The maps obtained in these situations have much in common with the Lorenz maps discussed earlier with $\delta < 1$ as the derivative is unbounded as x tends to zero from above, but they are also continuous unimodal maps and so general theorems such as Sharkovskii's theorem (which describes how the existence of a periodic orbit of a given period implies the existence of other periodic orbits in continuous maps) hold (see LOGISTIC EQUATION [III.19]). When applied to the special case of unimodal maps this restricts the order in which periodic orbits are created. However, while the logistic map has many windows of stable periodic orbits, each with its own period-doubling cascade and associated chaotic motion, the *stable* periodic orbits of the square root map (4) are much more constrained. For example, if $0 < a < \frac{1}{4}$, the stable periodic orbits form a period-adding sequence as μ decreases through zero. There is a parameter interval on which a period-n orbit is the only attractor, followed by a bifurcation creating a stable period-$(n+1)$ orbit so that the two stable orbits of period n and period $(n+1)$ coexist. Then there is a further bifurcation at which the period-n orbit loses stability and the stable period-$(n+1)$ orbit is the only attractor. This sequence of behavior is repeated with n tending to infinity as μ tends to zero.

8 Afterview

Bifurcation theory provides insights into why certain types of dynamics occur and how they arise. In cases such as period-doubling cascades it provides a framework in which to understand the changes in complexity of dynamics even if the behavior at a single parameter value might appear nonrepeatable. The number of different cases that may need to be considered can proliferate, and there is currently a nonuniformity of nomenclature that means that it is hard to tell whether a particular case has been studied previously in the literature. Given that the techniques are useful in any discipline that uses dynamic modeling, this aspect is unfortunate and leads to many reinventions of the same result. However, this only underlines the central role played by bifurcation theory in understanding the dynamics of mathematical models wherever they occur.

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IV.22 Symmetry in Applied Mathematics

Ian Stewart

We tend to think of the applied mathematical toolkit as a collection of specific techniques for precise calculations, each intended for a particular kind of problem, solving a system of algebraic or differential equations numerically, for example. But some of the most powerful ideas in mathematics are so broad that at first sight they seem too vague and nebulous to have practical implications. Among them are probability, continuity, and symmetry.

Probability started out as a way to capture uncertainty in gambling games, but it quickly developed into a vitally important collection of mathematical techniques used throughout applied science, economics, sociology—even for formulating government policy.

Continuity proved such an elusive concept that it was used intuitively for centuries before it could be defined rigorously; now it underpins calculus, perhaps the most widely used mathematical tool of them all, especially in the form of ordinary and partial differential equations. But continuity is also fundamental to topology, a relative newcomer from pure mathematics that is starting to demonstrate its worth in the design of efficient trajectories for spacecraft, improved methods for forecasting the weather, frontier investigations in quantum mechanics, and the structure of biologically important molecules, especially deoxyribonucleic acid (DNA).

Symmetry, the topic of this article, was initially a rather ill-defined feeling that certain parts of shapes or structures were much the same as other parts of those shapes or structures. It has since become a vital method for understanding pattern formation throughout the scientific world, with applications that range from architecture to zoology. Symmetry, it turns out, underlies many of the deepest aspects of the natural world. Our universe behaves the way it does because of its symmetries—of space, time, and matter. Both relativity and particle physics are based on symmetry principles. Symmetry methods shed light on difficult problems by revealing general principles that can help us find solutions. Symmetry can be static, dynamic, even

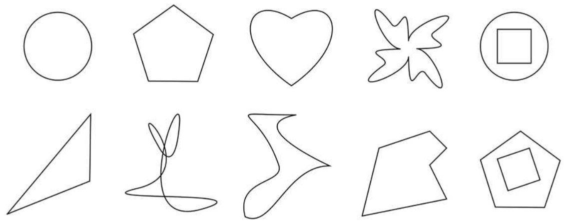

Figure 1 Which shapes are symmetric?

chaotic. It is a concept of great generality and deep abstraction, where beauty and power go hand in hand.

1 What Is Symmetry?

Symmetry is most easily understood in a geometric setting. Figure 1 shows a variety of plane figures, some symmetric, some not. Which are which?

