is a maximal torus T. Then (5.6) becomes

$$T \longrightarrow G \longrightarrow G/T \longrightarrow \mathrm{B}T. \tag{5.7}$$

Example 4.6 described the situation in the special case where $G = \mathrm{SU}(2)$. Of course, one might also consider more general principal torus bundles.

Example 5.10. We give an example of a principal T^2–bundle where the Eilenberg–Moore spectral sequence for the rational cohomology of the total space has a non-zero d_2 differential and the Leray–Serre spectral sequence has a non-trivial d_3 which does not arise from a transgression. Namely, a simply-connected 7-dimensional manifold Y can be constructed from the SU(2)–principal bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1$ whose characteristic class generates the top-dimensional cohomology $H^4(\mathbb{C}P^1 \times \mathbb{C}P^1)$, see [FOT08, Ex. 2.91]. The minimal model for Y is the differential graded algebra $S(a,b) \otimes \Lambda(u,v,t)$, with $d(a) = 0 = d(b)$, $d(u) = a^2$, $d(v) = b^2$, and $d(t) = ab$. A basis for $H^5(Y)$ is given by the Massey products $at - bu$ and $av - bt$. Ustinovskii [Ust14, Pf. of Thm. 2], pointing out a mistake in [Höf93, Prop. 5.1], studies the Leray–Serre spectral sequence for the principal T^2–bundle given by the 9-dimensional manifold $X = S^3 \times S^3 \times S^3$ over Y that is classified by the pair (a,b) of elements in $H^2(Y)$. We get a fibration sequence

$$T^2 \longrightarrow X \longrightarrow Y \longrightarrow BT^2,$$

and we would like to understand the differentials in the Eilenberg–Moore spectral sequence that computes the cohomology $H^\bullet(X)$. Let us write a' and b' for the generators of $H^2(BT^2)$ that are sent to a and b in $H^2(Y)$, respectively. A computation in the bar complex gives

$$d_2(([a'|b'] - [b'|a'])a) = \pm d_1([a']t - [b']u) = \pm[\,](at - bu),$$

which is a non-zero Massey product in $H^5(Y)$. This shows that the Eilenberg–Moore spectral sequence for the cohomology of the total space of a T^2–principal bundle can have a non-zero d_2. By comparison, the Leray–Serre spectral sequence for $H^\bullet(X)$ must have a non-trivial $d_3 \colon E_3^{2,2} \to E_3^{5,0}$ which cannot arise from a transgression. None of the assumptions for Theorem 5.2 are satisfied here.

The following result, which follows from our degeneracy criterion Theorem 5.2, generalizes all previous degeneracy results for the Eilenberg–Moore spectral sequence in situation (5.7), such as the one in [NNS99].

Theorem 5.11. *For any connected compact Lie group G with maximal torus T, the Eilenberg–Moore spectral sequence*

$$E_2^{p,q} \cong \mathrm{Tor}_{-p}^{H^\bullet(BT)}(K, H^\bullet(G/T))^q \Longrightarrow H^{p+q}(G)$$

for the principal fibration in (5.7) *always degenerates from its* E_2 *page on.*

Proof. We show that the assumptions of Theorem 5.2 are satisfied.

In general, the cohomology $H^\bullet(BT)$ of a torus is a polynomial algebra on n generators in degree 2, where $n = \dim(T) = \mathrm{rank}(G)$. Let us write $I \subseteq H^\bullet(BT)$ for the kernel of the induced homomorphism $H^\bullet(BT) \to H^\bullet(G/T)$. Kac showed [Kac86, Thm. 1] that the ideal I is generated by a regular sequence $h_1, h_2, \dots, h_n$ of maximal length n, which gives (5.3).

It follows that the classes $h_1, h_2, \dots, h_n$ generate a polynomial subalgebra [Smi95, Prop. 6.2.1], and that $H^\bullet(BT)$ is a free module over it [Smi95, Cor. 6.2.8]. The ideal I is called the *ideal of generalized invariants* (see [Kac86] again). Let R denote the image of $H^\bullet(BT)$ inside $H^\bullet(G/T)$. The algebra R is generated by $H^2(G/T)$ as the map $G/T \to BT$ is 2–connected. It turns out that $H^\bullet(G/T)$ is a free R–module [Kac86, (2)], and this gives (5.4). □

Theorem 5.12. *For any connected compact Lie group G with maximal torus T, the Leray–Serre spectral sequence*

$$E_2^{p,q} \cong H^p(G/T) \otimes_K H^q(T) \Longrightarrow H^{p+q}(G)$$

for the principal fibration in (5.7) *degenerates from its* E_3 *page on.*

Proof. The cohomology of the maximal torus T is an exterior algebra $\Lambda(t_1, \dots, t_n)$ over our field K, where $n = \dim(T) = \mathrm{rank}(G)$ and $\deg(t_i) = 1$ for $i = 1, 2, \dots, n$. We obtain for the E_2 page of the spectral sequence therefore

