Submaximally Symmetric Quaternionic Structures

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Almost Quaternionic Structures

An almost quaternionic structure (AQS) on a manifold M^{4n} is a smooth rank 4 sub-bundle of tangent space endomorphisms

$$Q \to \mathrm{End}(TM)$$

Such that for each $x \in M$, Q_x is isomorphic to the quaternions as an associative algebra.

$$Q_x \simeq \mathbb{H}$$

This means that the tangent space T_xM can be considered a pre-quaternionic vector space.

Almost Hypercomplex Structures

While an almost quaternionic structure Q is pointwise isomorphic to the algebra of quaternions, it is not generally trivial as an algebra bundle, so it is not always possible to fix the isomorphisms smoothly and globally. When this is possible, Q is called an almost hypercomplex structure, and

$$Q = \langle 1, I, J, K \rangle$$

for a choice of global anti-commuting almost complex structures I, J, K, which satisfy the usual quaternionic equation

$$I^2 = J^2 = K^2 = IJK = -1$$

Thus any manifold which admits an almost hypercomplex structure automatically admits an almost complex structure. However, there are almost quaternionic manifolds which do not admit any almost complex structures. On the other hand, all almost quaternionic structures are locally trivializable.

Quaternionic Projective Space

The first example of an almost quaternionic manifold which is not hypercomplex is the space of quaternionic lines in $\mathbb{H}^{n+1}$:

$$\mathbb{H}P^n = \{\mathbb{H} \cdot v : v \in \mathbb{H}^{n+1} - 0\}$$

This is the quaternionic projective space $\mathbb{H}P^n$. It inherits the symmetry group $PGl(n+1, \mathbb{H})$, which has Lie algebra $\mathfrak{sl}(n+1, \mathbb{H})$ and dimension $4(n+1)^2 - 1$, from the linear action on the quaternionic vector space $\mathbb{H}^{n+1}$.

$\mathbb{H}P^n$ is compact, and in particular $\mathbb{H}P^1$ is diffeomorphic to the sphere S^4, which is known to not admit any almost complex structure for topological reasons.

Quaternionic Connections

Let the almost quaternionic structure Q be fixed and ∇ be a connection.

Definition

We call ∇ a quaternionic connection if $\nabla\Gamma(Q) \subset \Omega^1(M) \otimes \Gamma(Q)$.

If there exists a quaternionic connection which is torsion free, we call Q a quaternionic structure (or $1-$integrable). There always exists a "minimal" torsion T^Q among torsions of quaternionic connections. This T^Q is called the structure tensor of Q.

Definition

A quaternionic connection ∇ with $T^\nabla = T^Q$ is called an Oproiu connection.

The Oproiu connections are not unique.

Quaternionic Invariants

The Oproiu connections share the same torsion T^Q, but their curvatures do not coincide. However:

Definition

The Weyl curvature W^{Op} of an Oproiu connection ∇^{Op} is independent of the choice of ∇^{Op}. Therefore, W^{Op} is an invariant of the quaternionic structure. We will call this the quaternionic Weyl curvature, and denote it by W^Q.

Symmetries of Almost- Quaternionic and Hypercomplex Structures

We will mainly consider local infinitesimal symmetries of almost quaternionic structures, meaning vector fields X which preserve the bundle Q:

$$\mathcal{L}_X(\Gamma(Q)) \subset \Gamma(Q)$$

We call these quaternionic symmetries. When the quaternionic structure is almost hypercomplex, and the symmetry additionally preserves each almost complex structure, the symmetry is called a hypercomplex symmetry . Both hypercomplex and quaternionic symmetries form Lie algebras, and the former is a subalgebra of the latter. Note that the dimension of the quaternionic symmetry algebra can be strictly greater than the dimension of the hypercomplex symmetry algebra even though Q is hypercomplex.

Maximal and Sub-maximal Models

Definition

By the symmetry dimension of some structure ϕ, we mean the dimension $\dim \mathfrak{aut}(\phi)$ of its algebra of infinitesimal automorphisms $\mathfrak{aut}(\phi)$.