The short answer is that the top row consists of symmetric shapes and the bottom row consists of asymmetric shapes. But there is more. Not only can shapes be symmetric, or not—they can have different *kinds* of symmetry. The heart (the middle of the top row) has the most familiar symmetry, one we encounter every time we look at ourselves in a mirror: *bilateral* symmetry. The left-hand side of the figure is an exact copy of the right-hand side, but flipped over. Three other shapes in the figure are bilaterally symmetric: the circle, the pentagon, and the circle with a square hole. The pentagon with a tilted square hole is not; if you flip it left–right, the pentagonal outline does not change but the square hole does because of the way it is tilted.

However, the circle has more than just bilateral symmetry. It would look the same if it were reflected in any mirror that runs through its center. The pentagon would look the same if it were reflected in any mirror that runs through its center and passes through a vertex.

What about the fourth shape in the top row, the flowery thing? If you reflect it in a mirror, it looks different, no matter where the mirror is placed. However, if you rotate it through a right angle about its center, it looks exactly the same as it did to start with. So this shape has *rotational* symmetry for a right-angle rotation. Thus primed, we notice that the pentagon also has rotational symmetry, for a rotation of 72°, and the circle has rotational symmetry for any angle whatsoever.

Most of the shapes on the bottom row look completely asymmetric. No significant part of any of them looks much like some other part of the same shape. The possible exception is the pentagon with a square hole. The pentagon is symmetric, as we have just seen, and so is the square. Surely combining symmetric shapes should lead to a symmetric shape? On the other hand, it does look a bit lopsided, which is not what we would expect from symmetry. With the current definition of symmetry, this shape is asymmetric, even though some pieces of it are symmetric.

In the middle of the nineteenth century mathematicians finally managed to define symmetry, for geometric shapes, by abstracting the common idea that unifies all of the above discussion. The background involves the idea of a *transformation* (or function or map). Some of the most common transformations, in this context, are rigid motions of the plane. These are rules for moving the entire plane so that the distances between points do not change. There are three basic types.

Translations. Slide the plane in a fixed direction so that every point moves the same distance.

Rotations. Choose a point, keep this fixed, and spin the entire plane around it through some angle.

Reflections. Choose a line, think of it as a mirror, and reflect every point in it.

These transformations do not exhaust the rigid motions of the plane, but every rigid motion can be obtained by combining them. One of the new transformations produced in this way is the *glide reflection*; reflect the plane in a line and then translate it along the direction of that line.

Having defined rigid motions, we can now define what a symmetry is. Given some shape in the plane, a symmetry of that shape is a rigid motion of the plane that leaves the shape *as a whole* unchanged. Individual points in the shape may move, but the end result looks exactly the same as it did to start with—not just in terms of shape, but also location.

For example, if we reflect the pentagon about a line through its center and a vertex, then points not on the line flip over to the other side. But they swap places in pairs, each landing where the other one started from, so the final position of the pentagon fits precisely on top of the initial position. This is false for the pentagon with a square hole, and this is the reason why that shape is not considered to possess symmetry.

The shapes in figure 1 are all of finite size, so they cannot possess translational symmetry. Translational symmetries require infinite patterns. A typical example is a square tiling of the entire plane, like bathroom tiles but continued indefinitely. If this pattern is translated sideways by the width of one tile, it remains unchanged. The same is true if it is translated upward by the width of one tile. It follows that, if the tiling is translated an integer number of widths in either of these two directions, it again remains unchanged. The symmetry group of translations is a *lattice*, consisting of integer combinations of two basic translations. The square tiling also has rotational symmetries, through multiples of 90° about the center of a tile or a corner where four tiles meet. Another type of rotational symmetry is rotation through 180° about the center of the edge of a tile. The square tiling pattern also has various reflectional symmetries.

Lattices in the plane can be interpreted as wallpaper patterns. In 1891 the pioneer of mathematical crystallography Yevgraf Fyodorov proved that there are exactly 17 different symmetry classes of wallpaper patterns. George Pólya obtained the same result independently in 1924. Lattice symmetries, often in combination with rotations and reflections, are of crucial importance in crystallography, but now the "tiling" is the regular atomic structure of the crystal, and it repeats in three-dimensional space along integer combinations of three independent directions. Again there may also be rotations and reflections. The physics of a crystal is strongly influenced by the symmetries of its atomic lattice. In the 1890s Fyodorov, Arthur Shönflies, and William Barlow proved that there are 230 symmetry types of lattice, or 219 if certain mirror-image pairs are considered to be the same.