$$E_2^{p,q} \cong H^p(G/T) \otimes_K \Lambda(t_1, t_2, \dots, t_n)^q.$$

The differential $\mathrm{d}_2\colon \mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$ is a bigraded derivation. It is, therefore, completely determined by its restrictions to the generators t_i and to the elements on the $q = 0$ row. Of course, by degree considerations, the differentials vanish in the 0–th row. On the other hand, the differential $\mathrm{d}_2\colon \mathrm{E}_2^{0,1} = \mathrm{H}^1(T) \to \mathrm{H}^2(G/T) \cong \mathrm{E}_2^{2,0}$ is the Borel transgression τ. This gives the following description for the differential

$$\mathrm{d}_2\colon \mathrm{E}_2^{p,q} \longrightarrow \mathrm{E}_2^{p+2,q-1}, \mathrm{d}_2(x \otimes t) = \pm(x \cdot \tau(t)) \otimes 1,$$

for all $x \in \mathrm{H}^\bullet(G/T)$ and $t \in \mathrm{H}^1(T)$. The E_2 page of the spectral sequence is now the Koszul resolution for the field K over the polynomial algebra $\mathrm{H}^\bullet(\mathrm{B}T)$ (see [Smi95, Sec. 6.2]). The E_3 page of the spectral sequence, therefore, is given as Tor, and we have

$$\mathrm{E}_3^{p,q} \cong \mathrm{Tor}_{-p}^{\mathrm{H}^\bullet(\mathrm{B}T)}(K, \mathrm{H}^\bullet(G/T))^q.$$

From the proof of Theorem 5.11, we know that $\mathrm{H}^\bullet(\mathrm{B}T)$ is a free module over the polynomial subalgebra generated by a regular sequence $h_1, h_2, \ldots, h_n$ for the ideal I, and the cohomology $\mathrm{H}^\bullet(G/T)$ of the flag manifold is a free R–module.

We note that we have

$$\mathrm{Tor}_0^{\mathrm{H}^\bullet(\mathrm{B}T)}(K, \mathrm{H}^\bullet(G/T)) \cong K \otimes_R \mathrm{H}^\bullet(G/T)$$

and, from [Ser89, Chap. IV, Prop. 2], we get:

$$\mathrm{Tor}^{\mathrm{H}^\bullet(\mathrm{B}T)}(K, R) \cong \Lambda(z_1, z_2, \ldots, z_n),$$

where the $z_1, z_2, \ldots, z_n$ are generators of total degree $\deg(z_i) = \deg(h_i) - 1$ for $i = 1, 2, \ldots, n$. We obtain the E_3 page of the spectral sequence:

$$\mathrm{E}_3^{\bullet,\bullet} \cong K \otimes_R \mathrm{H}^\bullet(G/T) \otimes_K \Lambda(z_1, z_2, \ldots, z_n).$$

We find that the E_3 page is generated as an algebra by classes on the zeroth and first rows. Hence, the differential d_3 vanishes on the generators. Similarly, all the higher differentials d_r for $r \geqslant 4$ do so, too. □

Remark 5.13. The preceding result, Theorem 5.12, would follow immediately from our Theorem 5.8 if the classes $h_1, h_2, \ldots, h_n$ in the proof were polynomial generators for $\mathrm{H}^\bullet(\mathrm{B}T)$, but that happens only in trivial cases. Therefore, we gave a direct proof.

Remark 5.14. Leray [Ler50] showed the degeneration of the Leray–Serre spectral sequence for the cohomology $\mathrm{H}^\bullet(G)$ of a connected compact Lie group G over the rationals $\mathbb{Q}$. Kac [Kac86] gave an argument for the degeneration in the case of finite fields $\mathbb{F}_p$, but his formula for the differential d_2 did not take into account the Borel transgression τ. Duan [Dua18] repaired this and gave an explicit description for τ to calculate the integral cohomology rings of compact Lie groups and Schubert varieties.

Remark 5.15. The arguments above for the degeneration of the Eilenberg–Moore and the Leray–Serre spectral sequences for compact Lie groups can also be adopted for the more general setting of connected finite loop spaces with maximal tori as well as for p–compact groups [DW94, Neu00].

Remark 5.16. The degeneration of the Eilenberg–Moore and the Leray–Serre spectral sequence over a field can also be obtained for Kac–Moody groups [Kac85, Kum02, Kit14]. Given a Kac–Moody group $\mathscr{G}$ over $\mathbb{C}$ associated to a generalized Cartan matrix, there exists a unitary form $\mathscr{K}$, given as the fixed point group under the action of a canonical anti-linear involution on $\mathscr{G}$. The inclusion of $\mathscr{K}$ into $\mathscr{G}$ is a homotopy equivalence. Given a maximal torus T for $\mathscr{K}$, we can study the associated sequence of fibrations

$$T \longrightarrow \mathscr{K} \longrightarrow \mathscr{K}/T \longrightarrow \mathrm{B}T.$$

Again the assumptions (5.3) and (5.4) are satisfied [Kac85, Kum02, Kit14] and our Theorem 5.2 implies that the Eilenberg–Moore spectral sequence for $\mathrm{H}^\bullet(\mathscr{K})$ degenerates from the E_2 page on, which was also shown directly by Kitchloo [Kit14, Thm 6.8]. It follows then with similar arguments as in the proof of Theorem 5.12 that the Leray–Serre spectral sequence for $\mathrm{H}^\bullet(\mathscr{K})$ degenerates from its E_3 page on (compare also [Kum85] and [Kum02, Ch. XI]).

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