Observation: The most symmetric model of a type of geometric structure (for fixed dimension) is often unique, and there is a significant gap between the symmetry dimension of the maximal model and the so called sub-maximal model. It is interesting to determine the size of this gap.

Definition

The sub-maximal problem is to compute the difference between maximal and sub-maximal symmetry dimension, and to realize a sub-maximal model.

Maximal Model of AQS

The maximal model of quaternionic geometry with $n > 1$ is $\mathbb{H}P^n$ with symmetry dimension $4(n+1)^2 - 1$. This is not hypercomplex, and if a choice of local hypercomplex structure is made, the local symmetry algebra shrinks to (at most) the quaternionic affine algebra $\mathfrak{sl}(n, \mathbb{H}) \ltimes \mathbb{H}^n$.

Remark

If we consider global symmetry groups instead of algebras, the sub-maximal model is $\mathbb{H}P^n$ with one point removed, whose symmetry group is the stabilizer of the point, yielding symmetry dimension $4(n+1)^2 - 1 - 4n = 4n^2 + 4n + 3$

Main Theorem

Theorem

The submaximal symmetry dimension of quaternionic or almost quaternionic structures with fixed quaternionic dimension $n > 1$ is $4n^2 - 4n + 9$. This dimension can be realized both by a quaternionic structure and by an almost quaternionic structure with vanishing Weyl curvature. Thus the symmetry gap is $12n - 6$.

There are homogeneous sub-maximal models, and in the quaternionic case the symmetry algebra $\mathfrak{s}$ is a deformation of the graded annihilator algebra

$$\mathfrak{a} = \Big((\mathfrak{so}(2) \oplus \mathbb{R} \oplus \mathfrak{gl}(n-2, \mathbb{H})) \ltimes \mathrm{Heis}(8n-12, \mathbb{H}) \oplus \mathfrak{sp}(1) \Big) \ltimes \mathbb{H}^n,$$

such that the rightmost copy of $\mathbb{H}^n$ is non-abelian and not a sub-algebra.

The Quaternionic Sub-maximal Model: I

The quaternionic sub-maximal model is actually hypercomplex. For $n = 2$, it is given by the block matrices I, J and IJ:

$$I = \begin{pmatrix} A_I & C_I \\ 0 & B_I \end{pmatrix}, \quad A_I = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$B_I = \frac{1}{\alpha} \begin{pmatrix} 0 & h_2^2 - h_3^2 - h_4^2 & -2\,h_2\,h_4 & 2\,h_2\,h_3 \\ -h_2^2 + h_3^2 + h_4^2 & 0 & 2\,h_2\,h_3 & 2\,h_2\,h_4 \\ 2\,h_2\,h_4 & -2\,h_2\,h_3 & 0 & h_2^2 - h_3^2 - h_4^2 \\ -2\,h_2\,h_3 & -2\,h_2\,h_4 & -h_2^2 + h_3^2 + h_4^2 & 0 \end{pmatrix}$$

$$C_I = \frac{1}{\alpha}\Bigg(\frac{1}{2} \begin{pmatrix} 0 & h_2(h_2^2 + 4h_3^2 + 4h_4^2) & h_4(2h_2^2 - h_3^2 - h_4^2) & -h_3(2h_2^2 - h_3^2 - h_4^2) \\ -h_2(h_2^2 - 2h_3^2 - 2h_4^2) & 0 & h_3(4h_2^2 + h_3^2 + h_4^2) & h_4(4h_2^2 + h_3^2 + h_4^2) \\ -h_3(2h_2^2 - h_3^2 - h_4^2) & -h_4(2h_2^2 + h_3^2 + h_4^2) & h_2(3h_3^2 + h_4^2) & 2h_2h_3h_4 \\ -h_4(2h_2^2 - h_3^2 - h_4^2) & h_3(2h_2^2 + h_3^2 + h_4^2) & 2h_2h_3h_4 & h_2(h_3^2 + 3h_4^2) \end{pmatrix} +$$

$$+ \begin{pmatrix} h_2h_5 + h_3h_7 + h_4h_8 & h_2h_6 - h_3h_8 + h_4h_7 & h_2h_7 - h_3h_5 - h_4h_6 & h_2h_8 + h_3h_6 - h_4h_5 \\ -h_2h_6 - h_3h_8 + h_4h_7 & h_2h_5 - h_3h_7 - h_4h_8 & -h_2h_8 + h_3h_6 - h_4h_5 & h_2h_7 + h_3h_5 + h_4h_6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\Bigg)$$

Here $\alpha = h_2^2 + h_3^2 + h_4^2$, and $h_1, \ldots, h_8$ are local coordinates.