Of course, no physical crystal can be of infinite extent. However, the idealization to infinite lattices can be a very accurate model for a real crystal because the size of the crystal is typically much larger than the lattice spacing. Real crystals differ from this ideal model in many ways: dislocations, where the lattice fails to repeat exactly; grain boundaries, where local lattices pointing in different directions meet; and so on. All applied mathematics involves a modeling step, representing the physical system by a simplified and idealized mathematical model. What matters is the extent to which the model provides useful insights. Its failure to include certain features of reality is not, of itself, a valid criticism. In fact, such a failure can be a virtue if it makes the analysis simpler without losing anything important.

Symmetries need not be rigid motions. Another geometric symmetry is dilation—change of scale. A logarithmic spiral, found in nature as a nautilus shell, remains unchanged if it is dilated by some fixed amount and also rotated through an appropriate angle.

Symmetry does not apply only to shapes; it is equally evident in mathematical formulas. For example, $x + y + z$ treats the three variables x, y, and z in exactly the same manner. But the expression $x^3 + y - 2z^2$ does not. The first formula is symmetric in x, y, and z; the second is not. This time the transformations concerned are *permutations* of the three symbols—ways to swap them around. However we permute them, $x + y + z$ stays the same. For instance, if we swap x and y but leave z the same, the expression becomes $y + x + z$. By the laws of algebra, this equals $x + y + z$. But the same permutation applied to the other expression yields $y^3 + x - 2z^2$, which is clearly different.

Symmetry thereby becomes a very general concept. Given any mathematical structure, and some class of transformations that can act on the structure, we define a symmetry to be any transformation that preserves the structure—that is, leaves it unchanged.

If a physical system has symmetry, then most sensible mathematical models of that system will have corresponding symmetries. (I say "most" because, for example, numerical methods cannot always incorporate all symmetries exactly. No computer model of a circle can be unchanged by all rotations. This inability of numerical methods to capture all symmetries can sometimes cause trouble.) The precise formulation of symmetry for a given model depends on the kind of model being used and its relation to reality.

2 Symmetry Groups

The above definition tells us that a symmetry of some structure (shape, equation, process) is not a thing but a transformation. However, it also tells us something deeper: structures may have several different symmetries. Indeed, some structures, such as the circle, have infinitely many symmetries. So there is a shift of emphasis from symmetry to symmetries, not symmetry as an abstract property of a structure but the *set of all symmetries* of the structure.

This set of transformations has a simple but vital feature. Transformations can be combined by performing them in turn. If two symmetries of some structure are combined in this way, the result is always a symmetry of that structure. It is not hard to see why: if you do not change something, and then you *again* do not change it . . . you do not change it. This feature is known as the "group property," and the set of all symmetry transformations, together with this operation of composition, is the *symmetry group* of the structure.

The shapes in figure 1 illustrate several common kinds of symmetry group. The rotations of the circle form the *circle group* or *special orthogonal group in the plane*, denoted by S^1 or $SO(2)$. The reflections and rotations of the circle form the *orthogonal group in the plane*, denoted by $O(2)$. The rotations of the pentagon form the *cyclic group* Z_5 of order 5 (the order of a group is the number of transformations that it contains). The rotations of the flower shape form the cyclic group Z_4 of order 4. There is an analogous cyclic group Z_n whose order is any positive integer n; it can be defined as the group of rotational symmetries of a regular n-sided polygon. If we also include the five reflectional symmetries of the pentagon, we obtain the *dihedral group* D_5, which has order 10. There is an analogous dihedral group D_n of order $2n$ for any positive integer n: the rotations and reflections of a regular n-sided polygon. Finally, we mention the *symmetric group* S_n of all permutations of a set with n elements, such as $\{1, 2, 3, \dots, n\}$. This has order $n!$, that is, $n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$.

Even asymmetric shapes have some symmetry; the identity transformation "leave everything as it is." The symmetry group contains only this trivial but useful transformation, and it is symbolized by **1**.

We mention one useful piece of terminology. Often one group sits inside a bigger one. For example, $SO(2)$ is contained in $O(2)$, and Z_n is contained in D_n. In such cases, we say that the smaller group is a *subgroup* of the bigger one. (They may also be equal, a trivial but sensible convention.)

The study of groups has led to a huge area of mathematics known as *group theory*. Some of it is part of abstract algebra, especially when the group is finite—that is, contains a finite number of transformations. Examples are the cyclic, dihedral, and symmetric groups. Another area, involving analysis and topology, is the theory of *Lie groups*, such as the circle group, the orthogonal group, and their analogues in spaces of any dimension. Here the main emphasis is on continuous families of symmetry transformations, which correspond to all choices of some real number. For example, a circle can be rotated through any real angle. Yet another important area is *representation theory*, which studies all the possible ways to construct a given group using matrices, linear transformations of some vector space.

One of the early triumphs of group theory in applied mathematics was Noether's theorem, proved by Emmy Noether in 1918. This applies to a special type of differential equation known as a Hamiltonian system, which arises in models of classical mechanics in the absence of frictional forces. Celestial mechanics—the motion of the planets—is a significant example. The theorem states that whenever a Hamiltonian system has a continuous family of symmetries, there is an associated conserved quantity. "Conserved" means that this quantity remains unchanged as the system moves.

The laws of nature are the same at all times: if you translate time from t to $t + \theta$, the laws do not look any different. These transformations form a continuous family of symmetries, and the corresponding conserved quantity is energy. Translational symmetry in space (the laws are the same at every location) corresponds to conservation of momentum. Rotations about some axis in three-dimensional space provide another continuous family of symmetries; here the conserved quantity is angular momentum about that axis. All of the conservation laws of classical mechanics are consequences of symmetry.

3 Pattern Formation

Symmetry methods come into their own, and nowadays are almost mandatory, in problems about pattern formation. Often the most striking feature of some natural or experimental system is the occurrence of patterns. Rainbows are colored circular arcs of light. Ripples caused by a stone thrown into a pond are expanding circles. Sand dunes, ocean waves, and the stripes on a tiger or an angelfish are all patterns that can be modeled using repeating parallel features. Crystal lattices are repeating patterns of atoms. Galaxies form vast spirals, which rotate without (significantly) changing shape—a group of symmetries combining time translation with spatial rotation.

Many of these patterns arise through a general mechanism called "symmetry breaking." This is applicable whenever the equations that model a physical system have symmetry. I say "equations" here, even though I have already insisted that the symmetries of the system should appear in the equations, because it is not unusual for the model equations to have *more* symmetry than the pattern under consideration. Our theories of symmetry breaking and pattern formation rest on the structure of the symmetry group and its implications for mathematical models of symmetric systems.

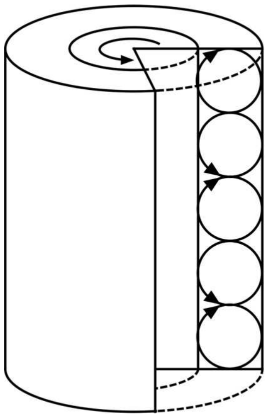

Figure 2 Taylor–Couette apparatus, showing flow pattern for Taylor vortices.

A classic example is Taylor–Couette flow, in which fluid is confined between two rotating cylinders (figure 2). In experiments this system exhibits a bewildering variety of patterns, depending on the angular velocities of the cylinders and the relative size of the gap between them. The experiment is named after Maurice Couette, who used a fixed outer cylinder and a rotating inner one to measure the viscosity of fluids. At the low velocities he employed, the flow is featureless, as in figure 3(a). In 1923 Geoffrey Ingram Taylor noted that when the angular velocity of the inner cylinder exceeds a critical threshold, the uniform pattern of Couette flow becomes unstable and instead a stack of vortices appears; see figures 2 and 3(b). The vortices spiral round the cylinder, and alternate vortices spin the opposite way in cross section (small circles with arrows). Taylor calculated this critical velocity and used it to test the NAVIER–STOKES EQUATIONS [III.23] for fluid flow.

Further experimental and theoretical work followed, and the apparatus was modified to allow the outer cylinder to rotate as well. This can make a difference because, in a rotating frame of reference, the fluid is subject to additional centrifugal forces. In these more general experiments, many other patterns were observed. Figure 3 shows a selection of them.