The Quaternionic Sub-maximal Model: J

$$J = \begin{pmatrix} A_J & C_J \\ 0 & B_J \end{pmatrix}, \quad A_J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$B_J = \frac{1}{\alpha} \begin{pmatrix} 0 & -2h_2h_3 & 2h_3h_4 & h_2^2 - h_3^2 + h_4^2 \\ 2h_2h_3 & 0 & h_2^2 - h_3^2 + h_4^2 & -2h_3h_4 \\ -2h_3h_4 & -h_2^2 + h_3^2 - h_4^2 & 0 & -2h_2h_3 \\ -h_2^2 + h_3^2 - h_4^2 & 2h_3h_4 & 2h_2h_3 & 0 \end{pmatrix}$$

$$C_J = \frac{1}{\alpha}\Big(\frac{1}{4} \begin{pmatrix} 3h_4(h_2^2 + h_3^2 + h_4^2) & h_3(h_2^2 - 5h_3^2 - 5h_4^2) & -6h_2h_3h_4 & -h_2(h_2^2 - 5h_3^2 + h_4^2) \\ h_3(5h_2^2 - h_3^2 - h_4^2) & -3h_4(h_2^2 + h_3^2 + h_4^2) & 3h_2(h_2^2 - h_3^2 + h_4^2) & -6h_2h_3h_4 \\ -h_2(h_2^2 - 5h_3^2 + h_4^2) & 2h_2h_3h_4 & h_3(h_2^2 - 5h_3^2 - h_4^2) & -h_4(3h_2^2 + 7h_3^2 + 3h_4^2) \\ 6h_2h_3h_4 & h_2(3h_2^2 + h_3^2 + 3h_4^2) & h_4(3h_2^2 - h_3^2 + 3h_4^2) & h_3(h_2^2 - h_3^2 - 5h_4^2) \end{pmatrix} +$$

$$+ \begin{pmatrix} h_2h_7 - h_3h_5 + h_4h_6 & -h_2h_8 - h_3h_6 - h_4h_5 & -h_2h_5 - h_3h_7 + h_4h_8 & h_2h_6 - h_3h_8 - h_4h_7 \\ 0 & 0 & 0 & 0 \\ h_2h_6 + h_3h_8 - h_4h_7 & -h_2h_5 + h_3h_7 + h_4h_8 & h_2h_8 - h_3h_6 + h_4h_5 & -h_2h_7 - h_3h_5 - h_4h_6 \\ 0 & 0 & 0 & 0 \end{pmatrix}\Big)$$

This sub-maximal model happens to be an invariant structure on an affine symmetric space, with the hypercomplex Obata connection coinciding with the canonical connection.

Cartan- and Parabolic Geometries

A Cartan geometry modelled on the homogeneous space G/H is a H-principal bundle $\mathcal{G} \to M$, equipped with a Cartan connection

$$\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$$

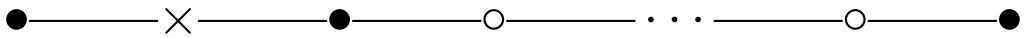
A parabolic geometry is a Cartan geometry modelled on G/P where G is a semi-simple Lie group and P is a parabolic subgroup. The main invariant of a (normal, regular) parabolic geometry is the harmonic curvature

$$\kappa_H \in C^\infty(\mathcal{G}, H^2(\mathfrak{g}_-, \mathfrak{g}))$$

a quantity derived from the Cartan curvature. From the point of view of symmetries, parabolic geometries have many useful properties. For example dim G is always the maximal symmetry dimension, and G/P is the maximal model, also called the flat model as it's locally characterized by $\kappa_H = 0$.