The most obvious symmetries of the Taylor–Couette system are rotations about the common axis of the cylinders. These preserve the structure of the apparatus. But notice that not all patterns have full rotational symmetry. In figure 3 only the first two—Couette flow and Taylor vortices—are symmetric under all rotations. Another family of symmetries arises if the system is

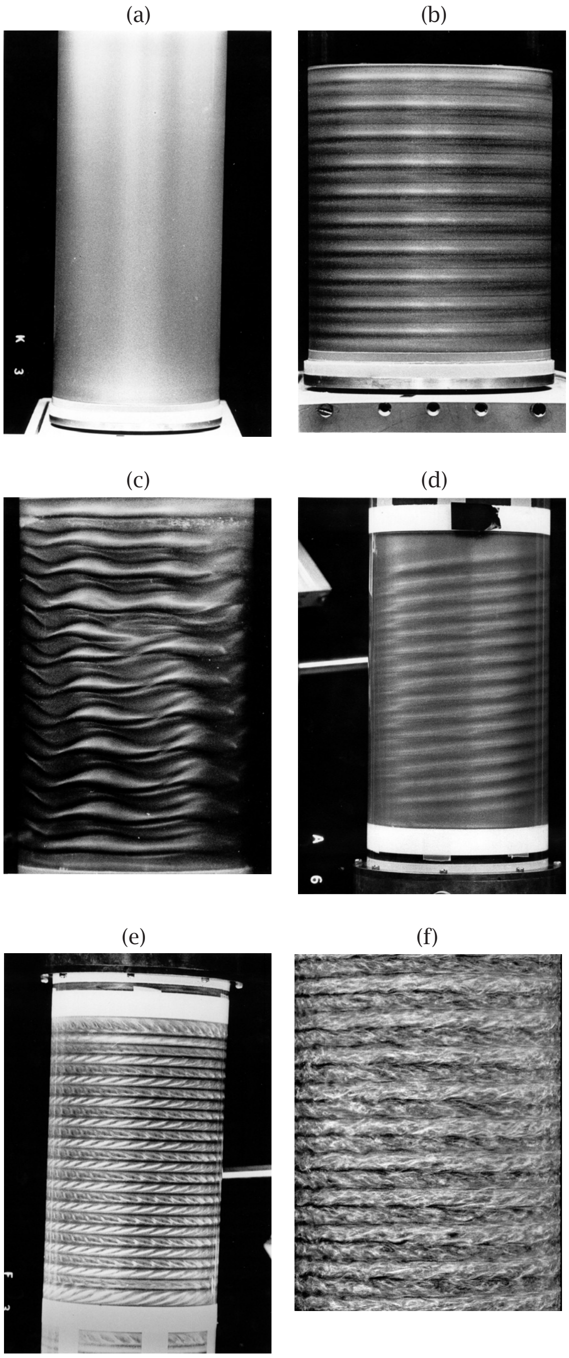

Figure 3 Some of the numerous flow patterns in the Taylor–Couette system: (a) Couette flow; (b) Taylor vortices; (c) wavy vortices; (d) spiral vortices; (e) twisted vortices; and (f) turbulent vortices. Source: Andereck et al. (1986).

modeled (as is common for some purposes) by two infinitely long cylinders but restricted to patterns that repeat periodically along their lengths. In effect, this wraps the top of the cylinder round and identifies it with the bottom. Mathematically, this trick employs a modeling assumption: "periodic boundary conditions" that require the flow near the top to join smoothly to the flow near the bottom. With periodic boundary conditions, the equations are symmetric under all vertical translations. But again, not all patterns have full translational symmetry. In fact, the only one that does is Couette flow. So all patterns except Couette flow break at least some of the symmetries of the system.

On the other hand, most of the patterns retain some of the symmetries of the system. Taylor vortices are unchanged by vertical translations through distances equal to the width of a vortex pair (see figure 2). The same is true of wavy vortices, spirals, and twisted vortices. We will see in section 7 how each pattern in the figure can be characterized by its symmetry group.

There are at least two different ways to try to understand pattern formation in the Taylor–Couette system. One is to solve the Navier–Stokes equations numerically. The computations are difficult and become infeasible for more complex patterns. They also provide little insight into the patterns beyond showing that they are (rather mysterious) consequences of the Navier–Stokes equations. The other is to seek theoretical understanding, and here the symmetry of the apparatus is of vital importance, explaining most of the observed patterns.

4 Symmetry of Equations

To understand the patterns that arise in the Taylor–Couette system, we begin with a simpler example and abstract its general features. We then explain what these features imply for the dynamics of the system. In a later section we return to the Taylor–Couette system and show how the general theory of dynamics with symmetry classifies the patterns and shows how they arise. This theory is based on a mathematical consequence of the symmetry of the system being modeled; the differential equations used to set up the model have the same symmetries as the system. A symmetry of an equation is defined to be a transformation of the variables that sends solutions of the equation to (usually different) solutions.

The appropriate formulation of symmetry for an ordinary differential equation is called equivariance.

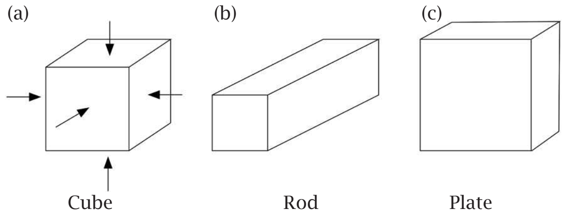

Figure 4 (a) Cube under compression (force at back not shown). (b) Rod solution. (c) Plate solution.

Suppose that Γ is a group of symmetries acting on the variables $x = (x_1, \dots, x_m)$ in the equation

$$\frac{dx}{dt} = F(x).$$

Then F is *equivariant* if

$$F(\gamma x) = \gamma F(x) \quad \text{for all symmetries } \gamma \text{ in } \Gamma.$$

It follows immediately that, if $x(t)$ is any solution of the system, so is $\gamma x(t)$ for any symmetry γ. In fact, this condition is logically equivalent to equivariance. In simple terms, solutions always occur in symmetrically related sets. We will see an example of this phenomenon below.

A central concept in symmetric dynamics is BIFURCATION [IV.21], for which the context is a family of differential equations—an equation that contains one or more parameters. These are variables that are assumed to remain constant when solving the equation but can take arbitrary values. In a problem about the motion of a planet, for example, the mass of the planet may appear as a parameter. Such a family undergoes a *bifurcation* at certain parameter values if the solutions change in a qualitative manner near those values. For example, the number of equilibria might change, or an equilibrium might become a time-periodic oscillation. Bifurcations have a stronger effect than, say, moving an equilibrium continuously or slightly changing the shape and period of an oscillation. They provide a technique for proving the existence of interesting solutions by working out when simpler solutions become unstable and what happens when they do.

To show how symmetries behave at a bifurcation point, we consider a simple model of the deformation of an elastic cube when it is compressed by six equal forces acting at right angles to its faces (figure 4(a)). We consider deformations into any cuboid shape with sides (a, b, c). When the forces are zero, the shape of the body is a cube, with $a = b = c$. The symmetry group consists of the permutations of the coordinate

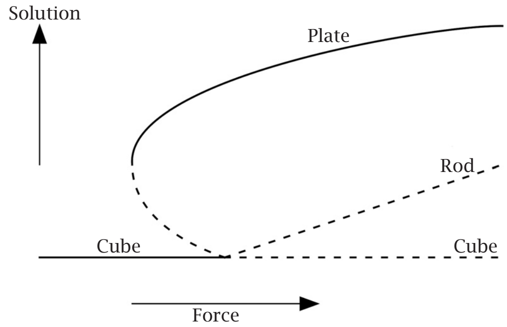

Figure 5 Bifurcation diagram (solid lines, stable; dashed lines, unstable).

axes, changing (a, b, c) to (a, c, b), (b, c, a), and so on. Physically, these rotate or reflect the shape, and they constitute the symmetric group S_3.

As the forces increase, always remaining equal, the cube becomes smaller. Initially, the sides remain equal, but analysis of a simplified but reasonable model shows that, when the forces become sufficiently large, the fully symmetric "cube" state becomes unstable. Two alternative shapes then arise: a *rod* shape (a, b, c) with two sides equal and smaller than the third, and a *plate* shape (a, b, c) with two sides equal and larger than the third (figure 4). The *bifurcation diagram* in figure 5 is a schematic plot of the shape of the deformed cube, plotted vertically, against the forces, plotted horizontally. The diagram shows how the existence and stability of the deformed states relate to the force. In this model only the plate solutions are stable.

In principle there might be a third shape, in which all three sides are of different lengths; however, such a solution does not occur (even unstably) in the model concerned.

Earlier, I remarked that solutions of symmetric equations occur in symmetrically related sets. Here, rod solutions occur in three symmetrically related forms, with the longer side of the rod pointing in any of the three coordinate directions. Algebraically, these solutions satisfy the symmetrically related conditions $a = b < c$, $a = c < b$, and $b = c < a$. The same goes for plate solutions.