Almost Quaternionic Structures as Parabolic Geometry

A manifold equipped with an almost quaternionic structure is equivalent to a parabolic geometry modelled on $\mathfrak{sl}(n+1, \mathbb{H})/P_2$, where P_2 is the stabilizer of a quaternionic line in $\mathbb{H}^{n+1}$. This can be summarized by the Satake diagram



Conjugacy classes of parabolic subalgebras are in bijection with $\mathbb{Z}$-gradings of the Lie algebra. The positive and negative graded subalgebras are nilpotent, and the semi-simple Levi-factor of the 0-graded subalgebra is given by the Satake diagram obtained by removing the crossed node. In the case of almost quaternionic structures, this is

$$\mathfrak{g}_0^{ss} = \mathfrak{sp}(1) \oplus \mathfrak{sl}(n, \mathbb{H})$$

This is important to us because the harmonic curvature κ_H takes its values in a certain $\mathfrak{g}_0^{ss}$-module.

The curvature and torsion modules

For a parabolic geometry, the $\mathfrak{g}_0^{ss}$-module of possible κ_H-values can be computed by an algorithm based on Kostant's version of the Bott-Borel-Weil theorem. For almost quaternionic structures this module decomposes into two irreducible components, $\kappa_H = \kappa_1 + \kappa_2$, which can be described as (real forms of) the following lowest weight modules:

Here the numbers over the nodes indicate the components in the fundamental weight basis for $\mathfrak{g}_0^{ss}$, and the number over the cross gives the eigenvalue of the center of $\mathfrak{g}_0$. The first module is recognized as encoding the structure tensor T^Q, and the second encodes the quaternionic Weyl curvature W^Q. Thus we call them the torsion module and the curvature module.

Orbits in the Curvature Module

The curvature module is trivial with respect to $\mathfrak{sp}(1)$, and can be written abstractly as the $\mathfrak{sl}(n, \mathbb{H})$-module

$$\mathbb{V} = S^3\mathbb{H}^{n*} \odot \mathbb{H}^n = S^3\mathbb{H}^{n*} \otimes \mathbb{H}^n / S^2\mathbb{H}^{n*}$$

The curvature must be invariant with respect to the symmetry algebra. Therefore we need to find a nonzero curvature $w \in \mathbb{V}$ such that $\dim \mathrm{Stab}(w)$ is maximal. This is equivalent to w being an element of a minimal G_0-orbit in $\mathbb{V}$.

Remark

For complex representations, the minimal orbit is unique and equal to the orbit of the highest weight vector due to Borel's fix-point theorem. This fails for real representations.

Computing the Minimal Curvature Orbits

Theorem

The element $S^3 1_1^* \otimes 1_n \in \mathbb{V}$ generates a minimal orbit. Here $1^* \in \mathbb{H}^*$ is the first real basis covector and $1_n \in \mathbb{H}$ is the last real basis vector.

Lemma

The minimal orbits are closed in $P\mathbb{V}$.

The algebra acting effectively on $\mathbb{V}$ is $\mathfrak{h} := \mathfrak{sl}(n, \mathbb{H})$. We choose a special parabolic subalgebra $\mathfrak{p}_{2,n-1}$, with grading

$$\mathfrak{h} = \mathfrak{h}_- \oplus (\mathfrak{sp}(1)^2 \oplus \mathbb{R}^2 \oplus \mathfrak{sl}(n-2, \mathbb{H})) \oplus \mathfrak{h}_+$$

We show that the minimal orbit must have a limit point in $\ker \mathfrak{h}_+$. This is an eigenspace of the grading element $z \in \mathfrak{h}_0$, which is central in $\mathfrak{h}_0$, hence a $\mathfrak{h}_0$-module. $\mathfrak{sl}(n-2, \mathbb{H})$ acts trivially on this, but it is an irreducible $\mathfrak{sp}(1)^2$-module of dimension 8 with weight $(1, 3)$. Thus the minimal orbits are those that intersect a minimal $Sp(1)^2$-orbit.