5 Symmetry Breaking

The symmetry group for the buckling cube model contains six transformations:

$$I\colon (a, b, c) \mapsto (a, b, c),$$
$$X\colon (a, b, c) \mapsto (a, c, b),$$
$$Y\colon (a, b, c) \mapsto (c, b, a),$$
$$Z\colon (a, b, c) \mapsto (b, a, c),$$
$$R\colon (a, b, c) \mapsto (b, c, a),$$
$$S\colon (a, b, c) \mapsto (c, a, b).$$

We can consider the symmetries of the possible states, that is, the transformations that leave the shape of the buckled cube unchanged. Rods and plates have square cross section, and the two lengths in those directions are equal. If we interchange those two axes, the shape remains the same. Only the cube state has all six symmetries S_3. The rod with $a = b < c$ is symmetric under the permutations that leave z fixed, namely $\{I, Z\}$. The rod with $a = c < b$ is symmetric under the permutations that leave y fixed, namely $\{I, Y\}$. The rod with $b = c < a$ is symmetric under the permutations that leave x fixed, namely $\{I, X\}$. The same holds for the plates. If a solution existed in which all three sides had different lengths, its symmetry group would consist of just $\{I\}$. All of these groups are subgroups of S_3.

Notice that the subgroups $\{I, X\}$, $\{I, Y\}$, $\{I, Z\}$ are themselves related by symmetry. For example, a solution (a, b, c) with $a = b < c$ becomes (b, c, a) with $b = c < a$ when the coordinate axes are permuted using R. In the terminology of group theory, these three subgroups are *conjugate* in the symmetry group.

The buckling cube is typical of many symmetric systems. Solutions need not have all of the symmetries of the system itself. Instead, some solutions may have smaller groups of symmetries—subgroups of the full symmetry group. Such solutions are said to *break* symmetry. For an equivariant system of ordinary differential equations, fully symmetric solutions may exist, but these may be unstable. If they are unstable, the system will find a stable solution (if it can), and this typically breaks the symmetry.

In general, the symmetry group of a solution is a subgroup of the symmetry group of the differential equation. Some subgroups may not occur here. Those that do are known as *isotropy subgroups*. For the buckling cube, the group S_3 has the five isotropy subgroups listed above. One subgroup is missing from that list, namely $\{I, R, S\}$. This is not an isotropy subgroup because a shape with these symmetries must satisfy the condition $(a, b, c) = (b, c, a)$, which forces $a = b = c$. This shape is the cube, which has additional symmetries X, Y, and Z. This situation is typical of subgroups that are not isotropy subgroups; the symmetries of such a subgroup force additional symmetries that are not in the subgroup.

A first step toward classifying the possible symmetry-breaking solutions of a differential equation with a known symmetry group is to use group theory to list the isotropy subgroups. Techniques then exist to find solutions with a given isotropy subgroup. Only one isotropy subgroup from each conjugacy class need be considered because solutions come in symmetrically related sets.

There are general theorems that guarantee the existence of solutions with certain types of isotropy subgroup, but they are too technical to state here. Roughly speaking, symmetry-breaking solutions often occur for large isotropy subgroups (but ones smaller than the full symmetry group). The theorems make this statement precise. In particular, they explain why rod and plate solutions occur for the buckling cube. Other general theorems help to determine whether a solution is stable or unstable.

6 Time-Periodic Solutions

A famous example of pattern formation is the Belousov–Zhabotinskii (or BZ) chemical reaction. This involves three chemicals together with an indicator that changes color from red to blue depending on whether the reaction is oxidizing or reducing. The chemicals are mixed together and placed in a shallow circular dish. They turn blue and then red. For a few minutes nothing seems to be happening; then tiny blue spots appear, which expand and turn into rings. As each ring grows, a new blue dot appears at its center. Soon the dish contains several expanding "target patterns" of rings. Unlike water waves, the patterns do not overlap and superpose. Instead, they meet to form angular junctions (figure 6(a)). Different target patterns expand at different rates, giving rings with differing thicknesses. However, each set of rings has a specific uniform speed of expansion, and the rings in that set all have the same width.

Another pattern can be created by breaking up a ring—by dragging a paperclip across it, for example. This new pattern curls up into a spiral (close to an Archimedean spiral with equally spaced turns), and the spiral slowly rotates about its center, winding more and more turns as it does so.

Neither pattern is an equilibrium, so the theorems alluded to above are not applicable. Instead, we can employ a different but related series of techniques. These apply to *time-periodic* solutions, which repeat their form at fixed intervals of time.

(a) (b)

Figure 6 Patterns in the BZ reaction: (a) snapshot of several coexisting target patterns; (b) snapshot of two coexisting spirals.

There are two main types of bifurcation in which the fully symmetric steady state loses stability as a parameter is varied, with new solutions appearing nearby. One is a steady-state bifurcation (which we met above), for which these new solutions are equilibria. The other is HOPF BIFURCATION [IV.21 §2], for which the new solutions are time-periodic oscillations. Hopf bifurcation can occur for equations without symmetry, but there is a generalization to symmetric systems. The main new ingredient is that symmetries can now occur not just in space (the shape of the pattern) and time (integer multiples of the period make no difference), but in a combination of both.

A single target pattern in the BZ reaction has a purely spatial symmetry: at any instant in time it is unchanged under all rotations about its center. It also has a purely temporal symmetry: in an ideal version where the pattern fills the whole plane, it looks identical after a time that is any integer multiple of the period.

A single spiral, occupying the entire plane, has no nontrivial spatial symmetry, but it has the same purely temporal symmetry as a target pattern. However, it has a further *spatiotemporal symmetry* that combines both. As time passes, the spiral slowly rotates without changing form. That is, an arbitrary translation of time, combined with a rotation through an appropriate angle, leaves the spiral pattern (and how it develops over time) unchanged.

For symmetric equations there is a version of the Hopf bifurcation theorem that applies to spatiotemporal symmetries when there is a suitable symmetry-breaking bifurcation. Again, its statement is too technical to give here, but it helps to explain the BZ patterns. In conjunction with the theory for steady-state bifurcation, it can be used to understand pattern formation in many physical systems.

7 Taylor–Couette Revisited

The Taylor–Couette system is a good example of the symmetry-breaking approach. A standard model, derived from the Navier–Stokes equations for fluid flow, involves three types of symmetry.

Spatial: rotation about the common axis of the cylinders through any angle.
Temporal: shift time by an integer multiple of the period.
Model: translation along the common axis of the cylinders through any distance (a consequence of the assumption of periodic boundary conditions).

Using bifurcation theory and a few properties of the Navier–Stokes equations, the dynamics of this model for the Taylor–Couette system can be reduced to an ordinary differential equation (the so-called center manifold reduction) in six variables. Two of these variables correspond to a steady-state bifurcation from Couette flow to Taylor vortices. This is steady in the sense that the fluid *velocity* at any point remains constant. The other four variables correspond to a Hopf bifurcation from Couette flow to spirals. These are the two basic "modes" of the system, and their combination is called a mode interaction.

Symmetric bifurcation theory's general theorems now prove the existence of numerous flow patterns, each with a specific isotropy subgroup. For example, if the rotational symmetry remains unbroken, but the group of translational symmetries breaks, a typical solution will have a specific translational symmetry, through some fraction of the length of the cylinder. All integer multiples of this translation are also symmetries of that solution. This combination of symmetries corresponds precisely to Taylor vortices; it leads to a discrete, repetitive pattern vertically, with no change in the horizontal direction.

A further breaking of the rotational symmetry produces a discrete set of rotational symmetries; these characterize wavy vortices. Symmetry under any rotation, if combined with a vertical translation through a corresponding distance, characterizes spiral vortices, and so on.

The only pattern in figure 3 that is not explained in this manner is the final one: turbulent vortices. This pattern turns out to be an example of symmetric chaos, chaotic dynamics that possesses "symmetry on average." At each instant the turbulent vortex state has no symmetry. But if the fluid velocity at each point is averaged over time, the result has the same symmetry as Taylor vortices. This is why the picture looks like Taylor vortices with random disturbances. A general theory of symmetric chaos also exists.

8 Conclusion

The symmetries of physical systems appear in the equations that model them. The symmetry affects the solutions of the equations, but it also provides systematic ways to solve them. One general area of application is pattern formation, and here the methods of symmetric bifurcation theory have been widely used. Topics include animal locomotion, speciation, hallucination patterns, the balance-sensing abilities of the inner ear, astrophysics, liquid crystals, fluid flow, coupled oscillators, elastic buckling, and convection.

In addition to a large number of specific applications, there are many other ways to exploit symmetry in applied mathematics. This article has barely scratched the surface.

Further Reading

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